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ADDITIVE FUNCTIONS ON THE SUM OF PRIMES ARE SET OF SQUARES FOR MULTIPLICATIVE FUNCTIONS

MOHD. FAREEDUDDIN*, E. KESHAVA REDDY

Department of Mathematics, JNTUA, Anantapur, Andhra Pradesh, India.

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ABSTRACT

If additive condition involves more than four squares, then such functionsr is uniquely determined.In this theorem, assume that a multiplicative functions g is 5-additive on positive squares.

$$g(a^2 + b^2 + c^2 + d^2 + e^2) = g(a^2) + g(b^2) + g(c^2) + g(d^2) + g(e^2)$$

for arbitrary positive integers a, b, c, d and e, then g is an identity function.

Keywords: *Additive and multiplicative functions, primes, arbitrary positive squares, arithmetic, harmonic function.*

1. INTRODUCTION

Arithmetic functions involves a real or complex valued function a(n) to determine on the natural numbers set that states some arithmetic property of n. The most significant arithmetic functions are additive and multiplicative functions. Acomplete additive arithmetic function involves f(p q) = f(p) + f(n) for all coprime natural numbers p and q; and complete multiplicative arithmetic functions involves f(p q) = f(p) f(q) for all coprime natural numbers p and q. Prime numbers are the main factor for the multiplicative structure of all natural numbers [1].

The Fundamental theorem of Arithmetic (FTA) is the important theorem that deals with each natural number n > 1 that can be denoted as a product of distinct prime's factors. Additive Prime number Theory concerned with the illustration of integers as sum of primes or of closely related integers. The primes are defined by multiplicative properties whereas the problem involves additive properties. Numerous methods are used in this Additive theory such as: elementary methods, analytic methods, probabilistic methods, harmonic and functional analysis [2].

All positive integers can be formulated uniquely as a multiple of prime numbers. The Dirichlet's theorem implies that gcd (a, b) = 1, then ax+b is a prime for infinitely many values of x. This theorem states that lower than x value involves an asymptotically x/log(x) primes [3].

2. LITERATURE REVIEW

Poo-Sung Park (2016) analysis about the Multiplicative functions involves withthe sum of squares. This paper considers the integer k which is larger than or equivalent to 4. The author mentions additive and multiplicative function to satisfy the theorem. The multiplicative function f for all positive integers' valuesinvolves $f(x^2{}_1 + x^2{}_2 + ... + x_k{}^2) = f(x_1)^2 + f(x_2)^2 + ... + f(x_k)^2$. Arithmetic function $f(x): N \to C$ multiplicative if $f(1) = 1$ and $f(ab) = f(a)f(b)$ such*a* and*b* are relatively prime. This paper explainsabout multiplicative operation that includes sum of fixed number of squares.

Dubickas. A and Sarka. P (2012) describes on the multiplicative property that sum of primes involves additive functions. The author analysis the function $f(x): N \to C$ proving $f(p_0) \neq 0$ for relatively one prime number p_0 and k ≥2 is an integer. This paper illustrates the theorem such that

$$f(p_1 + p_2 + ... + p_k) = f(p_1) + f(p_2) + ... + f(p_k)$$

for every prime numbers $p_1, p_2 .... p_k$ then f must be an identity *f(n)=n* for each n $\in$ N. This paper involves the depiction of integers as sum of primes and two auxiliary lemmas for functions being additive on sums of primes.

Corresponding Author: Mohd. Fareeduddin,*
Department of Mathematics, JNTUA, Anantapur, Andhra Pradesh, India.

Clark, L, Pexamined about the multiplicative functions of the arithmetic operations. The multiplicative function is mainproperty of the arithmetic operations. A multiplicative function *f* satisfying this condition $f(1) \neq 0$ is called as multiplicative. The author describes about the definition and basic operations of the multiplicative functions, multiplicativity theorem and CRT and the multiplicativity of the totient. Further the author discuss about the additive functions, sum of squares and perfect numbers.

