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GENERALIZED INVEX FUNCTIONS AND THEIR PROPERTIES IN CONTINUOUS TIME PROGRAMMING

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ABSTRACT

Notions of invexity as function and as set are generalized. The pre–invex functions with respect to η (without differentiability) have been generalized. In this paper, the class of pre–invex functions introduced under semi-continuity conditions for continuous time programming. Pre–invexity for a function can be achieved via an intermediate-point check. A characterization of a pre–invex function in terms of its relationship with an intermediate-point, pre–invexity and prequasi–invexity are extensions of the case of the treatment provided in the context of continuous time programming.

Key Words: *Pre–invex function with respect to η; Prequasi–invex function w.r.t . η; Upper semi-continuous function; Lower semi continuous function .*

Key No. : *90C46, 90C15*

1. INTRODUCTION:

Convexity plays a key role in many aspects of mathematical programming including sufficient optimality conditions and duality theorems. An invex functions is one of the generalized convex functions and this was introduced by Hanson [15]. For more details on convexity/ invexity in continuous time programming interested readers may consult [1-29].

2. MATERIAL AND METHODS:

Definition: [2.1] A set $S \subseteq I \times R^n \times R^n \times R^n \times R^n$ is said to be invex if there exist a vector function $\eta : S \to R^n$ such that for all

$$(t, x, \dot{x}, y, \dot{y}), (t, u, \dot{u}, y, \dot{y}) \in S, \lambda \in [0,1] \Rightarrow (t, u + \lambda\eta(t, x, \dot{x}, u, \dot{u}), \dot{u} + \lambda\frac{d}{dt}\eta(t, x, \dot{x}, u, \dot{u}), y, \dot{y}) \in S .$$

A convex set is an invex set by taking $\eta(t, x, \dot{x}, u, \dot{u}) = x - y$, $\frac{d}{dt}\eta(t, x, \dot{x}, u, \dot{u}) = (\dot{x} - \dot{u})$ and but one can see with an example that converse is not true.

Example: 2.2 Let $S = [t, -3-3t, -2-2t] \cap [t, -1-t, 2, 2t]$ and

$$\eta(t, x, \dot{x}, u, \dot{u}) = \begin{cases} x - y, if\, 2 \ge x \ge -1, 2 \ge y \ge -1 \\ x - y, if\, -3 \le x \le -2, -3 \le y \le -2 \\ -2 - y, if\, -1 \le x \le 2, -3 \le y \le -2 \\ -1 - y, if\, -3 \le x \le -2, -1 \le y \le 2 \end{cases} .$$

Let $S = [t, -3-3t, -2-2t] \cap [t, -1-t, 2, 2t]$ and

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$$\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})=\begin{cases}\dot{x}-\dot{y}, if\ 2t\ge x\ge -t, 2t\ge y\ge -t\\ \dot{x}-\dot{y}, if\ -3t\le x\le -2t, -3t\le y\le -2t\\ -2-\dot{y}, if\ -t\le x\le 2t, -3t\le y\le -2t\\ -1-\dot{y}, if\ -3t\le x\le -2t, -t\le y\le 2t\end{cases}$$

Then S is an invex set w. r. t. η, but it is obvious that S is not a convex set .

Definition: 2.3 Let $S\subseteq I\times R^n\times R^n\times R^n\times R^n$ be pre–invex set with respect to $\eta : S\times S\to R^n$.Let $f : S\to R$ we say that f is pre–invex_ if

$$f(t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})\le\lambda f(t,x,\dot{x},y,\dot{y})+(1-\lambda)f(t,u,\dot{u},y,\dot{y}).$$

For all $(t,x,\dot{x},y,\dot{y}),(t,u,\dot{u},y,\dot{y})\in S,\lambda\in[0,1]$.

Definition: [2.4] Condition C

Let $\eta : S\to I\times R^n\times R^n\times R^n\times R^n$ be a vector function defined on invex set $S\subseteq I\times R^n\times R^n\times R^n\times R^n$.We say that the function η satisfies condition C if , for any $(t,x,\dot{x},y,\dot{y}),(t,u,\dot{u},y,\dot{y})\in S,\lambda\in[0,1]$.

C_1: $\eta(t,u,\dot{u},u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})) = -\lambda\eta(t,x,\dot{x},u,\dot{u})$

C_2: $\eta(t,u,\dot{u},u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})) = (1-\lambda)\eta(t,x,\dot{x},u,\dot{u})$

Definition: [2.5] Condition D

Let $\eta : S\to I\times R^n\times R^n\times R^n\times R^n$ be a vector function defined on invex set $S\subseteq I\times R^n\times R^n\times R^n\times R^n$. We say that the function η satisfies condition D if, for any $(t,x,\dot{x},y,\dot{y}),(t,u,\dot{u},y,\dot{y})\in S,\lambda\in[0,1]$.

D_1: $\frac{d}{dt}\eta(t,u,\dot{u},u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})) = -\frac{d}{dt}\lambda\eta(t,x,\dot{x},u,\dot{u})$

D_2: $\frac{d}{dt}\eta(t,u,\dot{u},u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})) = (1-\lambda)\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})$

for all $\lambda\in[0,1]$ hold .

