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FRACTIONAL INTEGRATION OF MULTIVARIABLE H-FUNCTION AND M-SERIES VIA PATHWAY MODEL

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ABSTRACT

In this paper, we discuss a composition formula of the so-called pathway fractional integration operator due to Nair with a finite product of multivariable H-function and generalized M-series. Further some interesting results are obtained in terms of corollaries, which involve Meijer G-function, Mittag-Leffler function, Bessel-function, and hypergeometric function. Also we obtained two fractional integral formulas involving left-sided Riemann-Liouville fractional integral operators in section 5.All the results derived here are of general character and can yield number of results in theory of special functions.

Keywords: *Pathway fractional integral operator, Multivariable H-function, Generalized M-series, Mittag-Leffler function, Bessel-function, Hypergeometric function.*

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1. INTRODUCTION

The fractional integral operators involving various special functions have found significant importance and applications in various subfields of applicable mathematical analysis. In recent years, the fractional calculus has become on of the most rapidly growing parts of science as well as mathematics. Since last four decades, a number of researcher like Choi *et al.* [4], Saigo [18,19], Khan *et al.* [3], Kiryakova [11,12], Kilbas [23,24], Machado *et al.* [13,14], Kalla [5,6], Kalla and Saxena [8] have studied, in depth, the properties, applications and different extensions of various hypergeometric operators in view of fractional integration. Many applications of fractional calculus can be found, for example, in turbulence and fluid dynamics, fractional kinetic models, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, non-linear biological systems and astrophysics.

Let $f(x) = L(a,b), \alpha \in \mathbb{C}, \Re(\alpha) > 0$, then left sided Riemann-Liouville fractional integral operator defined as

$$\left(I_{0_+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt \tag{1}$$

For more details, we refer to Kiryakova [11], Kilbas-Srivastava-Trujillo [20] and Samko-Kilbas-Marichev [21] etc.

If $f(t)$ is replaced by $t^{\gamma} f(t)$ in (1), the above operator turns out to be Erdélyi-Kober fractional integral; if it is replaced by ${}_2F_1\left(\eta+\beta, -\gamma; \eta; 1-\frac{t}{x}\right) f(t)$, then (1) takes the form of Saigo hypergeometric fractional integral

$$\frac{\Gamma(\eta)}{x^{-\eta-\beta}} I_{0_+}^{\eta,\beta,\gamma} f(x) = \int_0^x (x-t)^{\eta-1}\, {}_2F_1\left(\eta+\beta, -\gamma; \eta; 1-\frac{t}{x}\right) f(t)\, dt$$

Many other operators generalized fractional calculus can be obtained if on the place of $f(t)$ one takes $\phi(t)$ as it is done in Kiryakova [11] for a Fox's *H*-function $\phi(t) = H_{m,m}^{m,0}(t)$.

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The pathway fractional integration operator, as an extension of (1), is defined as follows [2, p.239]:

$$\left(P_{0+}^{(\eta,\lambda,d)}f\right)(x) = x^{\eta}\int_0^{\left[\frac{x}{d(1-\lambda)}\right]}\left[1-\frac{d(1-\lambda)t}{x}\right]^{\frac{\eta}{1-\lambda}} f(t)dt, \tag{2}$$

where $f(x) \in L(a,b), \eta \in \mathbb{C},\ \Re(\eta) > 0$, d>0 and 'pathway parameters' $\lambda < 1$.

The pathway model is introduced by Mathai [25] and studied further by Mathai and Haubold [26, 27]. For real scalar λ, the pathway model for scalar random variables is represented by the following probability density function (p.d.f.):

$$f(x) = c|x|^{\gamma-1}\left[1-d(1-\lambda)|x|^{\delta}\right]^{\frac{\beta}{1-\lambda}}, \tag{3}$$
$$-\infty < x < \infty, \delta > 0, \beta \geq 0, 1-d(1-\lambda)|x|^{\delta} > 0, \gamma > 0,$$

where c is the normalization constant is as follows:

$$c = \frac{1}{2}\frac{\delta[d(1-\lambda)]^{\frac{\gamma}{\delta}}\,\Gamma\left(\frac{\gamma}{\delta}+\frac{\beta}{1-\lambda}+1\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\,\Gamma\left(\frac{\beta}{1-\lambda}+1\right)}, for\ \lambda < 1, \tag{4}$$

$$=\frac{1}{2}\frac{\delta[d(1-\lambda)]^{\frac{\gamma}{\delta}}\,\Gamma\left(\frac{\beta}{1-\lambda}\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma\left(\frac{\beta}{\lambda-1}-\frac{\gamma}{\delta}\right)},\ for\ \frac{1}{\lambda-1}-\frac{\gamma}{\delta} > 0,\ \lambda > 1, \tag{5}$$

$$= \frac{1}{2}\frac{\delta\ (\lambda\beta)^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)}\quad for\ \lambda \longrightarrow 1. \tag{6}$$

For $\lambda < 1$, it is a finite range density with $1-d(1-\lambda)|x|^{\delta} > 0$ and (3) remains in the extended generalized type-1 beta family. The pathway density in (3) for $\lambda < 1$, includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f.

