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A NOTE ON INCLINE ALGEBRAS

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ABSTRACT

In this paper, we prove that -

(1) In the definition of an incline algebra K with zero element 0, the conditions
(i) $a+0=a$ for all $a \in K$ and (ii) $a * 0 = 0 * a = 0$ for all $a \in K$ are equivalent and hence any one of them can be deleted.

(2) "Every irreducible ideal of an incline algebra is not prime" is shown by giving an example.

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0. INTRODUCTION

Sun Shin Ahn, young bae Jun and Hee Sik Kim [1] introduced and studied the concepts - Sub incline, ideal, quotients of an incline algebras, prime ideal, irreducible ideal and maximal ideal of an incline algebra and their properties.

1. PRELIMINARIES

Definition 1.1: [1]. Incline algebra: A system $(K,+,*)$, where K is a non empty set "$+$" and "$*$" are binary operations on K satisfying the following axioms is called an incline algebra.

(i) $x+y = y+x$ ($+$ is commutative)

(ii) $x+(y+z) = (x+y)+z$ ($+$ is associative)

(iii) $x*(y*z) = (x*y)*z$ ($*$ is associative)

(iv) $x*(y+z) = (x*y)+ (x*z)$ ($*$ is left distributive)

(v) $(y+z)*x = (y*x) + (z*x)$ ($*$ is right distributive)

(vi) $x+x = x$ ($+$ is idempotent)

(vii) $x+(x*y) = x$

(viii) $y+(x*y) = y$ for all $x, y, z \in K$

Definition 1.2: [1]. Let $(K,+,*)$ be an incline algebra.

(i) K is called commutative if
$x*y = y*x$ for all $x, y \in K$.

(ii) An element $0 \in K$ is called a zero element if
$x+0 = x$ and $x * 0 = 0 * x = 0$ for all $x \in K$

(iii) An element $1 \in K$ is called a multiplicative identity if
$x*1 = 1*x = x$ for all $x \in K$

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Clearly, every distributive lattice $(K, \vee, \wedge)$ is an incline algebra $(K, +, *)$ with $+ = \vee$ and $* = \wedge$. Every incline algebra is not a distributive as the following example shows

Example 1.3: Consider the system $(K, +, *)$ where $K = \{0,1\}$ and the binary operations $+$ and $*$ are given by

+	0	1
0	0	1
1	1	1

*	0	1
0	0	0
1	0	0

This system is an incline algebra but not a distributive lattice since $0 = 1 \wedge 1$(by the definition of $\wedge$) $\neq 1$.

Note 1.3.1: Let $(K, +, *)$ be an incline algebra. From axioms (i), (ii), (vi), $(K, +)$ is a semi lattice and hence the binary relation $\leq$ on K, defined by $"x \leq y \Leftrightarrow x + y = y"$ is a partial ordering on K, such that for any $x, y \in K$, $x \vee y = l.u.b\{x, y\}$ exists and $x \vee y = x + y$.

Definition 1.4: [1]. A sub incline of an incline (algebra) $(K, +, *)$ is a non empty subset M of K which is closed under the operations $+$ and $*$

i.e., $"x, y \in M \Rightarrow x + y \in M,\ x * y \in M"$.

Definition 1.5: [1]. A sub incline M of an incline algebra $(K, +, *)$ is called an ideal if $"x \in M,\ y \in K,\ y \leq x \Rightarrow y \in M"$

Note 1.5.1: An ideal M of an incline algebra K is called proper if $M \neq K$. By the definition, every ideal of an incline algebra is a sub incline. Converse is not true as the following example shows.

Example 1.6: Consider the incline algebra $(K, +, *)$ where $K = \{0, 1, a\}$ the binary operations $+$ and $*$ are given by

+	0	1	a
0	0	1	a
1	1	1	a
a	a	a	a

*	0	1	a
0	0	0	0
1	0	0	0
a	0	0	0

Here $M = \{0, a\}$ is clearly, a sub incline of K. Clearly $1 \leq a$ (since $1 + a = a$), $a \in M$, but $1 \notin M$. So, M is not an ideal of K.

Definition 1.7: [1]. A proper ideal I of an incline algebra $(K, +, *)$

(i) prime if

$"a, b \in K, a * b \in I \Rightarrow a \in I \text{ or } b \in I"$

(ii) maximal ideal if

$"N$ is an ideal of K, $I \subseteq N, \Rightarrow I = N \text{ or } N = K"$

(iii) an irreducible ideal if

$"A \cap B = I \Rightarrow A = I \text{ or } B = I"$ for any ideals A and B of K

Theorem 1.8: [1]. Let I be a proper ideal of an incline algebra K. The following statements are equivalent.

