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NUMERICAL SOLUTION OF THE FRACTIONAL KORTEWEG–DE VRIES (KdV) EQUATION BY q-HOMOTOPY ANALYSIS METHOD (q-HAM)

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ABSTRACT

In this paper, we consider the fractional Korteweg–de Vries (KdV) equation. A relatively new method called the q-homotopy analysis method (q-HAM) is adopted to obtain an analytical solution of the fractional Korteweg–de Vries (KdV) equation in series form. Our analysis shows the simplicity nature of the application of q-HAM to nonlinear fractional differential equations. The convergence rate of the method used is faster in the sense that just very few terms of the series solution are needed for a good approximation due to the presence of the auxiliary parameter h comparable to exact solutions. Numerical solution obtained by this method is compared with the exact solution. Our error analysis shows that the analytical solution converges very rapidly to the exact solution. Numerical results are obtained using the software Mathematica.

Keywords: *q-homotopy analysis method (q-HAM), fractional Korteweg–de Vries (KdV) equation, approximate numerical solutions, symbolic computation.*

1. INTRODUCTION

In recent decades, fractional calculus has found diverse applications in different scientific and technological fields [1, 2], such as thermal engineering, acoustics, electromagnetism, control, robotics, viscoelasticity, diffusion, edge detection, turbulence, signal processing, information sciences, communications, and many other physical processes and also in medical sciences. Fractional differential equations (FDEs) have also been applied in modeling many physical and engineering problems and fractional differential equations in nonlinear dynamic [3]. The importance of getting approximate and exact solutions of nonlinear fractional differential equations in mathematics and physics remains an important problem that requires the discovering new methods of approximate and exact solutions. However, finding the exact solutions to these non-linear fractional differential equations are difficult to obtain it [4]. Therefore, the numerical methods used to deal with these equations [5] and they have largely been using some semi analytical techniques to solve these equations such as, differential transform method [6, 7, 8], Laplace decomposition method [9], homotopy perturbation method [10], variational iteration method [11, 12] and homotopy analysis method (HAM) [13, 14, 15]. The HAM initially proposed by Liao in his Ph.D. and thesis [13] is a powerful method to solve the non-linear problems. In recent years, this method has been successfully employed to solve many types of non-linear problems in science and engineering [16, 17, 18]. The HAM contains a certain auxiliary parameter *h*, which provides us with a simple way to adjust and control the convergence region and the rate of convergence of the series solution. Many workers applied the HAM to solve fractional differential equations [19, 20]. El-Tawil and Huseen [21] established a method namely q-homotopy analysis method (q-HAM) which is a more general method of HAM, The q-HAM contains an auxiliary parameter n as well as h such that the case of $n=1$ (q-HAM; $n=1$) the standard homotopy analysis method (HAM) can be reached. In this paper, we have applied the q-homotopy analysis method (q-HAM) [22, 23] to solve the fractional Korteweg–de Vries (KdV) equation [24] with given initial condition. The main advantage of the method is the fact that it provides its user with an analytical approximation solution, in a rapidly convergent series with elegantly computed terms. The structure of this paper is organized as follows:

In section 2, we begin with the basic definition of Caputo's fractional derivative. In section 3, we give the basic concept of the q-homotopy analysis method (q-HAM). In section 4, we apply this method to solve the fractional Korteweg–de Vries (KdV) equation with given initial condition.

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2. PRELIMINARIES

This section is devoted to some definitions and some known results. Caputo's fractional derivative is adopted in this work.

Definition 2.1: The Riemann-Liouville's (RL) fractional integral operator of order $\alpha \geq 0$, of a function $f \in L^1(a,b)$ is given as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0, \ \alpha > 0, \tag{1}$$

where Γ is the Gamma function and $I^0 f(t) = f(t).$

Definition 2.2: The fractional derivative in the Caputo's sense is defined as [4]

$$D^{\alpha} f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \tag{2}$$

$where\ n-1 < \alpha \leq n,\ n \in \mathrm{N},\ t > 0$

Lemma 2.1: Let $t \in (a,b]$. Then

$$\left[I^{\alpha}(t-a)^{\beta}\right](t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha}, \quad \alpha \geq 0, \ \beta > 0. \tag{3}$$