Definition: [2.6] Let $\eta : S\to I\times R^n\times R^n\times R^n\times R^n$ be a vector function. A function $f : S\to R$ defined on invex set $S\subseteq I\times R^n\times R^n\times R^n\times R^n$ with respect to η at $(t,u,\dot{u},y,\dot{y})\in S$ on S is said to be pre–invex if for any $(t,x,\dot{x},y,\dot{y})\in S,\lambda\ge 0,(1-\lambda)\ge 0$, if the following inequality is satisfied,

$$f(t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})\le\lambda f(t,x,\dot{x},y,\dot{y})+(1-\lambda)f(t,u,\dot{u},y,\dot{y})$$

for all $(t,x,\dot{x},y,\dot{y}),(t,u,\dot{u},y,\dot{y})\in S,\lambda\in[0,1]$.

Definition: [2.7] Let $S\subseteq I\times R^n\times R^n\times R^n\times R^n$ be an invex set w.r.t. $\eta : S\to I\times R^n\times R^n\times R^n\times R^n$. We say that f is prequasi–invex function, if for all $(t,x,\dot{x},y,\dot{y}),(t,u,\dot{u},y,\dot{y})\in S,\lambda\in[0,1]$ and

$f(t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})\le\max\{f(t,x,\dot{x},y,\dot{y}),f(t,u,\dot{u},y,\dot{y})\}$, implies

$$f(t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})\le\lambda f(t,x,\dot{x},y,\dot{y})+(1-\lambda)f(t,u,\dot{u},y,\dot{y})$$

for all $(t,x,\dot{x},y,\dot{y}),(t,u,\dot{u},y,\dot{y})\in S,\lambda\in[0,1]$.

Definition: [2.8] f is strictly prequasi–invex, if for all, $(t,x,\dot{x},y,\dot{y}),(t,u,\dot{u},y,\dot{y})\in S,\lambda\in[0,1]$.

$$f(t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})\le \max\{f(t,x,\dot{x},y,\dot{y}),f(t,u,\dot{u},y,\dot{y})\}$$ Implies

$$f(t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})<\lambda f(t,x,\dot{x},y,\dot{y})+(1-\lambda)f(t,u,\dot{u},y,\dot{y}).$$

Definition: [2.9] Let f is semi-strictly prequasi–invex, if for all $(t,x,\dot{x},y,\dot{y}),(t,u,\dot{u},y,\dot{y})\in S$, $f(t,x,\dot{x},y,\dot{y})\ne f(t,u,\dot{u},y,\dot{y})$ for all $\lambda\in[0,1]$ and

$$f(t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})\le \max\{f(t,x,\dot{x},y,\dot{y}),f(t,u,\dot{u},y,\dot{y})\}$$, implies that

$$f(t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})\le\lambda f(t,x,\dot{x},y,\dot{y})+(1-\lambda)f(t,u,\dot{u},y,\dot{y})$$

for all $(t,x,\dot{x},y,\dot{y}),(t,u,\dot{u},y,\dot{y})\in S,\lambda\in[0,1]$.

Definitions: [2.10] Upper semi continuity and Lower semi continuity

Let S be a non-empty subset of $I\times R^n\times R^n\times R^n\times R^n$ .The function $f:S\to R$ is said to be upper semi continuous at $(t,u,\dot{u},y,\dot{y})\in S$ .If for every $\varepsilon>0$, there exists a $\delta>0$ such that for all $(t,x,\dot{x},y,\dot{y})\in S$ if $\|(t,x,\dot{x},y,\dot{y})-(t,u,\dot{u},y,\dot{y})\|<\delta$, then $f(t,x,\dot{x},y,\dot{y})<f(t,u,\dot{u},y,\dot{y})$.

If f is upper semi-continuous at $u\in S$ then f is said to be lower semi-continuous at $(t,u,\dot{u},y,\dot{y})\in S$. Interested reader may consult [1, 10-29].

3. THEORY/CALCULATIONS:

Semi– Continuity and pre–invex function

Lemma 3.1: Let f be a continuous real–valued function on a convex set $S\subseteq I\times R^n\times R^n\times R^n\times R^n$ if, $f(t,\frac{1}{2}x+\frac{1}{2}u,\frac{1}{2}\dot{x}+\frac{1}{2}\dot{u},y,\dot{y})\le\frac{1}{2}f(t,x,\dot{x},y,\dot{y})+\frac{1}{2}f(t,u,\dot{u},y,\dot{y})$,

for all $(t,x,\dot{x},y,\dot{y}),(t,u,\dot{u},y,\dot{y})\in S$, then f is convex function on S.

The importance of the lemma is that the convexity can be determined by checking a mid–point convexity under the continuity condition. the same spirit will be followed in this paper . In this section, we will present two conditions that determine the pre–invexity check under conditions of upper and lower semi–continuity, respectively.

Our development extends the convexity results in lemma 3.1 to pre–invexity results.

First we derive the following basic results in lemma 3.2 which will be used to prove theorems 3.1and 4.1.

Lemma: 3.1 Let $S\subseteq I\times R^n\times R^n\times R^n\times R^n$ be a invex set with respect to η which satisfies condition C and D. If $f:S\to R$ satisfies

$$f(t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})\le f(t,x,\dot{x},y,\dot{y})$$

for all $(t,x,\dot{x},y,\dot{y}),(t,u,\dot{u},y,\dot{y})\in S$, and there exist $\lambda\in[0,1]$ such that

$$f(t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})\le\lambda f(t,x,\dot{x},y,\dot{y})+(1-\lambda)f(t,u,\dot{u},y,\dot{y}) \quad (1)$$

then the set

$$A=\{\lambda\in[0,1]/ f(t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$$
$$\le\lambda f(t,x,\dot{x},y,\dot{y})+(1-\lambda)f(t,u,\dot{u},y,\dot{y}),\ for\ all\ (t,x,\dot{x},y,\dot{y}),(t,u,\dot{u},y,\dot{y})\in S\},$$

is dense in $[0,1]$.