Poo-Sung Park (2016) examined about themultiplicative functions that satisfies the k-Additive uniqueness of the set of squares. The author analyzed that there are several multiplicative functions *f* to prove this condition $f\left(p^2 + q^2\right) = f\left(p^2\right) + f\left(q^2\right)$ for all integers' p and q. This paper proves that function *f* willbe uniquely determined whenever additiveconditions involve more than two squares.

Bojan Basic (2014) analyzed about the depiction of Arithmetic functions that satisfies the sum-of-squares operation. The author describes the theorem $f(m^2+n^2) = f(m)^2 + f(n)^2$ for all *m*, $n \in$ N. Therefore all functions can be categorized into three families, such as $f \equiv 0$, $f(n) = \pm n$ and $f(n) = \pm \frac{1}{2}$.

Granville. A and Soundarajan (2006) described about the multiplicative functions and a Variation for the Zeta-Function. In this paper, severalvital outcomes in multiplicative number theory may be restatedcertainly in terms of multiplicative functions *f* that profess to be another multiplicative function *g*. This paper involves formalizing a distance that offers a degree of such pretentiousness and as outcome gets a curiouszeta-function inequality.

Bui Minh Phong (2004) expressed about the groupsillustrating the identity function. In this paper IN, IN_0 and P represents the group of positive integers, non-negative integers and prime numbers respectively. M represents the multiplicative functions set *f* such that $f(1) = 1$. Further, this is associated with the set B $\subset IN$ involves a non-negative numbers which can be characterized as a summation of two squares of numbers and S represents the group of all squares of positive numbers. The author proves that function *f* with $f(4)\, f(9) \neq 0$ and a positive integer L analyses the following condition $f(p^2+q^2+r) = f(p^2) + f(q^2+r)$ for all n, m$\in IN$, then *f(m)*=m for all positive integers n, (M, 2L) = 1.

Bui Minh Phong (2013) focused about the identity function. This paper analyses the following equation $f(n^2+m^2+p+q) = g(n^2+p) + g(m^2+q)$ such that p, qare non-negative integers with p + q > 0 and *f*, *g* are multiplicative functions. In this equation, the set (p, q) denotes the greatest common divisor of the integer's m, n. The author considers the two theorems to prove the identity functions.

Indlekofer. K and Bui Minh Phong (2006) describe about themultiplicative functions that includes additive uniqueness. This paper proved that a multiplicative function F and a positive integer k satisfy F(2)$\neq$ 0, F(5) $\neq$1 and $F(n^2+m^2+k+1) = F(n^2+k) + F(m^2+1)$ for all n, m $\in IN$, then F(n)= n for all positive integers n, (n,2) = 1.

3. DEFINITION AND BASIC PROPERTIES

The important basic property of arithmetic functions is multiplicativity.

An arithmetic functions *g* is called to be multiplicative if $g(1) \neq 0$ and $g\left(m_1, m_2\right) = g\left(m_1\right).g\left(m_2\right)$ for all relatively prime positive integers m_1 and m_2.

An arithmetic functions *g* is called to be additive if $g(1) \neq 0$ and $g\left(m_1, m_2\right) = g\left(m_1\right) + g\left(m_2\right)$ for all relatively prime integers $m_1$and m_2.

The equations is generally expressed as

$$g\left(m_1^2 + m_2^2 + ... + m^2{}_n\right) = g\left(m_1\right)^2 + g\left(m_2\right)^2 + ... + g(m_n)^2$$

for all positive integers.

4. THREE ADDITIVITY FUNCTION

Assume that s multiplicative function *g* is 3-additive on positive squares. The function is, $g(p^2+q^2+r^2) = g(p^2) + g(q^2) + g(r^2)$ for arbitrary positive integers p, q, r.

