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THE MIDDLE NUCLEUS EQUALS THE CENTER IN PRIME JORDAN RINGS

[1]M. MANJULA DEVI, [2]K. SUVARNA*

[1,2]Department of Mathematics,
Sri Krishnadevaraya University, Anantapuramu-515003 (A.P.), India.

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ABSTRACT

In this paper we show that in a Jordan ring R, for fixed n in the middle nucleus N_m, the additive subgroup B generated by all elements of the form (n,R,R) is an ideal of R. Then it is proved that R is either associative or the middle nucleus equals the center.

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INTRODUCTION

In [3] Oehmke and Sandler have proved that if R is a simple finite dimensional algebra of characteristic ≠ 2,3 then the nucleus N = the center C. Their proof depends on the known structure of simple Jordan algebras. We have a second proof of this result in [1] which is also valid for characteristic 3, using theorems on trace functions. Klein feld [2] proved that in a simple Jordan ring of char ≠ 2 the middle nucleus and center coincide. In this paper we show that in a Jordan ring R, for fixed n in the middle nucleus N_m, the additive subgroup B generated by all elements of the form (n, R, R) is an ideal of R. Then it is proved that R is either associative or the middle nucleus equals the center.

PRELIMINARIES

Let R be a Jordan ring. We know that a Jordan ring R is a nonassociative ring in which products are commutative, that is

$$(x, y) = 0 \text{ or } xy = yx, \tag{1}$$

and which satisfies the Jordan identity $(xy)x^2 = x(yx^2)$, for all x, y in R.

That is $(x, y, x^2) = 0$ (2)

In Schafer [4], he linearized (2) and obtained $2(x, y, zx) + (z, y, x^2) = 0$ for all $x, y, z \in R$ (3)

We use the right multiplication notation $xy = xR_y = yx$, where R_y is a linear transformation on commutative algebra. Then it is well known that the identity $R_{x(yz)-(xy)z} = (R_x\ R_z - R_z\ R_x)\ R_y - R_y\ (R_x\ R_z - R_z\ R_x)$ holds in R. It can be written as

$$\begin{aligned} w(x, y, z) &= (R_y (R_x R_z) - R_y (R_z R_x)) - R_y (R_x R_z - R_z R_x), \\ &= (wy (xz) - wy (zx)) - y ((wx)z - (wz)x), \\ &= (((wy)x)z - ((wy)z)x) - y ((xw)z - x(wz)), \\ &= ((x(wy))z - x((wy)z)) - y ((xw)z - x (wz)). \end{aligned}$$

Then $w(x, y, z) = (x, wy, z) - y (x, w, z)$.

$$\therefore (x, wy, z) = w (x, y, z) + y (x, w, z), \tag{4}$$

This identity is valid in a Jordan ring R.

The following identity is valid in any ring:

$$(wx, y, z) - (w, xy, z) + (w, x, yz) - w (x, y, z) - (w, x, y)z = 0 \tag{5}$$

Corresponding Author: K. Suvarna, [2]Department of Mathematics,*
Sri Krishnadevaraya University, Anantapuramu-515003 (A.P.), India.

Let N be the nucleus and C be the center of R.

The left nucleus N_l of R is defined as $N_l = \{n \in R / (n, R, R) = 0\}$.

The right nucleus N_r of R is defined as $N_r = \{n \in R / (R, R, n) = 0\}$.

The middle nucleus N_m of R is defined as $N_m = \{n \in R \,/\, (R, n, R) = 0\}$. (6)

By the nucleus N of a ring R, we mean the set of all elements n in R such that (n, R, R) = (R, n, R) = (R, R, n) = 0.

The center C of R is defined as $C = \{c \in N \,/\, (c, R) = 0\}$. (7)

Let R be the n- divisible if nx=0 implies x = 0 for all x in R and n a natural number.