Proof: Note that both $\lambda=0$ and $\lambda=1$ belong to set A based on the fact that $f(t,u,\dot{u},y,\dot{y})<f(t,x,\dot{x},y,\dot{y})$ and the assumption

$$f(t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})\le f(t,x,\dot{x},y,\dot{y})$$

Suppose that the hypotheses hold and A is not dense in $[0,1]$ then there exit a $\lambda_0\in[0,1]$ and a neighborhood $N(\lambda_0)$ of λ_0 such that the

$$N(\lambda_0)\cap A=\phi \tag{2}$$

Define $\lambda_1=\inf\{\lambda\in A/\lambda>\lambda_0\}$ (3)

$$\lambda_2=\inf\{\lambda\in A/\lambda\le\lambda_0\} \tag{4}$$

Then by (2) we have $0\le\lambda_2<\lambda_1\le 1$. Now since $\{\lambda,1-\lambda\}\in(0,1)$.

We can choose $u_1,u_2\in A$ with $u_1\ge\lambda_1$ and $u_2\le\lambda_2$ such that

$$\max\{\lambda,1-\lambda\}(u_1-u_2)<(\lambda_1-\lambda_2) \tag{5}$$

then $u_2\le\lambda_2<\lambda_1\le u_1$. Next let us consider $\overline{\lambda}=\lambda u_1+(1-\lambda)u_2$ from condition C and D, we have

$$[t,u+u_2\eta(t,x,\dot{x},u,\dot{u})+\lambda\eta(t,u+u_1\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+u_1\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),u+u_2\eta(t,x,\dot{x},u,\dot{u}),$$
$$\dot{u}+u_2\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),\dot{u}+u_2\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})+\lambda\frac{d}{dt}\eta(t,u+u_1\eta(t,x,\dot{x},u,\dot{u}),$$
$$\dot{u}+u_1\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),u+u_2\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+u_2\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),y,\dot{y}]$$

$$=[t,u+u_2\eta(t,x,\dot{x},u,\dot{u})+\lambda\eta(t,u+u_1\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+u_1\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),$$
$$u+u_1\eta(t,x,\dot{x},u,\dot{u})-(u_1-u_2)\eta(t,x,\dot{x},u,\dot{u}),$$
$$\dot{u}+u_1\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})-(u_1-u_2)\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),\dot{u}+u_2\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})$$
$$+\lambda\frac{d}{dt}\eta(t,u+u_1\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+u_1\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),$$
$$u+u_1\eta(t,x,\dot{x},u,\dot{u})-(u_1-u_2)\eta(t,x,\dot{x},u,\dot{u}),$$
$$\dot{u}+u_1\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})-(u_1-u_2)\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),y,\dot{y}]$$

$$=[t,u+u_2\eta(t,x,\dot{x},u,\dot{u})+\lambda\eta(t,u+u_1\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+u_1\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),$$

$$u+u_1\eta(t,x,\dot{x},u,\dot{u})+\frac{(u_1-u_2)}{u_1}\eta(t,u,\dot{u},u+u_1\eta(t,x,\dot{x},u,\dot{u})),$$

$$\dot{u}+u_1\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})+\frac{(u_1-u_2)}{u_1}\frac{d}{dt}\eta(t,u,\dot{u},u+u_1\eta(t,x,\dot{x},u,\dot{u})),\dot{u}+u_2\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})$$

$$+\lambda\frac{d}{dt}\eta(t,u+u_1\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+u_1\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),$$

$$u+u_1\eta(t,x,\dot{x},u,\dot{u})+\frac{(u_1-u_2)}{u_1}\eta(t,u,\dot{u},u+u_1\eta(t,x,\dot{x},u,\dot{u})),$$

$$\dot{u}+u_1\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})+\frac{(u_1-u_2)}{u_1}\frac{d}{dt}\eta(t,u,\dot{u},u+u_1\eta(t,x,\dot{x},u,\dot{u})),y,\dot{y}]$$

$$=[t,u+u_2\eta(t,x,\dot{x},u,\dot{u})+\lambda\eta(t,u+u_1\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+u_1\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),$$

$$u+u_1\eta(t,x,\dot{x},u,\dot{u})-\lambda\frac{(u_1-u_2)}{u_1}\eta(t,u,\dot{u},u+u_1\eta(t,x,\dot{x},u,\dot{u})),$$

$$\dot{u}+u_1\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})-\lambda\frac{(u_1-u_2)}{u_1}\frac{d}{dt}\eta(t,u,\dot{u},u+u_1\eta(t,x,\dot{x},u,\dot{u})),\dot{u}+u_2\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})$$

$$+\lambda\frac{d}{dt}\eta(t,u+u_1\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+u_1\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),$$

$$u+u_1\eta(t,x,\dot{x},u,\dot{u})-\lambda\frac{(u_1-u_2)}{u_1}\eta(t,u,\dot{u},u+u_1\eta(t,x,\dot{x},u,\dot{u})),$$

$$\dot{u}+u_1\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})-\lambda\frac{(u_1-u_2)}{u_1}\frac{d}{dt}\eta(t,u,\dot{u},u+u_1\eta(t,x,\dot{x},u,\dot{u})),y,\dot{y}]$$

$$=[t,u+(\lambda u_1+(1-\lambda)u_2)\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+(\lambda u_1+(1-\lambda)u_2)\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}]$$

$$=[t,u+\bar{\lambda}\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\bar{\lambda}\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}]$$

for all $(t,x,\dot{x},y,\dot{y}),(t,u,\dot{u},y,\dot{y})\in S$.

