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NORMALIZATION BERNSTEIN FOR SOLVING FRACTIONAL LINEAR VOLTERRA-INTEGRO DIFFERENTIAL EQUATION

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ABSTRACT

In this paper, the Petrov-Galerkian method (PGM) is proposed to obtain approximate solutions of linear fractional Volterra integro-differential equation of the second kind (LFVIDE$_s$) via the normalization Bernstein bases. The fractional derivatives are described in the Caputo sense. Some examples are given and the their results shown in tables and figures, the Petrov-Galerkian method (PGM) is very effective and convenient and overcome the difficulty of traditional methods.

Key words: *Petrov-Galerkian method; Fractional Derivative; Caputo Sense.*

1. INTRODUCTION

Integro-differential equations are encountered in various fields of sciences. It plays an important role in many branches of linear and non-linear functional analysis and their applications are in the theory of sciences, engineering and social sciences. Many problems can be modeled by fractional integro-differential equations from various sciences and engineering applications. Finding the approximate or exact solutions of fractional integro-differential equations is an important task. Save in a limited number, there is difficulty in finding the analytical solutions of fractional integro-differential equations. Therefore, there have been attempts to develop new methods for obtaining analytical solutions which reasonably approximate the exact solutions.

Let us consider the linear fractional Volterra integro-differential equation of the second kind (LFVIDEs):

$$D_*^{\alpha} u(x) = f(x) + \int_a^x k(x,t)u(t)dt, \quad u(0) = \beta, \ x \in [0,1] \tag{1}$$

Where D_*^{α} is Caputo fractional derivative, α is a parameter describing the order of the fractional derivative, $f(x), k(x,t)$ are given continuous functions and *u(x)* is the unknown function to be determined.

However, several numbers of algorithms for solving linear fractional Volterra integro-differential equation of the second kind (LFVIDE$_s$) have been investigated. Zhao and Neville [1] use collocation methods are used for solving the fractional Volterra integro-differential equations with weakly singular kernels. Abdon and Necde [2] present Picard method to find existence and uniqueness of the solution Volterra fractional integral equation of the second kind. Saleh, Amer, Nagdy and Alngar [3] applied homotopy perturbation method (HPM) and variational iteration method (VIM) to approximate solutions for nonlinear Volterra fractional integro-differential equations with boundary conditions. Vedat and Shaher [4] obtain approximate analytical solutions to fractional Volterra integro-differential equations using the generalized differential transform method. Emran, Ezadkhah and Shateyi [5] adopt a computational approach for solving a class of nonlinear Volterra integro-differential equations of fractional order which is based on the Bernoulli polynomials approximation. Vanni and Aminatae [6] employed Operational Tau matrix method to approximation for a general class of fractional integro-differential equations.

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In this paper, we show how the approximately methods which are based on the Petrov-Galerkian method (PGM) can be used to solve (LFVIDE's) to obtain approximate solutions via the normalization Bernstein bases.

1.1. Basic Definitions of Fractional Derivatives

In this section some basic definitions and properties of fractional calculus theory which are necessary for the formulation of the problem are given

Definition 1: A real function f(t), t > 0, is said to be in the space $C_\mu, \mu \in R$, if there exists a real number $p > \mu$, such that $f(t) = t^p h_1(t)$; where $f_1(t) \in (0, \infty)$, and it is said to be in space C^n_μ if and only if $f^n \, C_\mu, n \in N$.

Definition 2: The Riemann-Liouvill fractional integral operator of order α for a function in C_μ, where $\mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \ \alpha > 0$$
$$J^\alpha f(x) = f(x).$$

Definition 3: Let $f \in C_{-1}^m \, 1, m \in N \cup \{0\}$. Then the Caputo fractional derivatives of f(x) is defined as:

$$D^\alpha f(x) = \begin{cases} J^{m-\alpha} f^m(x), & m-1 < \alpha \leq m, m \in N \\ \frac{D^m f(x)}{Dx^m}, & \alpha = m \end{cases}$$

Hence, we have following properties

1. $J^\alpha J^v f = J^{\alpha+v} f, \ \alpha, v > 0, f \in C_\mu, \mu > 0$
2. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \alpha > 0, \gamma > -1, \ x > 0$
3. $J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^k(0^+) \frac{x^k}{k!}, x > 0, m-1 < \alpha \leq m$
4. $J^\alpha D^\alpha f(x) = f(x), x > 0, m-1 < \alpha \leq m$
5. $D^\alpha C = 0$, C is constant
6. $D^\alpha x^B = \begin{cases} 0 & \beta \in N_0, < [\alpha] \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} & \beta \in N_0, \beta \geq [\alpha] \end{cases}$

   where $[\alpha]$ denoted the smallest integer greater than or equal to α and $N_0 = \{0,1,2, \dots\}$.

1.2. The Derivative for Orthonormal Brnstein Polynomials

The Bernstein polynomials of *n*th degree are defined on the interval [0, 1] as [7].