For $\lambda > 1$, writing $1-\lambda = -(\lambda-1)$, we have

$$f(x) = c|x|^{\gamma-1}\left[1+d(\lambda-1)|x|^{\delta}\right]^{-\frac{\beta}{\lambda-1}}, \tag{7}$$
$$-\infty < x < \infty, \delta > 0, \beta \geq 0, \lambda > 1,$$

which is extended generalized type-2 beta models for real x. It includes the type -2 beta density, the F density, the Cauchy density and many more.

Here we consider only the case of pathway parameters $\lambda < 1$. For $\lambda \longrightarrow 1$ both (3) and (7) take the exponential form, since

$$\begin{aligned}\lim_{\lambda\longrightarrow 1} c|x|^{\gamma-1}\left[1-d(1-\lambda)|x|^{\delta}\right]^{\frac{\eta}{1-\lambda}} &= \lim_{x\longrightarrow 1} c|x|^{\gamma-1}\left[1-d\ \eta\ |x|^{\delta}\ \frac{(1-\lambda)}{\eta}\right]^{\frac{\eta}{1-\lambda}}\\ &= c|x|^{\gamma-1}\exp\left(-d\ \eta\ |x|^{\delta}\right).\end{aligned} \tag{8}$$

This includes the generalized Gamma-, the Weibull-, the Chi-square, the Laplace-, Maxwell-Boltzmann and other related densities. Therefore, the operator introduced in this paper can be related and applicable to a wide variety of statistical density.

When $\lambda \longrightarrow 1_{-}, \left[1-\frac{d(1-\lambda)t}{x}\right]^{\frac{\eta}{1-\lambda}} \longrightarrow e^{-d\eta t}$. Then, the operators (1) reduces to the Laplace integral transform of f with the parameters $\frac{d\eta}{x}$:

$$\left(P_{0+}^{(\eta,1)}f\right)(x) = x^{\eta}\int_0^{\infty} e^{-\frac{d\eta}{x}}\ f(t)dt = x^{\eta}\ L_f\left(\frac{d\eta}{x}\right). \tag{9}$$

2. MATHEMATICAL PREREQUISITES

The multivariable H-function is defined in terms of multiple Mellin-Barnes type contour integral as [1].

$$H[z_1,\dots,z_r] = H^{0,n:m_1,n_1;\dots;m_r,n_r}_{p,q:p_1,q_1;\dots;p_r,q_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix}\left(a_j;\alpha_j^{(1)},\dots,\alpha_j^{(r)}\right)_{1,p}:\left(c_j^{(1)};\gamma_j^{(1)}\right)_{1,p_1};\dots;\left(c_j^{(r)};\gamma_j^{(r)}\right)_{1,p_r}\\ \left(b_j;\beta_j^{(1)},\dots,\beta_j^{(r)}\right)_{1,q}:\left(d_j^{(1)};\delta_j^{(1)}\right)_{1,q_1};\dots;\left(d_j^{(r)};\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right.\right] \tag{10}$$

$$H[z_1,\dots,z_r] = \frac{1}{(2\pi\omega)^r}\int_{\mathcal{L}_1}\dots\int_{\mathcal{L}_r}\Theta(\xi_1\dots\xi_r)\left\{\prod_{i=1}^{r}\varphi_i\,(\xi_i)\,z_i^{\xi_i}\right\}d\xi_1\dots d\xi_r$$

where

$$\Theta(\xi_1 \dots \xi_r) = \frac{\left[\prod_{j=1}^{n} \Gamma\left(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} \xi_i\right)\right]}{\left[\prod_{j=n+1}^{p} \Gamma\left(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} \xi_i\right)\right]\left[\prod_{j=1}^{q} \Gamma\left(1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)} \xi_i\right)\right]} \tag{11}$$

$$\varphi_i\,(\xi_i) = \frac{\left[\prod_{j=1}^{m_i} \Gamma\left(d_j^{(i)} - \delta_j^{(i)} \xi_i\right)\right]\left[\prod_{j=1}^{n_i} \Gamma\left(1 - c_j^{(i)} + \gamma_j^{(i)} \xi_i\right)\right]}{\left[\prod_{j=n_i+1}^{p_i} \Gamma\left(c_j^{(i)} - \gamma_j^{(i)} \xi_i\right)\right]\left[\prod_{j=m_i+1}^{q_i} \Gamma\left(1 - d_j^{(i)} + \delta_j^{(i)} \xi_i\right)\right]} \tag{12}$$