Hence from (1) and $u_1,u_2\in A$, we obtain

$$=f[t,u+\bar{\lambda}\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\bar{\lambda}\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}]$$

$$=f[t,u+u_2\eta(t,x,\dot{x},u,\dot{u})+\lambda\eta(t,u+u_1\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+u_1\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),u+u_2\eta(t,x,\dot{x},u,\dot{u}),$$

$$\dot{u}+u_2\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),\dot{u}+u_2\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})+\lambda\frac{d}{dt}\eta(t,u+u_1\eta(t,x,\dot{x},u,\dot{u}),$$

$$\dot{u}+u_1\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),u+u_2\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+u_2\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),y,\dot{y}]$$

$$\le \lambda f[t, u + u_1\eta(t, x, \dot{x}, u, \dot{u}), \dot{u} + u_1 \frac{d}{dt}\eta(t, x, \dot{x}, u, \dot{u}), y, \dot{y}]$$

$$+(1-\lambda) f[t, u + u_2\eta(t, x, \dot{x}, u, \dot{u}), \dot{u} + u_2 \frac{d}{dt}\eta(t, x, \dot{x}, u, \dot{u}), y, \dot{y}]$$

$$\le \lambda\{u_1 f(t, x, \dot{x}, y, \dot{y}) + (1-u_1) f(t, u, \dot{u}, y, \dot{y})\}$$
$$+(1-\lambda)\{u_2 f(t, x, \dot{x}, y, \dot{y}) + (1-u_2) f(t, u, \dot{u}, y, \dot{y})\}$$

$$\le (\lambda u_1 + (1-\lambda)u_2) f(t, x, \dot{x}, y, \dot{y}) + \{1-(\lambda u_1 + (1-\lambda)u_2)\} f(t, u, \dot{u}, y, \dot{y})$$

$$= \bar{\lambda} f(t, x, \dot{x}, y, \dot{y}) + (1-\bar{\lambda}) f(t, u, \dot{u}, y, \dot{y})$$

ie, $\bar{\lambda} \in A$ if $\bar{\lambda} \ge \lambda_0$ then from (5) , we have $(\bar{\lambda} - u_2) = \lambda(u_1 - u_2) < (\lambda_1 - \lambda_2)$ and therefore $\bar{\lambda} < \lambda_1$, because $\bar{\lambda} \ge \lambda_0$ and $\bar{\lambda} \in A$ .This is contradiction of (3) , similarly $\bar{\lambda} \le \lambda_0$ leads to a contradiction of (4).Consequently, A is dense in $[0,1]$ which completes the proof .

Theorem: 3.2 suppose that $S \subseteq I \times R^n \times R^n \times R^n \times R^n$ is an open invex set w.r.t. $\eta : S \to R^n$ and η satisfies condition C and D. Assume that f is an upper semi –continuous real valued function on S and f satisfies

$$= f[t, u + \lambda\eta(t, x, \dot{x}, u, \dot{u}), \dot{u} + \lambda \frac{d}{dt}\eta(t, x, \dot{x}, u, \dot{u}), y, \dot{y}] = f(z_1) \le f(t, x, \dot{x}, y, \dot{y})$$

for all $(t, x, \dot{x}, y, \dot{y}), (t, u, \dot{u}, y, \dot{y}) \in S$ then f is a pre–invex function on S if and only if there exists $\lambda \in [0,1]$ such that

$$= f[t, u + \lambda\eta(t, x, \dot{x}, u, \dot{u}), \dot{u} + \lambda \frac{d}{dt}\eta(t, x, \dot{x}, u, \dot{u}), y, \dot{y}] \le \lambda f(t, x, \dot{x}, y, \dot{y}) + (1-\lambda) f(t, u, \dot{u}, y, \dot{y}).$$

Proof: The necessity follows directly from the definition of the pre–invex function.

We only need to prove the sufficiency. Suppose that the hypotheses hold and f is not pre–invex on S. Hence there exist

$(t, x, \dot{x}, y, \dot{y}), (t, u, \dot{u}, y, \dot{y}) \in S$ and $\lambda \in [0,1]$, Such that

$$= f[t, u + \lambda\eta(t, x, \dot{x}, u, \dot{u}), \dot{u} + \lambda \frac{d}{dt}\eta(t, x, \dot{x}, u, \dot{u}), y, \dot{y}] = f(z_{\bar{\lambda}}) \ge \bar{\lambda} f(t, x, \dot{x}, y, \dot{y}) + (1-\bar{\lambda}) f(t, u, \dot{u}, y, \dot{y}) \quad (6)$$

Let

$$z_{\bar{\lambda}} = [t, u + \lambda\eta(t, x, \dot{x}, u, \dot{u}), \dot{u} + \lambda \frac{d}{dt}\eta(t, x, \dot{x}, u, \dot{u}), y, \dot{y}]$$

$$A = \{\lambda \in [0,1] / f(t, u + \lambda\eta(t, x, \dot{x}, u, \dot{u}), \dot{u} + \lambda \frac{d}{dt}\eta(t, x, \dot{x}, u, \dot{u}), y, \dot{y})$$
$$\le \bar{\lambda} f(t, x, \dot{x}, y, \dot{y}) + (1-\bar{\lambda}) f(t, u, \dot{u}, y, \dot{y}), \textit{for all } (t, x, \dot{x}, y, \dot{y}), (t, u, \dot{u}, y, \dot{y}) \in S\}.$$

From the hypotheses in the theorem and lemma 1.1 A is dense then there exist a sequence $\{\lambda_n\}$ with $\lambda_n \in A$ such that $\lambda_n \to \bar{\lambda} (n \to \infty)$. Define $u_n = u + (\frac{\bar{\lambda} - \lambda_n}{1-\lambda_n})\eta(t, x, \dot{x}, u, \dot{u})$

and $\dot{u}_n = \dot{u} + (\frac{\bar{\lambda} - \lambda_n}{1-\lambda_n}) \frac{d}{dt}\eta(t, x, \dot{x}, u, \dot{u}), (t, u_n, \dot{u}_n, y, \dot{y}) \to (t, u, \dot{u}, y, \dot{y})(n \to \infty)$.

