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ON CONFORMAL CURVATURE TENSOR IN LORENTZIAN β-KENMOTSU MANIFOLDS

RIDDHI JUNG SHAH*

Department of Mathematics, Janata Cmpus, Nepal Sanskrit University, Nepal.

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ABSTRACT

In this paper we study the geometry of conformal curvature tensor in Lorentzian β-Kenmotsu manifolds. It is proved that conformally flat, φ-conformally flat and conformally recurrent Lorentzian β-Kenmotsu manifolds are η-Einstein manifolds.

Keywords: *β-Kenmotsu manifold, conformal curvature tensor, η-Einstein manifold, Einstein manifold.*

MSC 2010: *53C25, 53C50, 53D15.*

1. INTRODUCTION

In the Gray Hervella classification of almost Hermitiation manifolds, there appears a class W_4 of Hermitian manifolds which are closely related to the locally conformal Kaehler manifolds [1]. An almost contact metric manifold $M^n(\varphi,\xi,\eta,g)$ is said to be trans-Sasakian manifold if $(M\times R,J,G)$ belongs to the class W_4 of Hermitian manifolds where J is the almost complex structure on $M\times R$ defined by

(1.1) $$J\left(Z,f\frac{d}{dt}\right)=\left(\phi Z-f\xi,\eta(Z)\frac{d}{dt}\right)$$

for any vector field Z on M and smooth function f on $M\times R$ and G is the product metric on $M\times R$ [3]. This may be stated by the condition

(1.2) $$(\nabla_X\varphi)Y=\alpha\{g(X,Y)\xi-\eta(Y)X\}+\beta\{g(\varphi X,Y)\xi-\eta(Y)\varphi X\}$$

where α,β are smooth functions on M and we say that such a structure is the trans-Sasakian structure of type (α,β) [3,5].

Trans-Sasakian structure of type (0, 0), $(\alpha,0)$ and $(0,\beta)$ are the cosympletic, α-Sasakian and β-Kenmotsu manifold respectively. Kenmotsu manifolds defined in [2] are particular examples with $\beta=1$ i.e., Kenmotsu manifolds are the trans-Sasakian structure of type (0, 1). Lorentzian β-Kenmotsu manifolds have been studied by Prakasha *et al*. [4], Shreenivasa *et al*. [7] and others.

2. PRELIMINARIES

An n-dimensional differentiable manifold M is called Lorentzian β-Kenmotsu manifold if it admits a (1, 1) tensor field φ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

(2.1) $$\eta(\xi)=-1,\varphi^2(X)=X+\eta(X)\xi,\varphi\xi=0,\eta(\varphi X)=0,$$

(2.2) $$g(X,\xi)=\eta(X),$$

(2.3) $$g(\varphi X,\varphi Y)=g(X,Y)+\eta(X)\eta(Y),$$

(2.4) $$\nabla_X\xi=\beta[X-\eta(X)\xi],$$

(2.5) $$(\nabla_X\eta)(Y)=\beta[g(X,Y)-\eta(X)\eta(Y)],$$

(2.6) $$\eta(R(X,Y)Z)=\beta^2[g(X,Z)\eta(Y)-g(Y,Z)\eta(X)],$$

Corresponding Author: Riddhi Jung Shah*
Department of Mathematics, Janata Cmpus, Nepal Sanskrit University, Nepal.

(2.7) $$R(\xi, X)Y = \beta^2[\eta(Y)X - g(X,Y)\xi],$$

(2.8) $$R(X,Y)\xi = \beta^2[\eta(X)Y - \eta(Y)X],$$

(2.9) $$S(X,\xi) = (1-n)\beta^2\eta(X),$$

(2.10) $$Q\xi = (1-n)\beta^2\xi,$$

(2.11) $$S(\varphi X, \varphi Y) = S(X,Y) + (1-n)\beta^2\eta(X)\eta(Y).$$

where R, S and Q are the curvature tensor, the Ricci tensor and the Ricci operator respectively.

3. CONFORMALLY FLAT LORENTZIAN β-KENMOTSU MANIFOLDS

In this section we prove the following results

Theorem 3.1: A conformally flat Lorentzian β -Kenmotsu manifold $M^n(\varphi,\xi,\eta,g)$ is an η -Einstein manifold.

