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A COMPARATIVE ANALYSIS OF LANE-EMDEN EQUATION BY DIFFERENTIAL AND MULTISTEP DIFFERENTIAL TRANSFORM METHOD POWERED BY ADOMIAN POLYNOMIAL

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ABSTRACT

In this paper,we study the elliptic equation $\Delta\theta+\theta^n=0$ *with* $\theta(0)=0$ *and* $\theta'(0)=0$ *on real positive line, called Lane-Emden equation. We have solved the equation by Differential Transform and Multistep Differential Transform Methods Powered by Adomian Polynomial and then made a comparative analysis with the existing numerical results. Pade' approximation is used to improve the convergence in case of Differential Transform Method Powered by Adomian Polynomial.*

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Keywords: *Lane-Emden equation, Adomian Polynomial, Differential Transform Method, Multistep Differential Transform Method, Pade' Approximation.*

1. INTRODUCTION

In recent years, the studies of singular initial-value problems (IVPs) of the type

$$\Delta\theta+\theta^n=0, \qquad \theta(0)=1, \quad \theta'(0)=0 \tag{1}$$

have seeked the attention of many scientist form various branches [1, 2, 3, 4]. This equation shout special importance in astrophysics, because, for values of polytropic index n between 0 and 5, the equation approximate to a reasonable accuracy of the structure of many stellar models. Fundamental contribution towards the solution of this equation is made by Lane, Ritter, Kelvin, Emden and Fowler [1]. Chandrashekar [1] established the fact that the exact solutions for this equation exist only for the polytropic indices $n=0,1,5$ and until now those are the exact solutions available in the literature. For all other polytropic indices the contributions of various numerical techniques are remarkable. Semi-analytical approaches that have recently been applied in solving the Lane-Emden equations include the Adomian decomposition method [5, 6], differential transformation method [7], homotopy perturbation method [8], He's Energy Balance Method (HEBM) [9], homotopy analysis method [10, 11], power series expansions [12, 13, 14] and variational iteration method [15]. Generally, when all the above cited analytical approaches are used to solve Lane-Emden equation, a truncated power series solution of the true solution is obtained. This solution converges rapidly in a very small region $0<\xi<1$. For $\xi>1$ convergence is very slow and the solutions are inaccurate even when using a large number of terms. Convergence acceleration methods such as Padé approximations may be used to improve the convergence of the resulting series or to enlarge their domains of convergence. An important physical parameter associated with the Lane-Emden function is the location of its first positive real zero. The first zero of $\theta(\xi)$ is defined as the smallest positive value ξ_0 for which $\theta(\xi_0)=0$. This value is important because it gives the radius of a polytropic star. The analytic approaches on their own are not very useful in solving for ξ_0 because their region of convergence is usually less than ξ_0. Recent numerical methods that have been proposed include the Legendre Tau method [16] and the sinc-collocation method [17], the Lagrangian approach [18], and the successive linearization method [19]. Accurate results for the Lane-Emden function have previously been reported in [20] where the Runge-Kutta routine with self-adapting step was used to generate seven digit tables for Lane-Emden functions. These tables are now widely used as a benchmark for testing the accuracy of new methods of solving the Lane-Emden equations.

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The aim of this paper is to solve Lane-Emden equation by a new approach, Differential Transform Method Powered by Adomian Polynomial (DTMAP) and Mulltistep Differential Transform Method Powered by Adomian Polynomial (DTMAP) as a comparative study.

2. BASIC EQUATION

We begin with the equations of mass continuity and of hydrostatic equilibrium.

$$\frac{dm(r)}{dr} = 4\pi\rho(r)r^2 \tag{2}$$

$$\frac{dp(r)}{dr} = -\frac{Gm(r)}{r^2}\rho(r) \tag{3}$$

where $\rho(r) \geq 0$ is the density, $p(r)$ is the thermodynamical pressure and $m(r)$ is the mass inside radius r respectively. Since there are three unknowns (pressure, density, and mass as a function of radius) and only two equations, in order to get a solution we must either add more equations (i.e. energy generation and transfer) or introduce an additional assumption. For a polytrope, pressure and density are related by a power law of the form

$$p = K\rho^{\frac{(n+1)}{n}} \tag{4}$$

where $n \neq 0$ is called the polytropic index. K and n are constants. This set of three equations (2)-(4) can then be reduced to a single differential equation and solved. By eliminating the mass berween the equations (2) and (3), we obtain a single second order nonlinear differential equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho \tag{5}$$

We represent density in terms of a new dimensionless variable θ by

$$\rho = \rho_c\theta^n \tag{6}$$

and then rescale the radial variable by the constant α such that

$$r = \alpha\xi \quad and \quad \alpha = \sqrt{\frac{(n+1)K\rho_c^{\frac{1}{n-1}}}{4\pi G}}, \quad n \neq -1 \tag{7}$$

Relations (6) and (7) transform the equation (5) into the form of Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{dr}\left(\xi^2\frac{d\theta}{dr}\right) = -\theta^n \tag{8}$$

Equation (8) with initial condition $\theta(0) = 1, \theta^{'}(0) = 0$ has a wide range of applications in celestial mechnics and has beed studied extensively in the literature [1, 2, 4].