Now if we take w = x, x = n in (5), then

$(xn, y, z) - (x, ny, z) + (x, n, yz) - x(n, y, z) - (x, n, y)z = 0,$
$(xn, y, z) - (x, ny, z) - x(n, y, z) = 0$ from (6).
$(xn, y, z) = (x, ny, z) + x(n, y, z),$
$(xn, y, z) = n(x, y, z) + y(x, n, z) + x(n, y, z),$ using (4).
$(xn, y, z) = n(x, y, z) + x(n, y, z),$ using (6).
$(xn, y, z) = n(x, y, z) + x(n, y, z),$ or
$(nx, y, z) = n(x, y, z) + x(n, y, z).$ (8)

As a consequence of (4),

$$(x, ny, z) = n(x, y, z) \quad (9)$$

for arbitrary elements x, y, z in R and n in N_m.

MAIN RESULTS

Lemma 1: For fixed n in N_m, the additive subgroup B generated by all elements of the form (n, R, R) is an ideal of R.

Proof: We have to prove $B = \{(n, R, R) / n \in N_m\}$ is an ideal.

Let $b = (ax)y - a(xy)$, here $a \in N_m$, $x, y \in R$.

Let B be the subspace of R of all finite sums of elements of the form $(ax)y - a(xy)$. Then

$b^1 = ((wx)a)y - ((wx)y)a = (a(wx))y - a((wx)y)$ is in B. Also
$b^{11} = ((wy)a)x - ((wy)x)a,$
$b^{111} = ((xy)a)w - ((xy)w)a$ are in B.

By taking x=a, y=x, z=y in equation (4), we get

$(x, wy, z) = w(x, y, z) + y(x, w, z),$
$(a, wx, y) = w(a, x, y) + x(a, w, y),$
$(a.wx)y - a(wx.y) = w((ax)y - a(xy)) + x((aw)y - a(wy))$
$b^1 = wb + q$
$\therefore\ wb = b^1 - q$ (10)

Here $q = x((aw)y - a(wy)),$
$= x((aw)y) - x(a(wy)),$
$= ((x(aw))y - (x, aw, y)) - x(a(wy)).$

Using this and (4) we get $q = (x(aw))y - (a(x, w, y) + w(x, a, y)) - x(a(wy)),$
$q = (x(aw))y - a(x, w, y) - x(a(wy)),$
$= (x(aw))y - a((xw)y - x(wy)) - x(a(wy)),$
$= (x(aw))y - a((xw)y) + ((wy)x)a - ((wy)a)x,$
$q = (x(aw))y - a((xw)y) - b^{11}.$

Thus $q + b^{11} = (x(aw))y - a((xw)y),$
$= ((xa)w)y - a((xw)y),$
$= (w(xa))y - a((w, x, y) + w(xy)),$
$= (w, xa, y) + w((xa)y) - a(w,x,y) - a(w(xy)),$
$= (x(w, a, y) + w((xa)y) + (a,xy,w) - (a(xy)w),$ using (4),

$= w((xa)y) – (a(xy)w + (a, xy, w)$
$= wb + (a(xy))w – a((xy)w)$
$q+b^{11} = wb + b^{111},$
$b^{l}-wb+b^{ll} = wb+b^{l11}$, using (10).
$2wb = b^{l}+b^{ll}-b^{lll}.$
$\therefore\ wb = b^{l}+b^{ll}-b^{lll}.$

Since b^{l}, b^{ll} and b^{lll} are in B, wb is also in B. By (1) we have wb = bw.

This proves that B is an ideal of R.

We know that the following identities hold in a Jordan ring R :

$(x, y, z) = - (z, y, x)$ or $(x, y, x) = 0$ (11)

and $S (x, y, z) = (x, y, z) + (y, z, x) + (z, x, y) = 0.$ (12)

By taking z = n in (11) we get

$(x, y, n) = - (n, y, x).$ (13)

Now we take z = n, $n \in N_m$ in (12) we obtain

$(x, y, n) + (y, n, x) + (n, x, y) = 0,$
$(x, y, n) + (n, x, y) = 0,$
$(x, y, n) = - (n, x, y).$ (14)

Using this and (13) we get $(x, y, n) = (y, x, n).$ (15)

Similarly by taking x = n in (12) and using (13) we get

$(n, x, y) = (n, y, x).$ (16)

Let A consists of all finite sums of elements of the form (x, y, z) or of the form w(x, y, z).