For $i = 1,..,r$ and $\omega = \sqrt{-1}, \mathcal{L}_i$ represents the contour which start at the point $\tau_i - \omega\infty$ and goes out the point $\tau_i + \omega\infty$ with $\tau_i \in \Re = (-\infty, \infty), i = 1, \dots, r$ such that all the poles of $\Gamma\left(d_j^{(i)} - \delta_j^{(i)} \xi_i\right), j = 1, \dots, m_i\,; i = 1, \dots, r$ are separated from those of $\Gamma\left(1 - c_j^{(i)} + \gamma_j^{(i)} \xi_i\right), j = 1, \dots, n_i\,; i = 1, \dots, r$ and $\Gamma\left(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} \xi_i\right),\ j = 1, \dots, n$. Here, the integers n, p, q, m_i, n_i, p_i and q_i, satisfy the inequalities $0 \le n \le p; q \ge 0, 0 \le m_i \le q_i$ and $0 \le n_i \le p_i$, $i = 1, \dots, r$. Further we suppose that the parameters

$$\begin{aligned} &a_j, j = 1, \dots, p;\ c_j^{(i)}, j = 1, \dots, p_i;\ i = 1, \dots, r \\ &b_j, j = 1, \dots, q;\ d_j^{(i)}, j = 1, \dots, q_i;\ i = 1, \dots, r \end{aligned} \tag{13}$$

are complex numbers and the related coefficients

$$\begin{aligned} &\alpha_j^{(i)}, j = 1, \dots, p;\ i = 1, \dots, r;\ \ \gamma_j^{(i)}, j = 1, \dots, p_i;\ i = 1, \dots, r \\ &\beta_j^{(i)}, j = 1, \dots, q;\ i = 1, \dots, r;\ \ \delta_j^{(i)}, j = 1, \dots, q_i;\ i = 1, \dots, r \end{aligned} \tag{14}$$

are positive real numbers.

We observe that for $n = m = p = q = 0$, the multivariable H-function breaks up in to product of r H-function and consequently there holds the following result [7].

$$\begin{aligned} H[z_1, \dots, z_r] &= H_{0,0:p_1,q_1;\dots;p_r,q_r}^{0,0:m_1,n_1;\dots;m_r,n_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \left| \begin{matrix} -:\left(c_j^{(1)}; \gamma_j^{(1)}\right)_{1,p_1}; \dots; \left(c_j^{(r)}; \gamma_j^{(r)}\right)_{1,p_r} \\ -:\left(d_j^{(1)}; \delta_j^{(1)}\right)_{1,q_1}; \dots; \left(d_j^{(r)}; \delta_j^{(r)}\right)_{1,q_r} \end{matrix} \right.\right] \\ &= \prod_{i=1}^{r} H_{p_i,q_i}^{m_i,n_i} \left[z_i \left| \begin{matrix} \left(c_j^{(i)}, \gamma_j^{(i)}\right)_{1,p_i} \\ \left(d_j^{(i)}, \delta_j^{(i)}\right)_{1,q_i} \end{matrix} \right.\right] \end{aligned} \tag{15}$$

If we consider $n = m = p = q = 0$ and $r = 1$ then the multivariable H-function converts in to as single variable H-function and it becomes

$$H[z] = H_{p,q}^{m,n}\,[z] = H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_p; \alpha_p) \\ (a_q; \beta_q) \end{matrix} \right.\right] = \frac{1}{2\pi\omega} \int_{\mathcal{L}} \Theta(\xi)\, \mathrm{z}^{-\xi}\, \mathrm{d}\xi\,, \tag{16}$$

where

$$\Theta(\xi) = \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j \xi)\ \prod_{j=1}^{n} \Gamma(1 - a_j - \alpha_j \xi)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j \xi)\ \prod_{j=n+1}^{p} \Gamma(a_j + \alpha_j \xi)} \tag{17}$$

$m, n, p, q \in N_0$ with $1 \le n \le p, 1 \le m \le q$, $\alpha_j, \beta_j \in \Re_+$ and $a_j, b_j \in \Re\ (i = 1, \dots, p; j = 1,..,q)$ and $\mathcal{L}$ is a suitable contour which separating the poles of $\Gamma\left(b_j + \beta_j \xi\right)$ from poles $\prod_{j=1}^{n} \Gamma\left(1 - a_j - \alpha_j \xi\right)$.