Theorem 1: g (m) = m for $1 \le m \le 12$ and m = 25

Proof: It is trivial that $g(3) = 3$ and $g(1) = 1$. Solving the following equations

$$g(6) = g(2)\, g(3) = 3g(2) = g(1^2 + 1^2 + 2^2) = g(1) + g(1) + g(4) = 2 + g(4)$$

$$g(9) = g(1^2 + 2^2 + 2^2) = 1 + 2g(4)$$

$$g(11) = g(1^2 + 1^2 + 3^2) = 2 + g(9)$$

$$g(14) = g(2)\, g(7) = g(1^2 + 2^2 + 3^2) = 1 + g(4) + g(9)$$

$$g(18) = g(2)\, g(9) = g(1^2 + 1^2 + 4^2) = 2 + g(16)$$

$$g(21) = g(3)\, g(7) = 3g(7) = g(1^2 + 2^2 + 4^2) = 1 + g(4) + g(16)$$

$$g(22) = g(2)\, g(11) = g(2^2 + 3^2 + 3^2) = g(4) + 2g(9),$$

By this equation, obtain that

$$g(2) = 2,\ g(3) = 3,\ g(4) = 4,\ g(7) = 7,\ g(9) = 9,\ g(11) = 11,\ g(16) = 16$$

From this equalities

$$g(27) = g(3^2 + 3^2 + 3^2) = 3g(9) = 27 = g(1^2 + 1^2 + 5^2) = 2 + g(25)$$

By obtain that $g(25) = 25$.

Then the equalities

$$g(30) = g(2)\, g(3)\, g(5) = 6g(5) = g(1^2 + 2^2 + 5^2) = 5 + g(25) = 30$$

Yields $g(5)=5$. Also $g(8)=8$ follows the equalities

$$g(24) = g(3)g(8) = 3g(8) = g(2^2 + 2^2 + 4^2) = g(4) + g(4) + g(16) = 24.$$

Hence the 3-additivity equation is proved.

Theorem 2: $g(2^a) = 2^a$, $g(3^a) = 3^a$, $g(5^a) = 5^a$ for all positive integer a.

Proof:

$$g(3^{2x+1}) = g(3^1.3^{2x}) = g(3.3^{2x}) = g(3.(3^x)^2) = g((3^x)^2+(3^x)^2+(3^x)^2) = 3g(3^{2x})$$

$$g(3^{2x+2})=g(3^2.3^{2x})= g(9.3^{2x})= g(9.(3^x)^2) = g((3^x)^2+2^2(3^x)^2+2^2(3^x)^2) = g(3^x)^2[1+4+4] = 3^2g(3^{2x}).$$

Now consider $g(2^x)$. From the equalities

$$g(9.2^{2x}) = 9\, g(2^{2x}) = 9(g(2^x)^2) = g((2^x)^2 + 2^2(2^x)^2 + 2^2(2^x)^2) = g\,(2^{2x}) + 2g(2^{2x+2})$$

By this equations, get the term $g(2^{2x+1}) = 2.g(2^{2x})$ and $g(2^{2x+2}) = 2^2.g(2^{2x})$

Similarly, from the equalities

$$g(30.5^{2x}) = g(6)\, g(5^{2x+1}) = 6(g(5^{x+1}) = g((5^x)^2 + 2^2(5^x)^2 + 5^2(5^x)^2) = 5g(5^{2x}) + g(5^{2x+2})$$

This equation conclude that $g(5^x) = 5^x$

5. FOUR ADDITIVITY FUNCTIONS

In this theorem, assume that a multiplicative functions g is 4-additive on positive squares.

$$g(a^2 + b^2 + c^2 + d^2) = g(a^2) + g(b^2) + g(c^2) + g(d^2)$$

for arbitrary positive integers a, b, c, d.

Theorem: $g(m) = m$ for m=1, 3, 5, 9, 11, 29, 41.