Proof: Let us consider a Lorentzian β -Kenmotsu manifold $(M^n, g), n > 3.$ The Weyl conformal curvature tensor C of type (1, 3) on a Riemannian manifold is defined by [8]

(3.1) $$C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y].$$

For conformally flat β -Kenmotsu manifold we have $C(X,Y)Z = 0$ and then (3.1) reduces to

(3.2) $$R(X,Y)Z = \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{r}{(n-1)(n-2)}[g(Y,Z)X - -g(X,Z)Y].$$

Taking inner product by W in (3.2) we obtain

(3.3) $$\tilde{R}(X,Y,Z,W) = \frac{1}{n-2}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) + S(X,W)g(Y,Z) - S(Y,W)g(X,Z)] - \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)],$$

where $\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W).$

Putting $W = \xi$ in (3.3) and using (2.2), (2.6) and (2.9) we get

(3.4) $$S(Y,Z)\eta(X) - S(X,Z)\eta(Y) = -\left(\beta^2 + \frac{r}{n-1}\right)\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}.$$

Replacing $x_i = \xi$ in (3.4) and using (2.1), (2.2) and (2.9) we obtain

(3.5) $$S(X,Z) = \lambda_1 g(X,Z) + \lambda_2\eta(X)\eta(Z),$$

where $\lambda_1 = \beta^2 + \frac{r}{n-1}$ and $\lambda_2 = n\beta^2 + \frac{r}{n-1}$.

From (3.5) we conclude that the manifold is η -Einstein. This completes the proof of the theorem.

Now, taking an orthonormal frame field and contracting over X and Z in (3.5) we get

(3.6) $$r = -n(n-1)\beta^2,$$

where r is the scalar curvature. This leads to the following corollary:

Corollary 3.1: A conformally flat Lorentzian β-Kenmotsu manifold $M^n(\varphi,\xi,\eta,g)$ is of constant scalar curvature $r=-n(n-1)\beta^2$.

Again, in view of (3.5) and (3.6) we obtain

(3.7) $$S(X,Z)=(1-n)\beta^2 g(X,Z).$$

This implies that the manifold is an Einstein manifold. Hence we have next result

Theorem 3.2: A conformally flat β-Kenmotsu manifold $M^n(\varphi,\xi,\eta,g)$ is an Einstein manifold with scalar curvature $r=-n(n-1)\beta^2$.

4. φ-CONFORMALLY FLAT LORENTZIAN β-KENMOTSU MANIFOLDS

Definition 4.1: A Lorentzian β-Kenmotsu manifold $M^n(\varphi,\xi,\eta,g)$ is said to be φ-conformally flat if the condition

(4.1) $$g(C(\varphi X,\varphi Y)\varphi Z,\varphi W)=0$$

holds for any vector fields $X,Y,Z\in TM$ [6].

Theorem 4.1: A φ-conformally flat Lorentzian β-Kenmotsu manifold $M^n(\varphi,\xi,\eta,g)$ is an η-Einstein manifold.

Proof: Let us consider an n-dimensional Lorentzian β-Kenmotsu manifold M. Suppose that the condition (4.1) holds in M, then in view of (3.1) and (4.1) we obtain

(4.2) $$\begin{aligned}\tilde{R}(\varphi X,\varphi Y,\varphi Z,\varphi W)&=\frac{1}{n-2}[S(\varphi Y,\varphi Z)g(\varphi X,\varphi W)-S(\varphi X,\varphi Z)g(\varphi Y,\varphi W)\\&\quad+S(\varphi X,\varphi W)g(\varphi Y,\varphi Z)-S(\varphi Y,\varphi W)g(\varphi X,\varphi Z)]\\&\quad-\frac{r}{(n-1)(n-2)}[g(\varphi Y,\varphi Z)g(\varphi X,\varphi W)-g(\varphi X,\varphi Z)g(\varphi Y,\varphi W)]\end{aligned}$$

where $\tilde{R}(\varphi X,\varphi Y,\varphi Z,\varphi W)=g(R(\varphi X,\varphi Y)\varphi Z,\varphi W)$.

By virtue of (2.3), (2.6), (2.11) and (4.2) we get

(4.3) $$\begin{aligned}&\beta^2[g(X,Z)g(Y,W)-g(Y,Z)g(X,W)+g(X,Z)\eta(Y)\eta(W)\\&+g(Y,W)\eta(X)\eta(Z)-g(X,W)\eta(Y)\eta(Z)]\\&=\frac{1}{n-2}[S(Y,Z)g(X,W)+S(Y,Z)\eta(X)\eta(W)-(n-1)\beta^2 g(X,W)\eta(Y)\eta(Z)\\&-S(X,Z)g(Y,W)-S(X,Z)\eta(Y)\eta(W)+(n-1)\beta^2 g(Y,W)\eta(X)\eta(Z)\\&+S(X,W)g(Y,Z)+S(X,W)\eta(Y)\eta(Z)-(n-1)\beta^2 g(Y,Z)\eta(X)\eta(W)\\&-S(Y,W)g(X,Z)-S(Y,W)\eta(X)\eta(Z)+(n-1)\beta^2 g(X,Z)\eta(Y)\eta(W)]\\&-\frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W)+g(Y,Z)\eta(X)\eta(W)+g(X,W)\eta(Y)\eta(Z)\\&-g(X,Z)g(Y,W)-g(X,Z)\eta(Y)\eta(W)-g(Y,W)\eta(X)\eta(Z)].\end{aligned}$$