Since S is an open invex set then when η is large sufficiently, we have $(t,u_n,\dot{u}_n,y,\dot{y})\in S$ and from condition C, we have $[t,u_n+\lambda_n\eta(t,x,\dot{x},u,\dot{u}),\dot{u}_n+\lambda_n\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}]$

$$=(t,u+(\frac{\bar{\lambda}-\lambda_n}{1-\lambda_n})\eta(t,x,\dot{x},u,\dot{u})+\lambda_n\eta(t,x,\dot{x},u+(\frac{\bar{\lambda}-\lambda_n}{1-\lambda_n})\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+(\frac{\bar{\lambda}-\lambda_n}{1-\lambda_n})\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),$$

$$\dot{u}+(\frac{\bar{\lambda}-\lambda_n}{1-\lambda_n})\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})+\lambda_n\frac{d}{dt}\eta(t,x,\dot{x},u+(\frac{\bar{\lambda}-\lambda_n}{1-\lambda_n})\eta(t,x,\dot{x},u,\dot{u}),$$

$$\dot{u}+(\frac{\bar{\lambda}-\lambda_n}{1-\lambda_n})\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}))$$

$$=[t,u+\bar{\lambda}\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\bar{\lambda}\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}]=z_{\bar{\lambda}} \tag{7}$$

By the upper semi continuity of the f on S for any $\varepsilon>0$ then there exist $N>0$, such that the following holds when $n>N$.

$f[t,u_n+\lambda_n\eta(t,x,\dot{x},u,\dot{u}),\dot{u}_n+\lambda_n\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}]\le f(t,u,\dot{u},y,\dot{y})+\varepsilon$, therefore from (7) and $\lambda_n\in A$,

$$f(z_{\bar{\lambda}})=f[t,u_n+\lambda_n\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda_n\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}]$$

$\le\lambda_n f(t,x,\dot{x},y,\dot{y})+(1-\lambda_n)f(t,u,\dot{u},y,\dot{y})\to\bar{\lambda}f(t,x,\dot{x},y,\dot{y})+(1-\bar{\lambda})f(t,u,\dot{u},y,\dot{y}),(n\to\infty)$, since $\varepsilon>0$, may be arbitrarily small] which contradicts the inequality (6).

Hence f is pre–invex function on S.

Theorem: 3.3 Let S be a non empty invex set in R^n with respect to $\eta:S\to R^n$ where η satisfies condition C and D. Assume that $f:S\to R$ is a lower semi continuous function and that f satisfies

$$=f[t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}]\le f(t,x,\dot{x},y,\dot{y})$$

for all $(t,x,\dot{x},y,\dot{y}),(t,u,\dot{u},y,\dot{y})\in S$.

Then f is pre–invex function on S if and only if, for any $(t,x,\dot{x},y,\dot{y}),(t,u,\dot{u},y,\dot{y})\in S$ then there exists an $\alpha\in(0,1)$ such that

$$f[t,u+\alpha\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\alpha\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}]\le\alpha f(t,x,\dot{x},y,\dot{y})+(1-\alpha)f(t,u,\dot{u},y,\dot{y}) \tag{8}$$

Proof: The necessity follows directly from the definition of the pre–invex function. We only need to prove the sufficiency. By contradiction suppose that there exist $(t,x,\dot{x},y,\dot{y}),(t,u,\dot{u},y,\dot{y})\in S$ and $\lambda\in[0,1]$ such that

$$f[t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}]>\bar{\lambda}f(t,x,\dot{x},y,\dot{y})+(1-\bar{\lambda})f(t,u,\dot{u},y,\dot{y}) \tag{9}$$

Let $z_t=[t,u+t\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+t\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}],t\in(\bar{\lambda},1]$ and

$$B=\{z_t \in S / t\in(\overline{\lambda},1], f(z_t)=f(t,u+t\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+t\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$$
$$\leq tf(t,x,\dot{x},y,\dot{y})+(1-t)f(t,u,\dot{u},y,\dot{y}), a=\inf(t\in(\overline{\lambda},1]/z_t\in B\}$$

It is easy to check that $z_t \in B$ from the assumption in the theorem and that $z_{\overline{\lambda}} \not\subset B$ from (9) then $z_t \not\subset B$. Let $\lambda \leq t < a$ and there exist a sequence $\{t_n\}$ with $t_n \geq a$ and $z_{t_n} \in B$ such that $t_n \to a (n\to\infty)$. Since f is lower semi continuous function, we have

$$f(z_n)\leq \lim_{n\to\infty} f(z_{t_n})\leq \lim_{n\to\infty}[t_n f(t,x,\dot{x},y,\dot{y})+(1-t_n)f(t,u,\dot{u},y,\dot{y})]$$
$$=af(t,x,\dot{x},y,\dot{y})+(1-a)f(t,u,\dot{u},y,\dot{y}).$$

Hence $z_a \in B$.