3. SOLUTION OF LANE-EMDEN EQUATION BY DIFFERENTIAL TRANSFORM METHOD POWERED BY ADOMIAN POLYNOMIAL

Definition 3.1: *Let $y(x)$ be the original analytic function and differentiated continuously in the domain of interest. Then Differential Transform of y(x) is defined as:*

$$Y_k = \frac{1}{k!}\left[\frac{d^k}{dx^k}y(x)\right]_{x=0} \tag{9}$$

where $y(x)$ is the original function and Y_k is the transformed function.

Definition 3.2: *Differential inverse transform of Y_k is defined as :*

$$y(x) = \sum_{k=0}^{\infty} Y_k x^k \tag{10}$$

Combining (9) and (10) we may write

$$y(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!}\left[\frac{d^k y}{dx^k}\right]_{x=0} \tag{11}$$

This implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically. Instead, relative derivatives are calculated by a recurrence relation which are described by the transformed equations of the original functions. Some fundamental transformations, which can be readily obtained are listed in the following table.

Table -1: Fundamental Operations in DTM

Original form	Transformed form
$y(x) = w(x) \pm v(x)$	$Y_k = W_k \pm V_k$
$y(x) = \alpha w(x)$	$Y_k = \alpha W_k$
$u(x) = \frac{d^m}{dx^m} w(x)$	$Y_k = \frac{(k+m)!}{k!} W_{k+m}$
$y(x) = x^n$	$Y_k = \delta(k-n)$ where $\delta(k-n) = \begin{cases} 1 & \text{ifk = n} \\ 0 & \text{otherwise} \end{cases}$
$y(x) = w(x)v(x)$	$Y_k = \sum_{r=0}^{k} W_r V_{k-r}$

To illustrate the basic concepts of the Differential Transform Method powered by Adomian Polynomial(DTMAP), we consider a general nonlinear ordinary differential equation with initial conditions of the form

$$Dy(x) + Ny(x) = g(x) \qquad (12)$$

with initial conditions

$$\frac{d^i y(0)}{dx^i} = c_i, \quad i = 0,1,2,...m-1$$

where D is the m^{th} order linear differential operator $D = \frac{d^m}{dx^m}$, N represents the general nonlinear differential operator and g(x) is the source term.

According to DTM, we can construct the following iteration formula:

$$(k+1)(k+2)...(k+m)Y_{k+m} = G_k - NY_k$$

with initial condition

$$Y_i = c_i, \quad i = 0,1,2,...m-1$$

But, according to Differential Transform powered by Adomian Polynomial method, we construct the iteration formula as

$$(k+1)(k+2)...(k+m)Y_{k+m} = G_k - A_k \qquad (13)$$

with initial condition

$$Y_i = c_i, \quad i = 0,1,2,...m-1 \qquad (14)$$

The Adomian Polynomial A_k defined as

$$A_k = \frac{1}{k!} \frac{d^k}{d\lambda^k} [N(\sum_{i=0}^{k} \lambda^i Y_i(x,t)]|_{\lambda=0} \qquad (15)$$

is the decomposition of the nonlinear operator Ny. The general formula (15) can be decomposed as follows:

$$A_0 = N(Y_0)$$

$$A_1 = Y_1 N(Y_0)$$

$$A_2 = Y_2 N^{'}(Y_0) + \frac{1}{2!} Y_1^2 N^{''}(Y_0)$$

$$A_3 = Y_3 N^{'}(Y_0) + Y_1 Y_2 N^{''}(Y_0) + \frac{1}{3!} Y_1^3 N^{'''}(Y_0), ...$$

Substituting (14) and (15) into (13) and then by iteration we obtain the succeeding value of Y_k. Then, the inverse transformation of the set of values $\{Y_k\}_{k=0}^{n}$ gives the n-term approximation to solution as follow:

$$y_n(x)=\sum_{k=0}^{n} Y_k x^k \tag{16}$$

Therefore the exact solution of the problem is given by

$$y(x)=\lim_{n\to\infty} y_n(x) \tag{17}$$

Example 3.1: *Consider the Lane-Emden equation*

$$\theta''+\frac{2}{\xi}\theta' +\theta^n = 0 \tag{18}$$

subject to the initial conditions

$$\theta(0)=1, \quad \theta'(0)=0 \tag{19}$$

Multiplying both sides of equation (18) by ξ ,

$$\xi\theta''+2\theta' +\xi\theta^n = 0 \tag{20}$$

By using above theorems of DTM and the DTM powered by Adomian Polynomial method we obtained the following recurrence relation

$$Y_{k+1}=-\frac{A_{k-1}}{(k+1)(k+2)}, \quad k\ge 1 \tag{21}$$

where A_k represented the Adomian Polynomial applied for decomposing the nonlinear term such that

$$A_k=\frac{1}{k!}\frac{d^k}{d\lambda^k}\left(\sum_{i=0}^{k}\lambda^i Y_i\right)^n \tag{22}$$