Wright [22] defined generalized hypergeometric function by means of the series representation in the form

$${}_p\psi_q(z) = {}_p\psi_q \left[\begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + nA_i)}{\prod_{j=1}^{q} \Gamma(b_j + nB_j)} \frac{z^n}{n!} \tag{18}$$

where $z, a_j, b_j \in \mathbb{C}, A_i, B_j \in \Re_+\ (i = 1, \dots, p; j = 1,..,q)$,

$$\sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > -1 \tag{19}$$

Sharma and Jain [9] introduced the generalized M-series as the function defined by means of the power series:

$$\begin{aligned} {}_p^{\alpha}M_q^{\beta}(z) &= {}_p^{\alpha}M_q^{\beta}(a_1, \dots, a_p; b_1, \dots, b_q; z) = {}_p^{\alpha}M_q^{\beta}\left((a_j)_1^p; (b_j)_1^q; z\right) \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n, \dots, (a_p)_n}{(b_1)_n, \dots, (b_q)_n} \frac{z^n}{\Gamma(\alpha n + \beta)}, (z, \alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0) \end{aligned} \tag{20}$$

where $(a_j)_n, (b_j)_n$ are known Pochammer symbols. The series (20) is defined when none of the parameters $b_j's, j = 1,2, \dots, q$ is a negative integer or zero; if any numerator parameter a_j is a negative integer or zero, then the series terminates to a polynomial in z. The series in (20) is convergent for all z if $p \le q$, it is convergent for $|z| < \delta = \alpha^\alpha$ if $p = q + 1$ and divergent, if $p > q + 1$. When $p = q + 1$ and $|z| = \delta$, the series can converge on conditions depending on the parameters. Properties of M-series are further studied by Saxena [10], Chouhan and Sarswat [17] etc.

The generalized M-series (20) can be represented as a special case of Fox H-function (16) and Wright generalized hypergeometric function (18), as

$$
\begin{aligned}
{}_p^\alpha M_q^\beta\left((a_j)_1^p; (b_j)_1^q; z\right) &= \Lambda\ {}_{p+1}\psi_{q+1}\left[\begin{matrix}(a_1,1),\dots,(a_p,1),(1,1);\\(b_1,1),\dots,(b_q,1),(\beta,\alpha);\end{matrix} z\right]\\
&= \Lambda\, H^{1,p+1}_{p+1,q+1}\left[-z\left|\begin{matrix}(1-\alpha_j,1;1)_1^p,(0,1)\\(0,1),(1-\beta_j,1)_1^q,(1-\beta,\alpha)\end{matrix}\right.\right]
\end{aligned}
\tag{21}
$$

Where $\Lambda = \frac{\prod_{j=1}^q \Gamma(b_j)}{\prod_{j=1}^p \Gamma(a_j)}$.

The generalized Mittag-Leffler function, introduced by Prabhakar [32] may be obtain from (21) for $p = q = 1$; $a = \gamma \in \mathbb{C}; b = 1$, as

$$
E_{\alpha,\beta}^\gamma(z) = \sum_{m=0}^\infty \frac{(\gamma)_m}{\Gamma(\alpha m+\beta)}\frac{z^m}{m!} = \sum_{m=0}^\infty \frac{(\gamma)_m}{(1)_m}\frac{z^m}{\Gamma(\alpha m+\beta)} = {}_1^\alpha M_1^\beta(1;1;z) \tag{22}
$$

The new generalization Mittag-Leffler function in terms of H-function is defined as

$$
E_{\alpha,\beta}^\gamma(z) = \frac{1}{\Gamma(\gamma)}\, H^{1,1}_{1,2}\left[-z\left|\begin{matrix}(1-\gamma,1)\\(0,1),\ (1-\beta,\alpha)\end{matrix}\right.\right] \quad (\alpha,\beta,\gamma \in \mathbb{C};\ \Re(\alpha) > 0) \tag{23}
$$

The present paper is organized as follows. The composition of pathway integral operators (2) with multivariables H -function, Mittag-Lefter, Bessel- function are given in section 3. Section 4 investigates the pathway fractional integration of generalized M-series and finally the further special cases in section 5 and concluding remark are drawn in section 6.

3. FRACTIONAL INTEGRATION OF H-FUNCTION VIA PATHWAY MODEL

Recently pathway fractional integral operators involving the various special functions have been considered by many author (see, eq., [29, 30, 31]). Our main result in this section is based on the following assertion giving a composition formula of the pathway fractional integration operator (2) with a power function (see Nair [2, Lemma 1]).

Lemma 1: *Let* $\eta \in \mathbb{C}, \Re(\eta) > 0, \beta \in \mathbb{C}$ *and* $\lambda < 1$. *If* $\Re(\beta) > 0$ *and* $\Re\left(\frac{\eta}{1-\lambda}\right) > -1$, *then we have*

$$
\left\{P_{0+}^{(\eta,\lambda,d)}\left[t^{\beta-1}\right]\right\}(x) = \frac{x^{\eta+\beta}\Gamma(\beta)\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{[d(1-\lambda)]^\beta\,\Gamma\left(1+\frac{\eta}{1-\lambda}+\beta\right)} \tag{24}
$$

Now we are in a position to state and prove our main result, which is a composition formula of the pathway fractional integration operator (2) with a finite product of multivariable H-function (10) asserted by the following Theorem.