3. BASIC CONCEPTS OF Q-HOMOTOPY ANALYSIS METHOD

Considering the following differential equation of the form:

$$N[D_t^{\alpha} u(x,t)] - f(x,t) = 0, \tag{4}$$

where N is a nonlinear operator, D_t^{α} denote the Caputo's fractional derivative, $u(x,t)$ is an unknown function, x and t denote the space and time variables and $f(x,t)$ is a known function, respectively. To generalize the original homotopy method, the zeroth-order deformation equation is constructed as

$$(1-nq) L[\varphi(x,t;q) - u_0(x,t)] = q\hbar H(x,t)(N[D_t^{\alpha} \varphi(x,t;q)] - f(x,t)), \tag{5}$$

where $n \geq 1,\ q \in \left[0, \frac{1}{n}\right]$ denotes the so called embedding parameter, $\hbar \neq 0$ is a non-zero auxiliary parameter, $H(x,t)$ is a non-zero auxiliary function, L is an auxiliary linear operator, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $\varphi(x,t;q)$ is an unknown function. It is important to note that one has great freedom to choose the auxiliary things in q-HAM. Clearly, when $q = 0$ and $q = \frac{1}{n}$, it holds that:

$$\varphi(x,t;0) = u_0(x,t) \ \ and \ \ \phi\left(x,t;\frac{1}{n}\right) = u(x,t). \tag{6}$$

Thus, as q increases from 0 to $\frac{1}{n}$, the solution $\varphi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. If $u_0(x,t), L, h, H(x,t)$ are chosen approximately, the solution $\varphi(x,t;q)$ of equation (5) exists for $q \in \left[0, \frac{1}{n}\right]$. Expanding $\varphi(x,t;q)$ in Taylor's series about $q = 0$, we have:

$$\varphi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) q^m, \tag{7}$$

where $$u_m(x,t) = \frac{1}{m!} \left.\frac{\partial^m \varphi(x,t;q)}{\partial q^m}\right|_{q=0}. \tag{8}$$

We suppose that the auxiliary linear operator *L*, the initial guess u_0, the nonzero auxiliary function $H(x,t)$ and the nonzero auxiliary parameter $\hbar$ are properly chosen such that the above series (7) converges at $q=\frac{1}{n}$, and then we have:

$$u(x,t)=\varphi(x,t;\frac{1}{n})=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)\left(\frac{1}{n}\right)^m, \tag{9}$$

which must be one of the solutions of the original non-linear differential equation. Let the vector u_n be defined as follows:

$$\overrightarrow{u_n}=\{u_0(x,t),u_1(x,t),u_2(x,t),...,u_n(x,t)\}. \tag{10}$$

Differentiating the equation (5), *m*-times with respect to the (embedding) parameter q, then evaluating at $q=0$ and finally dividing them by $m!$ throughout, we obtain the *m-th* order deformation equation (Lioa [13]) as:

$$L[u_m(x,t)-\chi_m^* u_{m-1}(x,t)]=\hbar H R_m(\overrightarrow{u_{m-1}}(x,t)) \tag{11}$$

with initial conditions $u_m^{(k)}(x,t)=0, \quad k=0,1,2,3,...,m-1$

$$R_m(\overrightarrow{u_{m-1}}(x,t))=\frac{1}{(m-1)!}\left.\frac{\partial^{m-1}(N[D_t^{\alpha}\varphi(x,t;q)]-f(x,t))}{\partial q^{m-1}}\right|_{q=0} \tag{12}$$

and $$\chi_m^*=\begin{cases}0, & m\le 1,\\ n, & otherwise.\end{cases} \tag{13}$$

3.1. Remark: It should be emphasized that $u_m(x,t)$ *for* $m\ge 1$ is governed by the linear operator (11) with the linear boundary conditions that come from the original problem. The existence of the factor $\left(\frac{1}{n}\right)^m$, more chances for convergence may occur or even much faster convergence can be obtained better than the standard HAM. It should be noted that the cases of $(n=1)$ in equation (5), the standard HAM can be reached.

4. APPROXIMATE SERIES SOLUTION OF THE PROBLEM

In this section, we shall apply the q-HAM to obtain the series solution of the fractional Korteweg–de Vries (KdV) equation with given initial condition.