Similarly, let $z_t=[t,u+t\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+t\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}], t\in[0,\overline{\lambda}]$ and

$$D=\{z_t \in S / t\in[0,\overline{\lambda}), f(z_t)=f(t,u+t\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+t\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$$
$$\leq tf(t,x,\dot{x},y,\dot{y})+(1-t)f(t,u,\dot{u},y,\dot{y}), b=\sup(t\in[0,\overline{\lambda})/z_t\in D\}$$

It can be verified that

$$z_0=z\in D, z_{\overline{\lambda}}=[t,u+\overline{\lambda}\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\overline{\lambda}\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}]\not\subset D,$$

where (9) is used in second equation ,then $z_t \not\subset D$, $b<t\leq\overline{\lambda}$ and then there exist a sequence $\{t_n\}$ with $t_n \leq b$ and $z_{t_n}\in D$ such that $t_n \to b(n\to\infty)$. Since, f is lower semi continuous function, we have,

$$f(z_v)\leq f(z_0)\leq \lim_{n\to\infty} f(z_{t_n})\leq \lim_{n\to\infty}[t_n f(t,x,\dot{x},y,\dot{y})-(1-t_n)f(t,u,\dot{u},y,\dot{y})]$$
$$=bf(t,x,\dot{x},y,\dot{y})-(1-b)f(t,u,\dot{u},y,\dot{y}) \tag{10}$$

Hence $z_0 \in D$. From the definition of u and v we have $0\leq b<\overline{\lambda}<a\leq 1$.

Now from condition C and D,

$$[t,u+b\eta(t,x,\dot{x},u,\dot{u}),\lambda\eta(t,u+a\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),$$
$$u+b\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+b\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),\dot{u}+b\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})$$
$$+\lambda\frac{d}{dt}\eta(t,u+a\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),$$
$$u+b\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+b\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),y,\dot{y}]$$

$$= [t, u + b\eta(t,x,\dot{x},u,\dot{u}), \lambda\eta(t, u + a\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),$$

$$u + a\eta(t,x,\dot{x},u,\dot{u}) - (a-b)\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}) - (a-b)\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),$$

$$\dot{u} + b\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}) + \lambda\frac{d}{dt}\eta(t, u + a\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),$$

$$u + a\eta(t,x,\dot{x},u,\dot{u}) - (a-b)\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}) - (a-b)\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})), y, \dot{y}]$$

$$= [t, u + (\lambda a + (1-\lambda)b)\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + (\lambda a + (1-\lambda)b)\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}), y, \dot{y}]$$

$= [t, u + \bar{\lambda}\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + \bar{\lambda}\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}), y, \dot{y}]$ for all $(t,x,\dot{x},y,\dot{y}), (t,u,\dot{u},y,\dot{y}) \in S$. for all $\lambda \in [0,1]$.

Hence from the definition of a and b, we have

$$f[t, u + b\eta(t,x,\dot{x},u,\dot{u}), \lambda\eta(t, u + a\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),$$

$$u + b\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + b\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})), \dot{u} + b\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})$$

$$+\lambda\frac{d}{dt}\eta(t, u + a\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),$$

$$u + b\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + b\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})), y, \dot{y}]$$

$$= f[t, u + (\lambda a + (1-\lambda)b)\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + (\lambda a + (1-\lambda)b)\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}), y, \dot{y}]$$

$$> (\lambda a + (1-\lambda)b) f(t,x,\dot{x},y,\dot{y}) + \{1 - (\lambda a + (1-\lambda)b)\} f(t,u,\dot{u},y,\dot{y})$$

$$= \lambda\{af(t,x,\dot{x},y,\dot{y}) + (1-a) f(t,u,\dot{u},y,\dot{y})\} + (1-\lambda)\{bf(t,x,\dot{x},y,\dot{y}) + (1-b) f(t,u,\dot{u},y,\dot{y})\}$$

$$= \lambda f[t, u + a\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}), y, \dot{y}]$$

$$+(1-\lambda) f[t, u + b\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + b\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}), y, \dot{y}]$$

for all $\lambda \in [0,1]$.

This contradicts (6). In the above derivation (10) is used in the first equality and $a \in B$ and $b \in D$ are used in last inequality.

Corollary: 3.4 Let S be an non-empty invex set in $I \times R^n \times R^n \times R^n \times R^n$ with respect to $\eta : S \to R^n$, where η satisfies condition C. Assume that $f : S \to R$ is a lower semi continuous function and that f satisfies

$f[t, u + \lambda\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + \lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}), y, \dot{y}] \le f(t,x,\dot{x},y,\dot{y})$ for all $(t,x,\dot{x},y,\dot{y}), (t,u,\dot{u},y,\dot{y}) \in S$

and f is a pre–invex function on S if and only if there exist $\lambda \in [0,1]$ such that the following holds for every $(t,x,\dot{x},y,\dot{y}), (t,u,\dot{u},y,\dot{y}) \in S$, Then

$$f[t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}]\leq\lambda f(t,x,\dot{x},y,\dot{y})+(1-\lambda)f(t,u,\dot{u},y,\dot{y}),$$

From the results in this section under semi-continuity condition judging a function to be pre–invex or not can be reduced to checking intermediate point pre–invexity for the function.