Proof: Consider $g(1) = 1$ and $g(4) = 4$. Since

$$g(10) = g(2)g(5) = g(1^2+1^2+2^2+2^2) = 2g(1) + 2g(4) = 10$$

$$g(12) = g(4)g(3) = g(1^2+1^2+1^2+3^2) = 3g(1) + g(9) = 3 + g(9)$$

$g(15) = g(3)g(5) = g(1^2+1^2+2^2+3^2) = 2g(1) + g(4) + g(9) = 6 + g(9)$

$g(18) = g(2)g(9) = g(1^2+2^2+2^2+3^2) = g(1) + 2g(4) + g(9) = 9 + g(9)$

$g(20) = g(4)g(5) = 4g(5) = g(1^2+1^2+3^2+3^2) = 2g(1) + 2g(9) = 2 + 2g(9)$,

This infers that

$g(2) = 2, g(3) = 3, g(5) = 5, g(7) = 7, g(9) = 9$

Also, consider $g(16) = g(2^2+2^2+2^2+2^2) = 16$. Then $g(11) = 11$ from

$g(22) = g(2)g(11) = g(1^2+1^2+2^2+4^2) = 22$.

Similarly, $g(17) = 17$, $g(29) = 29$, and $g(41) = 41$.

Computation of $g(\mathrm{m}) = m$ inductively for other m except 2.4^k, 6.4^k and 14.4^k.

$g(2.4^k) = 2.4^k$

$g(6.4^k) = g(3)g(2.4^k) = 6.4^k$

$g(14.4^k) = g(7)g(2.4^k) =14.4^k$.

Hence the equation is proved.

6. PROOF OF THEOREM FOR FIVE ADDITIVITY FUNCTIONS

If more than four squares in the additive condition are involved, then such r is uniquely determined.In this theorem, assume that a multiplicative function$g$ is 5-additive on positive squares. $g(a^2 + b^2 + c^2 + d^2 + e^2) = g(a^2) + g(b^2) + g(c^2) + g(d^2) + g(e^2)$ for arbitrary positive integers a, b, c, d and e.Every positive integer can be denoted as sum of five positive squares except 1, 2, 3, 4, 6, 7, 9, 10, 12, 15, 18 and 33.

Theorem: $g(\mathrm{m}) = \mathrm{m}$ for $1 \leq \mathrm{m} \leq 16$.

Proof: Consider $g(1) = 1$ and $g(5) = 5$.

Solving the system of equations,

$g(8) = g(1^2 + 1^2 + 1^2 + 1^2 + 2^2) = 4 + g(4)$

$g(11) = g(1^2 + 1^2 + 1^2 + 2^2 + 2^2) = 3 + 2g(4)$

$g(13) = g(1^2 + 1^2 + 1^2 + 1^2 + 3^2) = 4 + g(9)$

$g(14) = g(2)g(7) = g(1^2 + 1^2 + 2^2 + 2^2 + 2^2) =2 + 3g(4)$

$g(16) = g(1^2 + 1^2 + 1^2 + 2^2 + 3^2) = 3 + g(4) + g(9)$

$g(20) = g(4)g(5) = 5\, g(4) = g(1^2 + 1^2 + 1^2 + 1^2 + 4^2) = 4 + g(16)$

$g(21) = g(3)g(7) = g(1^2 + 1^2 + 1^2 + 3^2 + 3^2) = 3 + 2g(9)$

$g(22) = g(2)g(11) = g(1^2 + 2^2 + 2^2 + 2^2 + 3^2) = 1 + 3\, g(4) + g(9)$

$g(24) = g(8)g(3) = g(1^2 + 1^2 + 2^2 + 3^2 + 3^2) = 2 + g(4) + 2g(9)$

$g(26) = g(2)g(13) = g(1^2 + 1^2 + 2^2 + 2^2 + 4^2) = 2 + 2\, g(4) + g(16)$

Thus the function $g(\mathrm{m}) = \mathrm{m}$ for $1 \leq \mathrm{m} \leq 16$.

It is clear that $g(18) = g(2)\, g(9) = 18$ and $g(33) = g(3)\, g(11) = 33$. Since other numbers can be expressible as sums of 5 positive squares, we can conclude that g is the identity function by induction.

CONCLUSION

This paper mainly discusses about the additive and multiplicative functions. The theorem concerned about functions with more than four squares in the additive condition is involved, and then such function is uniquely determined. Moreover three additivity, four additivity and five additivity theorem is proved for positive square numbers that involves multiplicative functions.

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