4.1. THE FRACTIONAL KORTEWEG-DE VRIES (KDV) EQUATION:

We first consider the fractional Korteweg–de Vries (KdV) equation given by [24] as,

$$D_t^{\alpha}u+6uu_x+u_{xxx}=0,\ t>0,\quad 0<\alpha\le 1, \tag{14}$$

with initial conditions $u(x,0)=\frac{1}{2}Sech^2\left[\frac{x}{2}\right].$ (15)

The true solutions for $\alpha=1$ of the equation (14) which is obtained by the MFRDTM [24] is given by

$$u(x,t)=\frac{1}{2}Sech^2\left[\frac{x-t}{2}\right]. \tag{16}$$

Let us now solve the equation (14) by the q-homotopy analysis method (q-HAM).

We choose the linear noninteger order operator as:

$$L[\varphi(x,t;q)]=D_t^{\alpha}\varphi(x,t;q), \tag{17}$$

with the property $L(c_1)=0,$ where c_1 is constant. Also, we use $u(x,0)=\frac{1}{2}Sech^2\left[\frac{x}{2}\right]$ as the initial approximation. From the equation (15), we define the non-linear fractional partial differential operator as:

$$N[\varphi(x,t;q)]=D_t^{\alpha}\varphi(x,t;q)+6\varphi(x,t;q)\varphi_x(x,t;q)+\varphi_{xxx}(x,t;q). \tag{18}$$

Using the above definitions, we construct the zero-order deformation equation:

$$(1-nq)L[\varphi(x,t;q)-u_0(x,t)] = q\hbar\, H(x,t)\, N[\varphi(x,t;q)]. \tag{19}$$

Choosing the $H(x,t)=1$, we define the *mth*-order deformation equation as:

$$L[u_m(x,t)-\chi_m^* u_{m-1}(x,t)] = \hbar\, R_m(\overrightarrow{u_{m-1}}(x,t)), \tag{20}$$

with the initial condition for $m \geq 1,\ u_m(x,0)=0,$

where $\chi_m^* = \begin{cases} 0, & m \leq 1, \\ n, & otherwise \end{cases}$

and

$$R_m(\overrightarrow{u_{m-1}}(x,t)) = D_t^{\alpha} u_{m-1}(x,t) + 6\sum_{j=0}^{m-1} u_j(x,t)(u_{m-1-j}(x,t))_x + (u_{m-1}(x,t))_{xxx}. \tag{21}$$

Now the solution of equation (20), for $m \geq 1$ becomes

$$u_m(x,t) = \chi_m^* u_{m-1}(x,t) + \hbar L^{-1}[R_m(\overrightarrow{u_{m-1}}(x,t))]. \tag{22}$$

It is straightforward to choose the initial approximation $u_0(x,t)=u(x,0)$ which is given by the equation (15). Therefore, using the q-HAM, we obtain the components of the solution successively as follows.

We, therefore, obtain: $u(x,0) = \frac{1}{2} Sech^2\left[\frac{x}{2}\right],$

$$\begin{aligned} u_1(x,t) &= \hbar\, D_t^{-\alpha}[D_t^{\alpha} u_0 + 6u_0(u_0)_x + (u_0)_{xxx}] \\ &= \hbar\, I_t^{\alpha}[D_t^{\alpha} u_0 + 6u_0(u_0)_x + (u_0)_{xxx}] = -4\hbar\, Csch[x]^3\, Sinh\left[\frac{x}{2}\right]^4 \frac{t^{\alpha}}{\Gamma(\alpha+1)}, \end{aligned} \tag{23}$$

$$\begin{aligned} u_2(x,t) &= n u_1 + \hbar\, D_t^{-\alpha}[D_t^{\alpha} u_1 + 6u_0(u_1)_x + 6u_1(u_0)_x + (u_1)_{xxx}] \\ &= n u_1 + \hbar\, I_t^{\alpha}[D_t^{\alpha} u_1 + 6u_0(u_1)_x + 6u_1(u_0)_x + (u_1)_{xxx}] \\ &= -4\hbar(\hbar+n)\ Csch[x]^3\, Sinh\left[\frac{x}{2}\right]^4 \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\hbar^2}{4}[-2+Cosh[x]]Sech\left[\frac{x}{2}\right]^4 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{aligned} \tag{24}$$