Theorem 1: *Let* $\lambda < 1, \mu > 0, d > 0, l > 0$, *the parameters* $z_i, \eta, \rho \in \mathbb{C}$ $(i = 1, \dots, r), \Re(\eta) > 0$ *and* $\Re\left(\frac{\eta}{1-\lambda}\right) > -1$, *then there holds the following formula:*

$$
\begin{aligned}
&P_{0+}^{(\eta,\lambda,l)}\{t^{\rho-1}\, H[d^\mu t^\mu z_1, \dots, d^\mu t^\mu z_r]\}\\
&= \frac{t^{\eta+\rho}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{[l(1-\lambda)]^\rho}\, H^{0,n+1:m_1,n_1;\dots;m_r,n_r}_{p+1,q+1:p_1,q_1;\dots;p_r,q_r}\left[\begin{matrix}\frac{d^\mu t^\mu z_1}{\{l(1-\lambda)\}}\\ \vdots \\ \frac{d^\mu t^\mu z_r}{\{l(1-\lambda)\}}\end{matrix}\left|\begin{matrix}(1-\rho;\mu,\dots,\mu),\left(a_j;\alpha_j^{(1)},\dots,\alpha_j^{(r)}\right)_{1,p}\\ \left(b_j;\beta_j^{(1)},\dots,\beta_j^{(r)}\right)_{1,q},\left(-\frac{\eta}{1-\lambda}-\rho;\mu,\dots,\mu\right)\end{matrix}\right.\right.\\
&\qquad\left.\begin{matrix}:\left(c_j^{(1)};\gamma_j^{(1)}\right)_{1,p_1};\dots;\left(c_j^{(r)};\gamma_j^{(r)}\right)_{1,p_r}\\ :\left(d_j^{(1)};\delta_j^{(1)}\right)_{1,q_1};\dots;\left(d_j^{(r)};\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right]
\end{aligned}
\tag{25}
$$

Proof: Let $\mathfrak{I}$ be the left-hand side of (25). By applying (10) and using (2) to left hand side of (25), then we have

$$\mathfrak{I} = \frac{1}{(2\pi\omega)^r} \int_{\mathcal{L}_1} \dots \int_{\mathcal{L}_r} \Theta(\xi_1 \dots \xi_r) \left\{\prod_{i=1}^{r} \varphi_i(\xi_i) \left(d^\mu z_i\right)^{\xi_i}\right\} d\xi_1 \dots d\xi_r \ \times P_{0+}^{(\eta,\lambda,l)}\left\{t^{\rho+\mu\sum_{i=1}^{r}\xi_i - 1}\right\}$$

Now we apply Lemma 1 to use (24) with β replaced by $(\rho + \mu \sum_{i=1}^{r} \xi_i)$ to pathway integral, after a little simplification, we obtain the following result

$$\mathfrak{I} = \frac{1}{(2\pi\omega)^r} \int_{\mathcal{L}_1} \dots \int_{\mathcal{L}_r} \Theta(\xi_1 \dots \xi_r) \left\{\prod_{i=1}^{r} \varphi_i(\xi_i) \left(d^\mu z_i\right)^{\xi_i}\right\} d\xi_1 \dots d\xi_r$$
$$\times\, t^{\eta+\rho+\mu\sum_{i=1}^{r}\xi_i} \left(\frac{\Gamma(\rho+\mu\sum_{i=1}^{r}\xi_i)\ \Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{[l(1-\lambda)]^{\rho+\mu\sum_{i=1}^{r}\xi_i}\ \Gamma\left(\rho+\frac{\eta}{1-\lambda}+\mu\sum_{i=1}^{r}\xi_i+1\right)}\right)$$

then, in view of (10), interpreting the involved Mellin-Bernes contour integrals in term of Multivariable H-function, we readily obtain the right hand side of (25).

Setting $n = p = q = 0$ in (25) is seen to break the multivariable H-function in (15) into an r-times product of H-functions yield an interesting result asserted by the following Corollary 1.

Corollary 1: *Let $\lambda < 1, \mu > 0, d > 0, l > 0$, the parameters $z_i, \eta, \rho \in \mathbb{C}\ (i = 1, \dots, r), \Re(\eta) > 0$ and $\Re\left(\frac{\eta}{1-\lambda}\right) > -1$, then there holds the following formula:*

$$P_{0+}^{(\eta,\lambda,l)}\left\{t^{\rho-1}\prod_{i=1}^{r} H_{p_i,q_i}^{m_i,n_i}\left[d^\mu t^\mu z_i \left|\begin{matrix}\left(c_j^{(i)};\gamma_j^{(i)}\right)_{1,p_i}\\ \left(d_j^{(i)};\delta_j^{(i)}\right)_{1,q_i}\end{matrix}\right.\right]\right\}$$
$$= \frac{t^{\eta+\rho}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{[l(1-\lambda)]^\rho} H_{1,1:\{p_i,q_i\}}^{0,1:\{m_i,n_i\}}\left[\frac{d^\mu t^\mu z_i}{\{l(1-\lambda)\}}\left|\begin{matrix}(1-\rho;\mu,\dots,\mu):\ \left(c_j^{(i)};\gamma_j^{(i)}\right)_{1,p_i}\\ \left(-\frac{\eta}{1-\lambda}-\rho;\mu,\dots,\mu\right):\left(d_j^{(i)};\delta_j^{(i)}\right)_{1,q_i}\end{matrix}\right.\right] \tag{26}$$

If we observe that for $n = p = q = 0$ and $r = 1$, the multivariable H-function reduces in to single variable H-function and consequently there holds the following Corollary.