Relationship with pre–invex functions in [12] Nikodem obtained an inserting results, hence there analogous result in Continuous Time Programming is as follows;

Theorem: 3.5 Let $S\subseteq R^n$ be an invex set in R^n w.r.t. $\eta: S\times S\to R^n$, where η satisfies condition C .Then real-valued function f is pre–invex on S if and only if it is a prequasi–invex function on S and there exist a $\lambda\in[0,1]$ such that ,

$$f[t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}]\leq\lambda f(t,x,\dot{x},y,\dot{y})+(1-\lambda)f(t,u,\dot{u},y,\dot{y}),$$

for all $(t,x,\dot{x},y,\dot{y}),(t,u,\dot{u},y,\dot{y})\in S$ (11)

Proof: The necessity is easy to verify. We only need to prove the sufficiency for every $x,y\in S$.

Let $z_\lambda=(t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$

for all $\lambda\in[0,1]$. Two different situations where $f(t,x,\dot{x},y,\dot{y})=f(t,u,\dot{u},y,\dot{y})$ or $f(t,x,\dot{x},y,\dot{y})\neq f(t,u,\dot{u},y,\dot{y})$ will be considered separately in the following.

(I) $f(t,x,\dot{x},y,\dot{y})=f(t,u,\dot{u},y,\dot{y})$. We need to show that

$$f[t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}]\leq\lambda f(t,x,\dot{x},y,\dot{y})+(1-\lambda)f(t,u,\dot{u},y,\dot{y}),$$

for all $(t,x,\dot{x},y,\dot{y}),(t,u,\dot{u},y,\dot{y})\in S$, for all $\lambda\in[0,1]$.

By contradiction, suppose that there exist $\beta\in(0,1]$ such that

$$f(z_\beta)=f(t,u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$$

$$>\beta f(t,x,\dot{x},y,\dot{y})+(1-\beta)f(t,u,\dot{u},y,\dot{y})=f(t,x,\dot{x},y,\dot{y})=f(t,u,\dot{u},y,\dot{y}) \tag{12}$$

Assume that $0<\alpha<\beta\leq1$. Let $j=\frac{(\beta-\alpha)}{(1-\alpha)}$ from condition C and D we have

$$(t,u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$$

$$=(t,u+j\eta(t,x,\dot{x},u,\dot{u})+\alpha\eta(t,x,\dot{x},u+j\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+j\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),$$

$$\dot{u}+j\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})+\alpha\eta(t,x,\dot{x},u+j\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+j\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),y,\dot{y})$$

from (11) and (12), we obtain

$$f(t,u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})\leq\alpha f(t,x,\dot{x},y,\dot{y})+(1-\alpha)f(t,u,\dot{u},y,\dot{y})$$

$$<f(t,u+a\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}) \tag{13}$$

In the second inequality the fact that $f(t,x,\dot{x},y,\dot{y}) < f(t,u+j\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+j\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$ is used, otherwise this leads to a contradiction to (12) . On the other hand let $s=\frac{(\beta-u)}{(\beta)}$ from condition C and D, we have

$$(t,u+j\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+j\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$$
$$=(t,u+\beta\eta(t,x,\dot{x},u,\dot{u})+s\eta(t,y,\dot{y},u+j\eta(t,x,\dot{x},u,\dot{u}),$$
$$\dot{u}+j\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})$$
$$+s\eta(t,x,\dot{x},u+j\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+j\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),y,\dot{y})$$

Hence from the prequasi–invexity of f and

$f(t,u,\dot{u},y,\dot{y}) < f(t,u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$, we get

$$f(t,u+j\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+j\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$$

$$=f(t,u+\beta\eta(t,x,\dot{x},u,\dot{u})+s\eta(t,y,\dot{y},u+j\eta(t,x,\dot{x},u,\dot{u}),$$
$$\dot{u}+j\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})$$
$$+s\eta(t,x,\dot{x},u+j\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+j\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),y,\dot{y})$$

$\leq f(t,u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$ which contradicts the inequality (13).

(b) Assume that $0<\alpha<\beta<1$. Let $k=\frac{\beta}{\alpha}>\beta$. From condition C and D, we have

$$(t,u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$$
$$=(t,u+k\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+k\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}) \tag{14}$$

From (11) and (14) as well as (12), we obtain

$$f(t,u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$$
$$\leq\alpha f(t,u+k\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+k\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})+(1-\alpha)f(t,u,\dot{u},y,\dot{y})$$
$$<f(t,u+k\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+k\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}) \tag{15}$$

Let $m=\frac{(u-\beta)}{(1-\beta)}$ from condition C and D, we have

$$(t,u+m\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+m\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$$

$$=(t,u+\beta\eta(t,x,\dot{x},u,\dot{u})+m\eta(t,x,\dot{x},u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),$$

$$\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})+m\frac{d}{dt}\eta(t,x,\dot{x},u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}))y,\dot{y}).$$

From the prequasi–invexity of f and

$$f(t,x,\dot{x},y,\dot{y})<f(t,u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}),$$

we get , $f(t,u+k\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+k\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$

$$=f(t,u+\beta\eta(t,x,\dot{x},u,\dot{u})+m\eta(t,x,\dot{x},u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),$$

$$\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})+m\frac{d}{dt}\eta(t,x,\dot{x},u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}))y,\dot{y})$$

$\le f(t,u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$ which contradicts the inequality (15).