$$\begin{aligned} u_3(x,t) &= n u_2 + \hbar\, D_t^{-\alpha}[D_t^{\alpha} u_2 + 6u_0(u_2)_x + 6u_1(u_1)_x + 6u_2(u_0)_x + (u_2)_{xxx}] \\ &= n u_2 + \hbar\, I_t^{\alpha}[D_t^{\alpha} u_2 + 6u_0(u_2)_x + 6u_1(u_1)_x + 6u_2(u_0)_x + (u_2)_{xxx}] \\ &= -4\hbar(\hbar+n)^2\, Csch[x]^3\, Sinh\left[\frac{x}{2}\right]^4 \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{\hbar^2(\hbar+n)}{2}(-2+Cosh[x])Sech\left[\frac{x}{2}\right]^4 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad - \frac{\hbar^3}{8} Csch[x]^3\, Sinh\left[\frac{x}{2}\right]^6 Tanh\left[\frac{x}{2}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{57\hbar^3}{16} Sech\left[\frac{x}{2}\right]^7 Sinh\left[\frac{x}{2}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ &\quad + \frac{27\hbar^3}{32} Sech[x]^7\, Sinh\left[\frac{3x}{2}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{3\hbar^3}{32} Sech\left[\frac{x}{2}\right]^7 Sinh\left[\frac{5x}{2}\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}. \end{aligned} \tag{25}$$

From the equations (23), (24) and (25), we obtain $u_1(x,t), u_2(x,t)$ *and* $u_3(x,t)$ similarly by putting $m = 4, 5, \ldots$ in equation (22), we can obtain $u_i(x,t),\ i \geq 4$ by using the Methematica. Therefore, the four-terms approximate series solution to the problem (14) in terms of convergence parameter h *and* n is given by

$$u(x,t,n,h) \cong u_4(x,t,n,h) = \sum_{i=0}^{4} u_i(x,t,n,h)\left(\frac{1}{n}\right)^i. \tag{26}$$

4.2. Remark: It should be emphasized that $u_m(x;t)$ for $m>1$, is governed by the linear operator (7) with the linear boundary conditions that come from the original problem. The existence of the factor $\left(\frac{1}{n}\right)^i$ gives more chances for better convergence, faster that the solution obtained by the standard HAM. Of course, when $n=1$, we are in the case of the standard HAM.

4.3. The *h*-curve:
The question that comes to the mind when the following this method of solution is how one chooses the auxiliary parameter *h* to get a good approximate solution. The answer is in the *h*-curve. Apparently, our choice in the plots can be seen directly from the graph, the range of which is by drawing a horizontal line on the curve parallel to x-axis. Fig. 1 is made with $n=1,\ and\ \alpha=1.$

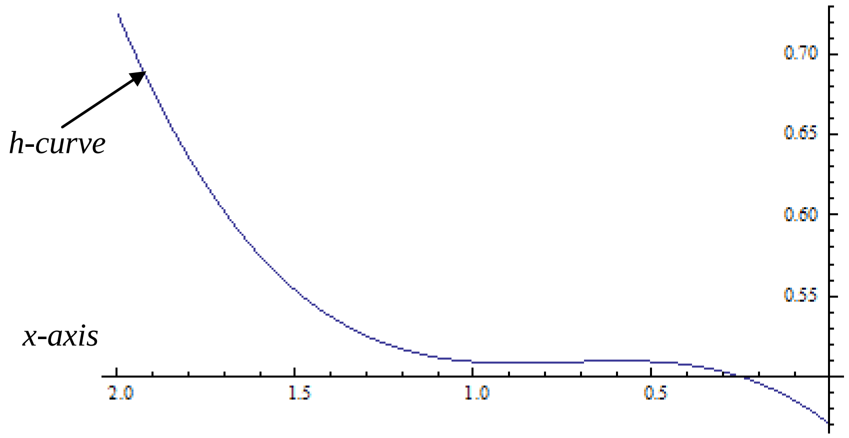

Figure-1: The *h*-curve of $u(x,t,n,h)$ of the four-terms approximate series solution of the equation (26) obtained by q-HAM for fixed the value of $n=1,\ and\ \alpha=1.$

5. NUMERICAL ANALYSIS

In this section, we give some numerical results using series solution obtained above. Comparison is made with the exact solution for a special case using the four-terms series solution. We also seen the graph displaying the best choice of h for fast convergence and the effects of different fractional order α on the solution obtained.