Corollary 2: *Let $\lambda < 1, \mu > 0, d > 0, l > 0$, the parameters $z, \eta, \rho \in \mathbb{C}, \Re(\eta) > 0, \Re(\rho) > 0$ and $\Re\left(\frac{\eta}{1-\lambda}\right) > -1$, then we obtain*

$$P_{0+}^{(\eta,\lambda,l)}\left\{t^{\rho-1}H_{p,q}^{m,n}(d^\mu t^\mu z)\right\} = \frac{t^{\eta+\rho}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{[l(1-\lambda)]^\rho} \times H_{p+1,q+1}^{m,n+1}\left[\frac{d^\mu t^\mu z}{\{l(1-\lambda)\}}\left|\begin{matrix}(1-\rho,\mu),(a_p,\alpha_p)\\ \left(-\frac{\eta}{1-\lambda}-\rho,\mu\right),(b_q,\beta_q)\end{matrix}\right.\right] \tag{27}$$

Setting $\alpha_i = \beta_j = 1$ in the Corollary 2, yields the following result.

Corollary 3: *Let $\lambda < 1, \mu > 0, d > 0, l > 0$, the parameters $z, \eta, \rho \in \mathbb{C}, \Re(\eta) > 0, \Re(\rho) > 0$ and $\Re\left(\frac{\eta}{1-\lambda}\right) > -1$, then we obtain*

$$P_{0+}^{(\eta,\lambda,l)}\left\{t^{\rho-1}G_{p,q}^{m,n}(d^\mu t^\mu z)\right\} = \frac{t^{\eta+\rho}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{[l(1-\lambda)]^\rho} \times G_{p+1,q+1}^{m,n+1}\left[\frac{d^\mu t^\mu z}{\{l(1-\lambda)\}}\left|\begin{matrix}(1-\rho,\mu),(a_i)_1^p\\ \left(-\frac{\eta}{1-\lambda}-\rho,\mu\right),(b_j)_1^q\end{matrix}\right.\right] \tag{28}$$

Interestingly, on setting $m = 1, p = 1, n = 1, q = 2, \mu = 1, z = -Y, a_1 = 1-\gamma, \alpha_1 = 1,\ b_1 = 0, \beta_1 = 1,\ b_2 = 1-\beta, \beta_2 = \alpha$, then Corollary 2 and consequently there holds the following result.

Corollary 4: *Let $\lambda < 1, d > 0, l > 0$, the parameters $\alpha, \beta, \gamma, \eta, \rho \in \mathbb{C}, \Re(\alpha) > 0, \Re(\eta) > 0, \Re(\rho) > 0$ and $\Re\left(\frac{\eta}{1-\lambda}\right) > -1$, then we have*

$$P_{0+}^{(\eta,\lambda,l)}\left\{t^{\rho-1}E_{\alpha,\beta}^{\gamma}(Ytd)\right\} = \frac{t^{\eta+\rho}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{[l(1-\lambda)]^\rho\,\Gamma(\gamma)} \times H_{2,3}^{1,2}\left[\begin{matrix}(1-\rho,1),(1-\gamma,1) & ; & -Ytd\\ (0,1),\left(-\frac{\eta}{1-\lambda}-\rho,1\right),(1-\beta,\alpha) & ; & \overline{\{l(1-\lambda)\}}\end{matrix}\right] \tag{29}$$

Further, if we set $m = 1, n = 0, p = 0, q = 0, \mu = 2, b_1 = 0, \beta_1 = 1, b_2 = v, \beta_2 = 1, \rho = \sigma + \vartheta$ in Corollary 2, then we obtain the following Corollary.

Corollary 5: Let the condition of Corollary 2 be satisfied then the equation (26) reduces to the following result.

$$P_{0+}^{(\eta,\lambda,l)}\{t^{\rho-1}J_v(dtz)\} = \left(\frac{zd}{2}\right)^v \frac{t^{\eta+\sigma+\vartheta}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{[l(1-\lambda)]^{\sigma+\vartheta}} \times {}_1\psi_2\left[\begin{matrix}(\sigma+\vartheta,2) & ; & -d^2t^2z^2 \\ (\vartheta+1,1),\left(1+\frac{\eta}{1-\lambda}+\sigma+\vartheta,2\right) & ; & \overline{4\{l(1-\lambda)\}}\end{matrix}\right] \quad (30)$$

where $J_v(z)$ is the ordinary Bessel function of first kind (Olver [15]).