From Eq.(14), the initial conditions given in Eq.(19) can be transformed as

$$Y_0=1, \quad Y_1=0 \tag{23}$$

Substituting Eq.(22) and Eq.(23) into Eq.(21) and then by direct iteration steps we obtain the following:

$$\begin{aligned}
&Y_0=1,\quad Y_1=0,\quad Y_2=-\frac{1}{3!},\quad Y_3=0,\quad Y_4=\frac{n}{5!},\quad Y_5=0,\quad Y_6=-\frac{n(8n-5)}{3(7!)},\\
&Y_7=0,\quad Y_8=\frac{n\left(122n^2-183n+70\right)}{9(9!)},\quad Y_9=0,\\
&Y_{10}=-\frac{n\left(5032n^3-12642n^2+10805n-3150\right)}{45(11!)},\quad Y_{11}=0,\\
&Y_{12}=\frac{n\left(183616n^4-663166n^3+915935n^2-574850n+138600\right)}{135(13!)},\quad Y_{13}=0\\
&Y_{14}=\frac{n\left(-21625216n^5+103178392n^4-200573786n^3+199037015n^2-101038350n+21021000\right)}{945(15!)}
\end{aligned} \tag{24}$$

The Pade' approximant for the solution is

$$\theta^{[5,5]}=\frac{\dfrac{\left(178n^2-951n+1250\right)\xi^2}{108(17n-50)}+\dfrac{\left(1290n^3-10849n^2+29100n-24500\right)\xi^4}{45360(17n-50)}+1}{\dfrac{\left(178n^2-645n+350\right)\xi^2}{108(17n-50)}+\dfrac{\left(86n^3-321n^2+190n\right)\xi^4}{3024(17n-50)}+1} \tag{25}$$

Chandrashekar [1] shows that equation (18) has exact solution only for the values of n as (0, 1, 5). For $n=0$, $n=1$ and $n=5$ we get the exact solution

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2$$

$$\theta(\xi) = \frac{\sin\xi}{\xi}$$

and $$\theta(\xi) = \frac{1}{\sqrt{1+\frac{\xi^2}{3}}}$$

Table -2: Comparison of radius ξ_1 obtained by our method and the numerical result [1]

n	ξ_1	**Numerical**
1	2.44949	2.44949
0.5	2.75226	2.7528
1	3.14159	3.14159
1.5	3.65398	3.65378
2	4.35287	4.35287
2.5	5.3474	5.35528
3	6.8936	6.89685
2.25	8.09456	8.01894
2.5	9.87529	9.53581
4	21.8937	14.97155
4.5	6465.83	31.83646

Table-3: Comparison of $\theta(\xi)$ obtained as Pade approximation by our method and the numerical result [20] for $n=1.5$

ξ	$P_{1.5}$	**Numerical**
0	1.0000	1.0000
1	0.84517	0.84517
2	0.495937	0.495937
3	40.158858	0.158858

Table-4: Comparison of $\theta(\xi)$ obtained as Pade approximation (P_3 **)** by our method and the numerical result [20] for $n=3$

ξ	P_3	**Numerical**
0	1.0000	1.0000
0.25	0.98968	0.98975
0.50	0.959839	0.95987
0.75	0.913542	0.91355
1.00	0.855058	0.85505
1.25	0.78898	0.78897
1.50	0.719502	0.71948
1.75	0.649988	0.64990
2.00	0.582851	0.58282
2.50	0.461127	0.46109
3.00	0.359226	0.35921
3.5	0.276263	0.27629
4.0	0.209281	0.20942
4.5	0.155067	0.15529
5.0	0.110813	0.11110
5.0	0.043698	0.04411
6.9011	0.017784	0.0000

Table-5: Comparison of $\theta(\xi)$ obtained as Pade approximation by our method and the numerical result [20] for $n = 4.5$

ξ	$\theta(\xi)$	**Numerical**
0	1.0000	1.0000
2	0.639654	0.639654
4	0.36078	0.3605326
6	0.233668	0.231356
8	0.1685385	0.1617233
10	0.131611	0.1198299

It is observed from the Table 2 that with DTMAD and Pade' approximation we obtain quite good results for $0 \le n \le 3$. Also in Table 3 we present the comparison of values of θ for various values of ξ for $n = 1.5$ with the numerical solution and observed that DTMAP with Pade' approximation always converges for smaller region and it has slow convergent rate or completely divergent in wider region. We observe that the results obtained by using DTMAP and Pade approximation have good agreement with the numerical results for $0 \le n \le 3$. For polytrope greater than 3 only for smaller values of ξ approximately 2, the values agree with numerical results. For $\xi > 2$ the rate of convergence is very slow. To overcome this drawback and to obtain efïficient solutions which agree for any polytrope and for all values ξ, in the following section we have solved the Lane-Emden equation by multistep DTMAP and Pade' approximation.