5.1. Comparison of the approximate solution with exact solution
Exact solution is known in the case of $\alpha=1$ and so we present the numerical result (four-terms series solution) obtained by the q-homotopy analysis method and the exact solution of equation the (14) under some conditions.

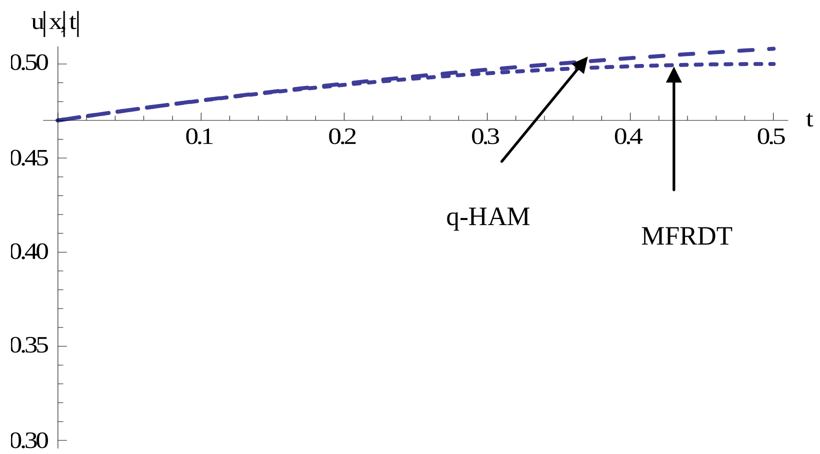

Figure-2: The seven-terms approximation solution of the q-HAM plot of $u(x,t)$ for $h=0.88,\ n=1$ and $\alpha=1$ against the exact solution obtained by MFRDTM.

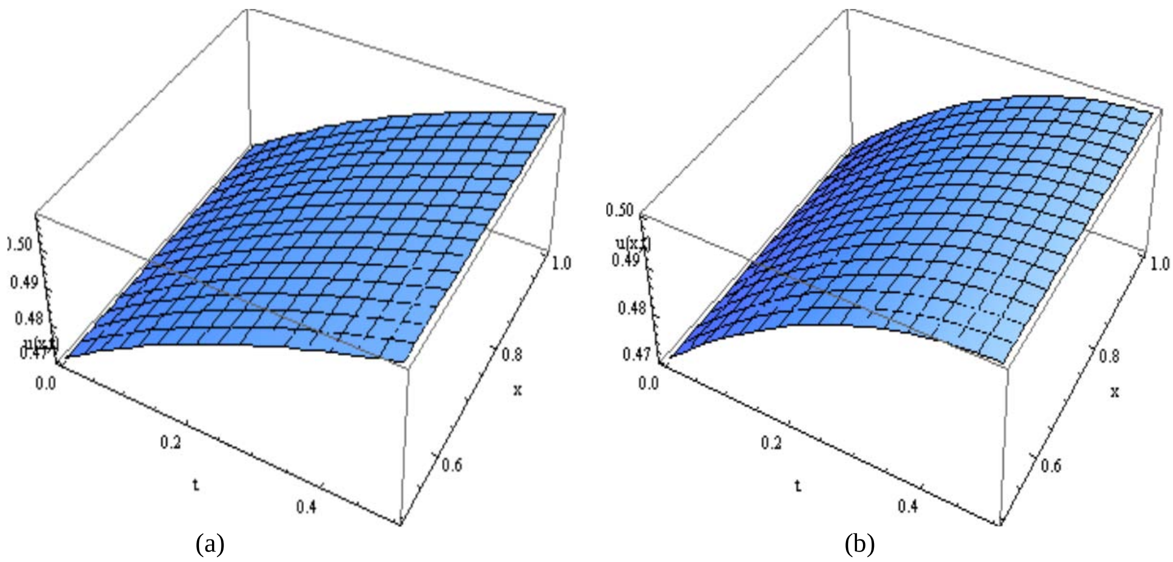

Figure-3: Fig. 3(a) is the exact solution (16) obtained by MFRDTM and fig. 3(b) is the four-terms approximate series solution (26), obtained by the q-HAM for $h=-0.95, n=1, \alpha=1.$

5.2. Remark: It should be noted that we have used only the four-terms of the series solution obtained by the q-homotopy analysis method to make fig.2 as against the solution obtained by the modified fractional reduced differential transform method [MFRDTM]. Fig.2 and 3 shows a perfect match with exact solution. This shows the effectiveness of the homotopy analysis method over other analytical methods due to the ability to control or choose appropriately the auxiliary parameter *h*.