Corollary 6: Let the condition of corollary 2 be satisfied and on setting $= 1, n = 1, p = 1, q = 2, a_1 = 1-a, \alpha_1 = 1, b_1 = 0, \beta_1 = 0, b_2 = 1-c, \mu = 1, \beta_2 = 1\ z = -Y$, then equation (26) reduces to the following formula:

$$P_{0+}^{(\eta,\lambda,l)}\{t^{\rho-1}{}_1F_1(a;c;dtY)\} = \frac{t^{\eta+\rho}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)\Gamma(\rho)}{[l(1-\lambda)]^{\rho}\ \Gamma\left(\frac{\eta}{1-\lambda}+\rho+1\right)} \times {}_2F_2\left[\begin{matrix}a,(\rho) & ; & Ytd \\ c,\left(\frac{\eta}{1-\lambda}+\rho+1\right) & ; & \overline{\{l(1-\lambda)\}}\end{matrix}\right] \quad (31)$$

Provided $\Re(\eta) > 0, \Re(\rho) > 0$ *and* $\Re\left(\frac{\eta}{1-\lambda}\right) > -1$.

4. FRACTIONAL INTEGRATION OF GENERALIZED M-SERIES VIA PATHWAY MODEL

In this section we consider composition of the pathway fractional integral $P_{0+}^{(\eta,\lambda,l)}$ given by (2) with the generalized M-series (20).

Theorem 2: *Let* $\lambda < 1, \mu > 0, d > 0, l > 0$, *the parameters* $z, \alpha, \beta, \gamma, \eta, \rho \in \mathbb{C}$ *and* $\Re(\eta) > 0$, $\Re(\alpha) > 0, \Re(\rho) > 0\ \Re\left(\frac{\eta}{1-\lambda}\right) > -1$, *then the following formula holds:*

$$P_{0+}^{(\eta,\lambda,l)}\left\{t^{\rho-1}\,{}_p^{\alpha}M_q^{\beta}(d^\mu t^\mu z)\right\} = \frac{t^{\eta+\rho}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{[l(1-\lambda)]^{\rho}} \frac{\sum_{j=1}^{q}\Gamma(b_j)}{\sum_{i=1}^{p}\Gamma(a_i)}$$
$$\times H_{p+2,q+3}^{1,p+2}\left[-\frac{d^\mu t^\mu z}{\{l(1-\lambda)\}}\middle|\begin{matrix}(1-\rho,\mu),(1-a_i)_1^p,(0,1) \\ \left(-\frac{\eta}{1-\lambda}-\rho,\mu\right),(0,1),(1-\beta,\alpha),\left(1-b_j\right)_1^q\end{matrix}\right] \quad (32)$$

The converges condition for the validity (32) are as follows

(i) $0 \le m \le q$, $0 \le n \le p$ for $a_i, b_j \in \mathbb{C}$ and for $\alpha_i, \beta_j \in \Re_+ = (0,\infty)$ $(i = 1,\dots,p; j = 1,\dots,q)$.

(ii) $\mathcal{L} = \mathcal{L}_{i\gamma\infty}$ is a contour starting at the point $\gamma - i\infty$ and going to $\gamma + i\infty$ where $\gamma \in \Re\ (-\infty,\infty)$, such that all the poles of $\Gamma(b_j+\beta_j\xi), j = 1,\dots,m$ are separated from those $\Gamma(1-a_j-\alpha_j\xi), i = 1,\dots,n$.

The integral converges if $\alpha > 0, |\arg z| < \frac{1}{2}\pi\alpha, \alpha \neq 0$ and $\sigma = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p}\alpha_j + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}\beta_j$.

Proof: For convenience, let the left hand side of (32) be denoted by $\Im$. Applying (20) and using (2) to left hand side of (32), then we have

$$\Im = \sum_{m=0}^{\infty}\frac{(a_1)_m\dots(a_p)_m}{(b_1)_m\dots(b_q)_m}\frac{(d^\mu z)^m}{\Gamma(\alpha m+\beta)} \times \left[P_{0+}^{(\eta,\lambda,l)}(t^{\rho+\mu m-1})\right]$$

Now we apply Lemma 1 to use (24) with β replaced by $(\rho+\mu m)$ to pathway integral, after a little simplification, we obtain

$$\Im = \frac{t^{\eta+\rho}\,\Gamma\left(1+\frac{\eta}{1-\lambda}\right)}{[l(1-\lambda)]^{\rho}}\frac{\sum_{j=1}^{q}\Gamma(b_j)}{\sum_{i=1}^{p}\Gamma(a_i)}\sum_{m=0}^{\infty}\frac{\Gamma(a_i+m)}{\Gamma(b_j+m)}\frac{\Gamma(\rho+\mu m)}{\Gamma(\alpha m+\beta)} \times \frac{1}{\Gamma\left(\frac{\eta}{1-\lambda}+\rho+\mu m+1\right)}\left[\frac{d^\mu t^\mu z}{l(1-\lambda)}\right]^m$$

which, upon using definition of H-function (16), yields (32).