Then A is an ideal in any arbitrary ring.

Theorem 1: If R is a 2- and 3- divisible prime Jordan ring, then either R is associative or the middle nucleus equals the center.

Proof: We take y=n, $n \in N_m$ in (5). Then

We get $- (w, xn, z) + (w, x, nz) = (w, x, n)z.$

By taking w = z in this equation and using (11) we get

$(z, x, nz) = (z, x, n) z.$ (17)

By taking x = z, y = z, z = n, w = x in (4), then we obtain

$(z, xz, n) = x (z, z, n) + z(z, x, n)$. Using this and (15) we get
$(xz, z, n) = x (z, z, n) + z(z, x, n).$ (18)

By taking x=z, y=x, z=n in (3) then we get $2(z, x, nz) + (n, x, z^2) = 0.$ (19)

Now we take w=x, x=z, y=z, z=n in (5), we obtain

$(xz, z, n) – (x, z^2, n) + (x, z, zn) –x (z, z, n) – (x, z, z)n = 0,$
$(xz, z, n) – (x, z^2, n) + (x, z, zn) = x(z, z, n) + (x, z, z)n.$

Using this, (15), (11), (19) and (17) we get

$(xz, z, n) – (z^2, x, n) + (x, z, zn) = x(z, z, n) + (x, z, z)n,$
$(xz, z, n) + (n, x, z^2) + (x, z, zn) = x (z, z, n) + (x, z, z)n.$
$- (xz, z, n) – 2(z, x, nz) + (x, z, zn) = x(z, z, n) + (x, z, z) n.$
$(xz, z, n) -2(z, x, nz) + (x, z, n)z = x (z, z, n) + (x, z, z)n.$

Now using (15), (17) and (18) we get

$(xz, z, n) -2(z, x, nz) + (z, x, n)z = x (z, z, n) + (x, z, z)n,$
$(xz, z, n) – (z, x, nz) = x(z, z, n) + (x, z, z)n,$
$(xz, z, n) – x (z, z, n) – (z, x, nz) = (x, z, z)n,$
$(x, z, z)n = 0.$

By using (12) and (11), we obtain

$(z, z, x)n = 0$ and $(z, x, z)n = 0$.

$\therefore\ (x, z, z)n = (z, x, z)n = (z, z, x)n = 0.$ (20)

By linearizing (20), we get

$(x, y, z)n = - (x, z, y)n = (z, x, y)n = - (y, x, z)n = (y, z, x)n.$

From (12) we have $S(x, y, z) = 0$.

So $S(x, y, z)n = 0$. Then

$(x, y, z)n + (y, z, x)n + (z, x, y)n = 0,$

$3(x, y, z)n = 0.$

Since R is 3- divisible, $(x, y, z)n = 0$. (21)

Using this in (4), we get

$(x, ny, z) = n(x, y, z) + y(x, n, z)$

$(x, ny, z) = 0.$

Using this in (5), we obtain

$(wn, y, z) = w(n, y, z).$ (22)

By forming the associator in (21) with r, s, where $r, s \in R$.

We have $((x, y, z)n, r, s) = 0$.

$(x, y, z)(n, r, s) = 0$, using (22).

That is, $AB = 0$.

Since R is prime, either $A=0$ or $B=0$.

If $A = 0$ then R is associative.

If $B = (n, r, s) = 0$, then from (11), it follows that $(s, r, n) = 0$.

Thus n is in the nucleus N and satisfies $(n, r) = 0$, by 1.

So $n \in C$. That is, $n \in N_m$ implies that $n \in C$.

Hence $N_m = C$.

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