5.3. Solution plots with different fractional values of α

Here, we give the solution plots of the four-terms series solution (26) of the equation (14) using the MATHEMATICA obtained by q-homotopy analysis method (q-HAM). This shows the effect of the different fractional values of α on the obtained solution (26) in figure 4 and 5.

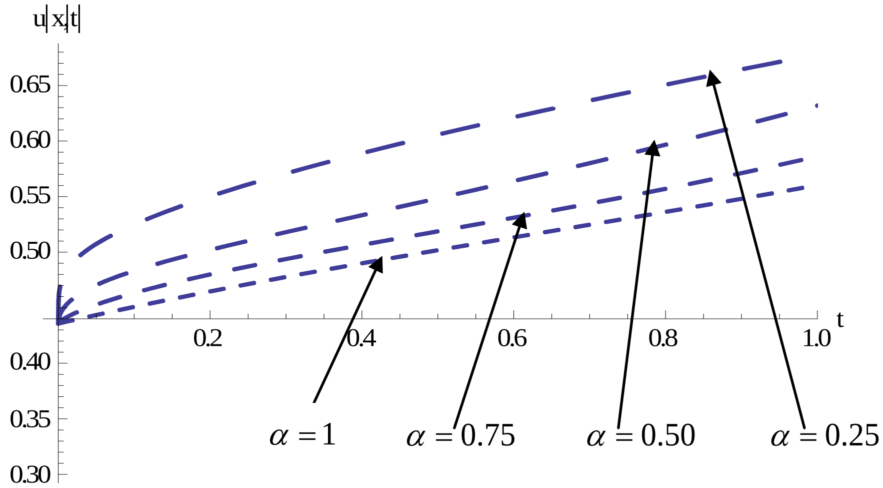

Figure-4: The q-HAM solution plot of Eq. (14) for different fractional values of α with fixed $x=0.75,\ h=-0.88\ and\ n=1.$

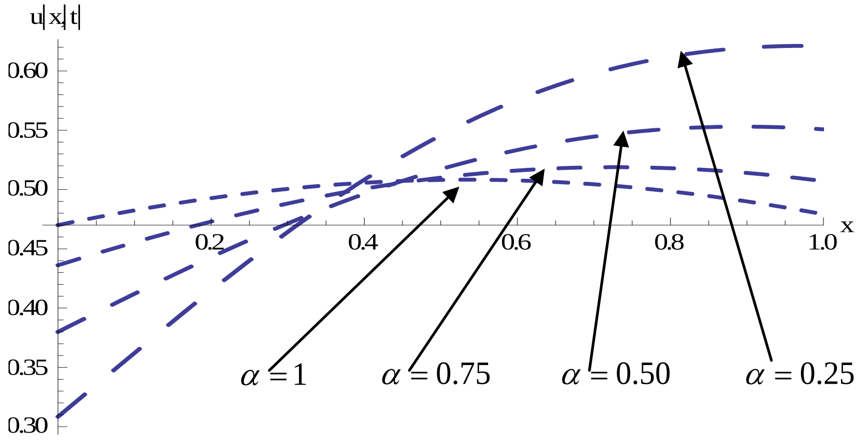

Figure-5: The q-HAM solution plot of Eq. (14) for different fractional values of α with fixed $t = 0.5,\ h = -0.88\ and\ n = 1.$