(a) I is an irreducible ideal.

(b) I is prime.

(c) $A \cap B \subseteq I \Rightarrow A \subseteq I \text{ or } B \subseteq I$ for any ideals A and B of K

2. MAIN RESULTS OF THE PAPER

We begin with the following

Theorem 2.1: Let $(K, +, *)$ be an incline algebra. For any $x, y \in K$, $x * y$ is a lower bound of $\{x, y\}$

i.e., $x * y \leq x,\ x * y \leq y$.

Proof: Let $x, y \in K$. Now, $x + x * y = x$ (by (vii) of def 1.1)

$\Rightarrow x * y + x = x$ (by (i)of def1.1)

$\Rightarrow x * y \leq x$

$y + x * y = y$ (by (viii) of def 1.1)

$\Rightarrow x * y + y = y$ (by (i) of def 1.1)

$\Rightarrow x * y \leq y$

Hence, $x * y$ is a lower bound of $\{x, y\}$.

Note 2.1.1: Interchanging x and y in theorem 2.1, we have that for any $x, y \in K$, $y * x$ is also a lower bound of $\{x, y\}$.

Theorem 2.2: Let K be an incline algebra. Let $0 \in K$. Then, the following statements are equivalent.

(i) $a + 0 = a$ for all $a \in K$:

(ii) 0 is the least element of K i:e, $0 \leq a$ for all $a \in K$

(iii) $a * 0 = 0 = 0 * a$ for all $a \in K$

Proof:

(i) $\Rightarrow$ (ii): Trivial by the definition of $\leq$.

(ii) $\Rightarrow$ (iii): Assume (ii). Let $a \in K$. By theorem 2.1, $a * 0 \leq 0$, $0 * a \leq 0$. Since 0 is the least element of K, We have $0 \leq a * 0$ and $0 \leq 0 * a$.

Hence $a * 0 = 0 = 0 * a$.

(iii) $\Rightarrow$ (i): Assume (iii) . For any $a \in K$,

$a = a + a * 0$ (by (vii) of definition 1.1)

$= a + 0$ (by our assumption).

Hence the theorem.

Note 2.2.1: Since (i) and (iii) are equivalent in theorem 2.2, we can retain any one of "$a + 0 = a$ for all $a \in K$" and "$a * 0 = 0 * a = 0$ for all $a \in K$" in the definition 1.2 (ii) of zero element in the preliminaries.

Theorem 2.3: Let I be a proper ideal of an incline algebra K. Consider the following statements.

(a) I is an irreducible ideal.

(b) I is prime.

(c) $A \cap B \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$ for any ideals A and B of K Then, (b) $\Rightarrow$ (c) $\Rightarrow$ (a) holds.

Proof:

(b) $\Rightarrow$ (c): Assume (b). Suppose (c) fails i.e., there exist ideals A, B of K such that $A \not\subset I, B \not\subset I$ and $A \cap B \subseteq I$. So, there exist elements x, y in K suchthat $x \in A - I$, $y \in B - I$. By theorem 2.1, $x * y \leq x$ and $x * y \leq y$. Since A and B are ideals, $x * y \in A \cap B$. Since $A \cap B \subseteq I$, we have that $x * y \in I$.

Since I is prime (by our assumption), either $x \in I$ or $y \in I$, a contradiction. Hence (c) holds.

(c) $\Rightarrow$ (a): Trivial.

Note 2.3.1: In [1], it is prove that the statements (a),(b) and (c) of theorem 2.3 are equivalent(see theorem 1.8. in the preliminaries). But this is not true as the following example shows.

Example 2.4: Consider the incline algebra $(K,+,*)$ of the example 1.6. Clearly, $I=\{0\}$ and $J=\{0,1\}$ are the only proper ideals of K. Clearly, I and J are irreducible ideals of K. I is not a prime ideal since $a\notin I$ and $a*a=0\in I$. Similarly, J is not a prime ideal.

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REFERENCES

[1] Sun Shin Ahn, young bae Jun, and Hee Sik Kim, Ideals and quotients of incline algebras, Comm. Korean math. Soc. 16 (2001), 573-583.

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