4. SOLUTION OF LANE-EMDEN EQUATION BY MULTIST EP DIFFERENTIAL TRANSFORM METHOD POWERED BY ADOMIAN POLYNOMIAL

We consider the initial value problem (IVP) (12) defined in section 3. Let $[0,T]$ be the interval over which we want to find the solution of the IVP. In actual application of DTM, the approximate solution of the IVP (12) can be expressed by the finite series (16) for $x \in [0,T]$.

In the multistep approach, the interval $[0,T]$ is divided into M subintervals $[x_{m-1}, x_m]$, $m = 1,2,3,...M$ for equal step size $h = T/M$ by using the nodes $x_m = mh$. The main idea of the multistep DTMAP to equation (12) is as follows. First apply the DTMAP to equation (12) over the interval $[0, x_1]$ and we shall obtain the following approximate solution.

$$y_1(x) = \sum_{k=0}^{K} Y_{1k} x^k \tag{26}$$

using the initial conditions $y_1^{(i)}(0) = c_i$. For $m > 2$ and at each of the subinterval $[x_{m-1}, x_m]$ we shall use the initial conditions $y_m^{(i)}(x_{m-1}) = y_{m-1}^{(i)}(x_{m-1})$ and apply DTMAP to equation (12) over the interval $[x_{m-1}, x_m]$. The process is repeated and generates a sequence of approximate solutions $y_m(x)$, $m = 1,2,...M$ for the solution $y(t)$.

$$y_m(x) = \sum_{k=0}^{K} Y_{mk} (x - x_{m-1})^k \tag{27}$$

where $n = K.M$. In fact, the multistep DTMAP assumes the following solution.

$$y(x) = \begin{cases} y_1(x), & x \in [0, x_1] \\ y_1(x), & x \in [x_1, x_2] \\ \vdots & \\ y_M(x), & x \in [x_{M-1}, x_M] \end{cases}$$

In the next section, we shall solve Lane-Emden equation by using multistep DTMAP and see that the obtained series solution has a wide range of convergence.

According to multistep DTMAP, taking $N = K.M$, the series solution for Lane-Emden equation (20) is given by,

$$\theta(\xi) = \begin{cases} \sum_{i=0}^{K} Y_{1i}\xi^i, & \xi \in [0, \xi_1] \\ \sum_{i=0}^{K} Y_{2i}(\xi - \xi_1)^i, & \xi \in [\xi_1, \xi_2] \\ \vdots & \\ \sum_{i=0}^{K} Y_{Mi}(\xi - \xi_{M-1})^i, & \xi \in [\xi_{M-1}, \xi_M] \end{cases}$$

where $Y_{ji}, j = 1,2, M$ satisfy the following recurrence relation. such that $Y_{i0} = Y_{(i-1)0}$

$$A_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}(\sum_{i=0}^{k}\lambda^i Y_{jk})^n, \quad j = 1,2, M \tag{28}$$

We have solved equation (20) for $n = 3$ and $n = 4.5$ by multistep DTMAP. Results are presented in the following Table 6. It is to be noted that multistep DTMAP results are obtained when $K = 10$, $M = 100$ and $T = 20$. The Numerical values are talen from [1,20].

Table-6: Comparison of $\theta(\xi)$ obtained as Pade approximation by our method and the numerical result [18] for $n = 3$

ξ	$\theta(\xi)$	**Numerical**
0	1.0000	1.0000
0.25	0.98975	0.98975
0.50	0.95987	0.95987
0.75	0.91355	0.91355
1.00	0.85505	0.85505
1.25	0.78897	0.78897
1.50	0.71948	0.71948
1.75	0.64990	0.64990
2.00	0.58282	0.58282
2.50	0.46109	0.46109
3.00	0.35921	0.35921
3.5	0.27629	0.27629
4.0	0.20941	0.20942
4.5	0.15524	0.15529
5.0	0.11113	0.11110
6.0	0.04423	0.04411
6.9011	0.00628	0.0000

5. CONCLUSION

In this work, we propose DTMAP and Multistep DTMAP to solve Lane-Emden equation. Even though DTMAP is used in the presence of Pade' approximation, it is observed that Multistep DTMAP improves the convergence of the series solution.Table 4, 5, 6 show the advantage of Multistep DTMAP over DTMAP.

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