Table-1: Absolute errors for u(x, t) obtained by the four-terms approximate series solution (26) the equation (14) obtained the q-HAM against with the exact solution obtained by MFRDTM [24] for $n = 1\ and\ h = -0.95$

t	x	$\alpha = 1$	$\alpha = 0.75$	$\alpha = 0.50$	$\alpha = 0.25$
.01	-3	6.4341854E-8	1.9513911E-3	7.8501271E-3	2.4518746E-2
	-2	1.4465059E-7	3.8500738E-3	1.576909E-2	4.9665235E-2
	-1	1.7481382E-7	4.4864765E-3	1.9548745E-2	7.7083517E-2
	0	0.0000000	0.0000000	0.0000000	0.0000000
	1	1.2350665E-7	4.3880173E-3	1.8153032E-2	6.1261918E-2
	2	2.0084081E-7	3.9583612E-3	1.7302939E-2	6.7066096E-2
	3	1.1208361E-7	2.0431764E-3	9.1512339E-3	3.9267895E-2
.02	-3	3.1070078E-9	2.9774284E-3	1.0377445E-2	2.8903930E-2
	-2	7.1011681E-8	5.9132759E-3	2.1049732E-2	5.7875504E-2
	-1	6.6448348E-8	7.0280876E-3	2.7218635E-2	9.5709924E-2
	0	0.0000000	0.0000000	0.0000000	0.0000000
	1	2.7269763E-7	6.7578813E-3	2.4441339E-2	7.3353075E-2
	2	2.9403599E-7	6.2104572E-3	2.4104253E-2	8.2463946E-2
	3	1.8776022E-7	3.2293205E-3	1.2966486E-2	4.9745348E-2
.03	-3	2.8630366E-7	3.7519086E-3	1.2071311E-2	3.1876547E-2
	-2	3.8130087E-7	7.4954524E-3	2.4649702E-2	6.3103487E-2
	-1	1.1900928E-6	9.0750797E-3	3.2995673E-2	1.0897626E-1
	0	0.0000000	0.0000000	0.0000000	0.0000000
	1	1.6579801E-6	8.5903517E-3	2.8850929E-2	8.1623805E-2
	2	1.1399201E-7	8.0285838E-3	2.9208179E-2	9.3186198E-2
	3	1.4277332E-7	4.2037871E-3	1.5935114E-2	5.7374956E-2
.04	-3	8.6912112E-7	4.3796942E-3	1.3351203E-2	3.4200079E-2
	-2	1.3709774E-6	8.7973092E-3	2.7397237E-2	6.6967149E-2
	-1	3.6612943E-6	1.0843835E-2	3.7816055E-2	1.1964957E-2
	0	0.0000000	0.0000000	0.0000000	0.0000000
	1	4.5026128E-6	1.0112703E-2	3.2318003E-2	8.8105309E-2
	2	5.0666361E-7	9.6014813E-3	3.3444148E-2	1.0166012E-2
	3	1.0724971E-7	5.0612894E-3	1.8476598E-2	6.3606173E-2
.05	-3	1.8353558E-6	4.9067225E-3	1.4377785E-2	3.6139437E-2
	-2	3.0550361E-6	9.9061081E-3	2.9611832E-2	7.0027067E-2
	-1	7.9440418E-6	1.2426918E-2	4.2039288E-2	1.2873086E-2
	0	0.0000000	0.0000000	0.0000000	0.0000000
	1	9.2777139E-6	1.1423800E-2	3.5202075E-2	9.3513577E-2
	2	1.7371032E-6	1.1009503E-2	3.7131662E-2	1.0875979E-2
	3	6.4680586E-7	5.8418999E-3	2.0751599E-2	6.8969615E-2

A very good agreement between the results of the q-HAM and the exact solutions is observed in Figures 2, 3 and Table 1, which confirms the validity of the q-HAM.

6. CONCLUSION

In this paper, we have successfully applied q-homotopy analysis method (q-HAM) to obtain an approximation of the analytic solution of the fractional Korteweg–de Vries (KdV) equation. In this method, the solution is found in the form of a convergent series with easily computed terms. The results obtained by the q-homotopy analysis method (q-HAM) are compared with the modified fractional reduced differential transform method (MFRDTM) solution, which show a very good agreement, even using only few terms of the recursive relations. In general, this method provides highly accurate numerical solutions and can be applied to a wide class of nonlinear problems. Also, the method avoids linearization and physically unrealistic assumptions. The results demonstrate reliability and efficiency of the q-homotopy analysis method (q-HAM). The fact that this technique solves the linear and nonlinear problems can be considered as a clear advantage of this algorithm over the decomposition method. Finally, we conclude that the q-HAM can be considered as a nice refinement in existing numerical techniques and have wide applications in different fields of sciences.

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