(II) $f(t,x,\dot{x},y,\dot{y})\neq f(t,u,\dot{u},y,\dot{y})$, in this case, we need to show that

$$f[t,u+\lambda\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\lambda\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}]\le\lambda f(t,x,\dot{x},y,\dot{y})+(1-\lambda)f(t,u,\dot{u},y,\dot{y}) \quad \text{for all } \lambda\in A. \tag{16}$$

Assume that $f(t,x,\dot{x},y,\dot{y})<f(t,u,\dot{u},y,\dot{y})$ then from (16) and the density of A there exist $(t,u,\dot{u},y,\dot{y})\in A$ with $(t,u,\dot{u},y,\dot{y})<(t,u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$ such that

$$f(t,u+k\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+k\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})\le kf(t,x,\dot{x},y,\dot{y})+(1-k)f(t,u,\dot{u},y,\dot{y})$$

$$\le f(t,u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y}) \tag{17}$$

Let $n=\frac{(\beta-u)}{(1-u)}$. It is obvious that $0<t<1$ .From condition C and D, we have

$$(t,u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$$

$$=(t,u+a\eta(t,x,\dot{x},u,\dot{u})+a\eta(t,x,\dot{x},u+n\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),$$

$$\dot{u}+a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})+n\frac{d}{dt}\eta(t,x,\dot{x},u+a\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}))y,\dot{y}).$$

(a_1) $f(t,x,\dot{x},y,\dot{y})<f(t,u+a\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$. From the prequasi–invexity of f, we get, $f(t,u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$

$= f(t, u + a\eta(t,x,\dot{x},u,\dot{u}) + a\eta(t,x,\dot{x},u + n\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),$

$\dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}) + n\frac{d}{dt}\eta(t,x,\dot{x},u + a\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}))\, y, \dot{y})$

$\leq f(t, u + a\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}), y, \dot{y})$ which contradicts the inequality (17).

(a_2) If $f(t,x,\dot{x},y,\dot{y}) > f(t, u + a\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}), y, \dot{y})$ from the prequasi–invexity of f and $f(t,x,\dot{x},y,\dot{y}) < f(t,u,\dot{u},y,\dot{y})$ as well as (16) we get

$f(t, u + \beta\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + \beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}), y, \dot{y})$

$= f(t, u + a\eta(t,x,\dot{x},u,\dot{u}) + a\eta(t,x,\dot{x},u + n\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),$

$\dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}) + n\frac{d}{dt}\eta(t,x,\dot{x},u + a\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}))\, y, \dot{y})$

$\leq f(t,u,\dot{u},y,\dot{y}) < kf(t,x,\dot{x},y,\dot{y}) + (1-k) f(t,u,\dot{u},y,\dot{y})$

$< f(t, u + \beta\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + \beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}), y, \dot{y})$ which is a contradiction .

(b) Assume that $f(t,u,\dot{u},y,\dot{y}) < f(t,x,\dot{x},y,\dot{y})$ then from (16) and the density of A there exist $(t,u,\dot{u},y,\dot{y}) \in A$ with

$(t,u,\dot{u},y,\dot{y}) < (t, u + \beta\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + \beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}), y, \dot{y})$ such that

$$f(t, u + a\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}), y, \dot{y}) \leq af(t,x,\dot{x},y,\dot{y}) + (1-a) f(t,u,\dot{u},y,\dot{y})$$

$$< f(t, u + \beta\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + \beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}), y, \dot{y}) \tag{18}$$

Let $p = \frac{(u-\beta)}{u}$. It is obvious that $0 < p < 1$ and from condition C and D, we have

$(t, u + \beta\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + \beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}), y, \dot{y})$

$= (t, u + a\eta(t,x,\dot{x},u,\dot{u}) + p\eta(t,x,\dot{x},u + n\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),$

$\dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}) + p\frac{d}{dt}\eta(t,x,\dot{x},u + a\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}))\, y, \dot{y})$

.

(a_1) If $f(t,u,\dot{u},y,\dot{y}) \leq f(t, u + \beta\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + \beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}), y, \dot{y})$. From the prequasi–invexity of

f , we get $f(t, u + \beta\eta(t,x,\dot{x},u,\dot{u}), \dot{u} + \beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}), y, \dot{y})$

$$= f(t,u+a\eta(t,x,\dot{x},u,\dot{u})+p\eta(t,x,\dot{x},u+n\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),$$

$$\dot{u}+a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})+p\frac{d}{dt}\eta(t,x,\dot{x},u+a\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}))y,\dot{y})$$

$\leq f(t,u+a\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$ which contradicts the inequality (18).

(a_2) If $f(t,u,\dot{u},y,\dot{y}) > f(t,u+a\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$ from the prequasi–invexity of f and $f(t,u,\dot{u},y,\dot{y}) < f(t,x,\dot{x},y,\dot{y})$ as well as (16)we get

$$f(t,u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$$

$$= f(t,u+a\eta(t,x,\dot{x},u,\dot{u})+p\eta(t,x,\dot{x},u+n\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})),$$

$$\dot{u}+a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u})+p\frac{d}{dt}\eta(t,x,\dot{x},u+a\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+a\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}))y,\dot{y})$$

$$\leq f(t,u,\dot{u},y,\dot{y}) < \beta f(t,x,\dot{x},y,\dot{y})+(1-\beta)f(t,u,\dot{u},y,\dot{y})$$

$$< f(t,u+\beta\eta(t,x,\dot{x},u,\dot{u}),\dot{u}+\beta\frac{d}{dt}\eta(t,x,\dot{x},u,\dot{u}),y,\dot{y})$$

Which is a contradiction .This completes the proof.

4. RESULTS AND DISCUSSION:

Our results reflects the complete satisfaction about pre–invexity for a function, which can be achieved via an intermediate-point check. A characterization of a pre–invex function in terms of its relationship with an intermediate-point, pre–invexity and prequasi–invexity are extensions of the case of the treatment provided in the context of continuous time programming.

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