Let $\{e_i : i=1,2,\dots,n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Putting $X=W=e_i$ in (4.3) and taking summation over $i, 1\le i\le n$, we get

(4.4) $$S(Y,Z)=\left[-\frac{(n^3-4n^2+4n-1)\beta^2-r}{(n-1)^2}\right]g(Y,Z)+\left[-\frac{n(n-1)\beta^2+r}{(n-1)^2}\right]\eta(Y)\eta(Z).$$

This implies that

(4.5) $$S(Y,Z)=ag(Y,Z)+b\eta(Y)\eta(Z)$$

where $a=-\dfrac{(n^3-4n^2+4n-1)\beta^2-r}{(n-1)^2}$ and $b=-\dfrac{n(n-1)\beta^2+r}{(n-1)^2}$.

From (4.5), it follows that the manifold is η-Einstein. Hence the theorem is proved.

Taking an orthonormal frame field at any point of the manifold and contracting over Y and Z in (4.4), we get

(4.6) $$r = -n(n-1)\beta^2.$$

By virtue of (4.6) and (4.4), we obtain

(4.7) $$S(Y,Z) = -(n-1)\beta^2 g(Y,Z).$$

This implies that the manifold is Einstein.

Thus we can state the following result:

Theorem 4.2: A φ-conformally flat Lorentzian β-Kenmotsu manifold $M^n(\varphi,\xi,\eta,g)$ is an Einstein manifold with scalar curvature $r = -n(n-1)\beta^2$.

5. CONFORMALLY RECURRENT LORENTZIAN β-KENMOTSU MANIFOLDS

Definition 5.1: A non-flat Riemannian manifold M is said to be conformally recurrent if the conformal curvature tensor C satisfies the condition

(5.1) $$\nabla C = A \otimes C,$$

where A is an everywhere non-zero 1-form.

We now define a function f on M by $f^2 = g(C,C)$, where the metric g is extended to the inner product between the tensor fields in the standard fashion. Then we know that

$$f(Yf) = f^2 A(Y).$$

So from this equation we have

(5.2) $$Yf = fA(Y), \quad \text{since } f \neq 0.$$

From (5.2), we obtain

(5.3) $$X(Yf) = \frac{(Yf)}{f}(Xf) + (XA(Y))f.$$

Similarly,

(5.4) $$Xf = fA(X),$$

From which we get

(5.5) $$Y(Xf) = \frac{(Xf)}{f}(Yf) + (YA(X))f.$$

From (5.3) and (5.5), we obtain

$$X(Yf) - Y(Xf) = \{(XA(Y)) - YA(X)\}f.$$

Therefore we get

$$(\nabla_X\nabla_Y - \nabla_Y\nabla_X)f = \{XA(Y) - YA(X)\}f.$$

Adding $-\nabla_{[X,Y]}f$ and using definition of recurrent on right side we obtain

(5.6) $$(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})f = \{XA(Y) - YA(X) - A[X,Y]\}f.$$

Since the left hand side of (5.6) is identically zero and $f \neq 0$ on M by our assumption we obtain

(5.7) $$dA(X,Y) = 0.$$

This implies that the 1-form A is closed.

Now, from (5.1), we get

$$(\nabla_U \nabla_V C)(X,Y)Z = \{UA(V) + A(U)A(V)\}C(X,Y)Z.$$

Hence from (5.7), we obtain

$$\left(R(X,Y).C\right)(U,V)\,Z = \left[2d\,A(X,Y)\right]C\,(U,V)\,Z = 0.$$

Therefore for a conformal recurrent manifold, we have

(5.8) $$R(X,Y).C = 0$$

for all $X, Y \in TM$.

Equation (5.8) implies that the manifold is conformally semi-symmetric. This completes the proof of the theorem.

It is known that a conformally semi-symmetric Lorentzian β -Kenmotsu manifold is an η -Einstein manifold [7].

This leads to the following theorem:

Theorem 5.2: A conformally recurrent Lorentzian β -Kenmotsu manifold $M^n(\varphi, \xi, \eta, g)$ is an η -Einstein manifold.

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