Several illustrative examples of Theorem 7 involving appropriately chosen special values of parameters can also be derived fairly easily.

5. FURTHER SPECIAL CASES

By setting $\lambda = 0, l = 1$ and $\eta \longrightarrow \eta - 1$ in (25) and (32) respectively, and applying the following easily derivable relation:

$$\left(P_{0+}^{(\eta-1,0,1)}f\right)(t) = \int_0^t (t-\tau)^{\eta-1}f(\tau)\,d\tau = \Gamma(\eta)\ \left(I_{0+}^{\eta}f\right)(t), \qquad (\Re(\eta) > 0)$$

We obtain two fractional integral formulas involving left-sided Riemann-Liouville fractional integral operator stated in the next Corollaries below.

Corollary 7: *Let* $\lambda < 1, \mu > 0, d > 0, l > 0,$ *the parameters* $z_i, \eta, \rho \in \mathbb{C}\ (i = 1, \dots, r), \Re(\eta) > 0, \Re(\rho) > 0$ *and* $\Re\left(\frac{\eta}{1-\lambda}\right) > -1,$ *then the following relation*

$$I_{0+}^{\eta}\{t^{\rho-1}\, H[d^{\mu}t^{\mu}z_1, \dots, d^{\mu}t^{\mu}z_r]\}$$
$$= t^{\rho+\eta-1}\,\Gamma(\eta)\, H^{0,n+1:m_1,n_1;\dots;m_r,n_r}_{p+1,q+1:p_1,q_1;\dots;p_r,q_r}\left[\begin{matrix}(1-\rho;\mu,\dots,\mu),\left(a_j;\alpha_j^{(1)},\dots,\alpha_j^{(r)}\right)_{1,p}\\ \left(b_j;\beta_j^{(1)},\dots,\beta_j^{(r)}\right)_{1,q},(1-\eta-\rho;\mu,\dots,\mu)\end{matrix}\right.$$
$$\left.\begin{matrix}:\left(c_j^{(1)};\gamma_j^{(1)}\right)_{1,p_1};\dots;\left(c_j^{(r)};\gamma_j^{(r)}\right)_{1,p_r}\\ :\left(d_j^{(1)};\delta_j^{(1)}\right)_{1,q_1};\dots;\left(d_j^{(r)};\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\middle|\begin{matrix}d^{\mu}t^{\mu}z_1\\ \vdots\\ d^{\mu}t^{\mu}z_r\end{matrix}\right] \tag{33}$$

holds.

Corollary 8: *Let* $\lambda < 1, \mu > 0, d > 0, l > 0,$ *the parameters* $z, \alpha, \beta, \gamma, \eta, \rho \in \mathbb{C}$ *and* $\Re(\eta) > 0,$ $\Re(\alpha) > 0, \Re(\rho) > 0\ \Re\left(\frac{\eta}{1-\lambda}\right) > -1.$*Then the following formula relation:*

$$I_{0+}^{\eta}\left\{t^{\rho-1}\,{}_{p}^{\alpha}M_q^{\beta}(d^{\mu}t^{\mu}z)\right\} = t^{\eta+\rho-1}\,\Gamma(\eta)\frac{\sum_{j=1}^{q}\Gamma(b_j)}{\sum_{i=1}^{p}\Gamma(a_i)}$$
$$\times H^{1,p+2}_{p+2,q+3}\left[-d^{\mu}t^{\mu}z\,\middle|\,\begin{matrix}(1-\rho,\mu),(1-a_i)_1^p,(0,1)\\ (1-\eta-\rho,\mu),(0,1),(1-\beta,\alpha),\left(1-b_j\right)_1^q\end{matrix}\right] \tag{34}$$

holds.

Remark: It is noted that if we set $\lambda = 0, l = 1$ and $f(t)$ is replaced by ${}_2F_1\left(\eta+\beta,-\gamma;\eta;1-\frac{t}{x}\right)f(t)\,dt,$ (2) yields the Saigo fractional integral operator. Thus we can obtain the generalizations of left-sided fractional integrals, like Saigo, Erdélyi-Kober (see [28]; see also [20]), and so on, by suitable substitutions. Therefore, the result presented here easily shown to be converted to those corresponding to the above well known fractional operators.

5. CONCLUDING REMARKS

We conclude this investigation by remarking that the result obtained here are in general character and useful in deriving various integral formulas in the theory of the pathway fractional integral formula. In this paper we have presented composition formula of the pathway fractional integration involving Fox H-function and M-Series. Results derived in this paper are very significant and may find application in the solution of fractional order differential equations that are arising in theory of special functions. Pathway fractional integral operator of several special functions given as special cases of our main results.

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