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \ \binom{n}{i} = \frac{\boldsymbol{n}!}{\boldsymbol{i}!\,(\boldsymbol{n}-\boldsymbol{i})!} \quad for \ i = 0,1,2, \dots, n$$

The representation of the orthonormal Bernstein Polynomials, denoted by $\boldsymbol{b_{i,n}(x)}$ here, was discovered by analyzing the resulting orthonormal polynomials after applying the Gram-Schmidt process on sets of Bernstein polynomials of degree $B_{i,n}(x)$, [8]. Then the following sets of orthonormal polynomials $b_{i,n}(x)$, $0 \leq i \leq n$. For $n = 3$, the four orthonormal Bernstein polynomials are given as:

$$b_{0,3}(x) = \sqrt{7}\,(1-x)^3,$$
$$b_{1,3}(x) = \sqrt{20}\left(3x\,(x-1)^2 + \frac{1}{2}(x-1)^3\right),$$
$$b_{2,3}(x) = \frac{10}{\sqrt{3}}\left(-3x(x-1)^2 - 3x^2(x-1) - \frac{3}{10}(x-1)^3\right)$$
$$b_{3,3}(x) = 4\left(3x\,(x-1)^2 + \frac{9}{2}x^2(x-1) + \frac{1}{4}\,(x-1)^3 + x^3\right)$$

2. CONVERGENCE OF THE PETROV-GALERKIN METHOD

In this section we introduce the (PGM) for Eq. (2). For the proof of all results in this section we can use the same manner used in [9], but for Eq. (2). Let X be a Banach space with the norm $\|.\|$ and let X^* denote its dual space. Assume $K: X \to X$ is a compact linear operator. We rewrite this eq. (1) in operator from as:

$$D_*^\alpha u - Ku = f, \ f \in X \tag{2}$$

where $u \in X$ is the unknown to be determined. The Peterov-Galerkin method (PGM) used for the numerical solutions of eq. (2). The Petrov-Galerkin methods (PGM) interpolate between the Galerkin method and the collocation method. For this purpose for each positive integer n, we assume that $X_n \subset X$, $Y_n \subset X^*$, and X_n, Y_n are finite dimensional vector spaces with $\dim X_n = \dim Y_n$, then X_n , Y_n satisfy condition (H): for each $x \in X$ and $y \in X^*$, there exists $x_n \in X_n$ and

$y_n \in Y_n$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. when The peterov-Galerkin method(PGM) for Eq.(2) is a numerical method for finding $u_n \in X$ such that

$$\langle D_*^{\alpha} u_n - K u_n, y_n \rangle = \langle f, y_n \rangle \text{ for all } y_n \in Y_n \quad (3)$$

It is clear that the Petrov-Galerkin method (PGM) is closely related to a generalized best approximation from X_n to $x \in X$ with respect to Y_n,. Given $x \in X$, an element $P_n x \in X_n$ is called a generalized best approximation from X_n to x with respect to Y_n if it satisfies the equation

$$\langle x - P_n x, y_n \rangle = 0 \quad \text{for all } y_n \in Y_n \quad (4)$$

Similarly, given $y \in X^*$, an element $p_n' y \in Y_n$ is called best approximation from Y_n *to* y with respect Y_n to y if it satisfies the equatio

$$\langle x_n, y - p_n' y \rangle = 0 \quad \text{for all } x_n \in X_n .$$

Now will demonstrates for each $x \in X$ has a unique generalized best approximation

Proposition 1: For each $x \in X$, the generalized best approximation from X_n to x with respect to Y_n exists uniquely if and only if

$$Y_n \cap X_n^{\perp} = \{0\} \quad (5)$$

Under this condition, P_n is a projection; i.e., $P_n^2 = P_n$

Assume that, for each n, there is a linear operator $\prod_n : X_n \to Y_n$ with $\prod_n X_n = Y_n$, and satisfying the condition

(H-1) $\|x_n\| \le C_1 \langle x_n, \prod_n x_n \rangle^{\frac{1}{2}}$ for all $x_n \in X_n$,

(H-2) $\|\prod_n x_n\| \le C_2 \|x_n\|$ for all $x_n \in X_n$,

Where C_1 and C_2 are positive constants independent of n. if a pair of sequence $\{X_n\}$ and $\{Y_n\}$ satisfy (H-1) and (H-2), we call $\{X_n, Y_n\}$ a regular pair.

For each $x \in X$, let $Q_n x$ be a best approximation from X_n to x, that is, $Q_n x \in X_n$ satisfies the equation

$$\|x - Q_n x\| = \inf_{x_n \in X_n} \|x - x_n\|.$$

If a regular Pair $\{X_n, Y_n\}$ satisfies dim X_n = dim Y_n and condition (H), then the corresponding generalized projection P_n satisfies:

(1) for all $x \in X$, $\|P_n x - x\| \to 0$ as n$\to \infty$

(2) there is a constant $C > 0$ such that, $\|P_n\| < C$, n = 1,2,..

(2) for some constant $C > 0$ independent of n, $\|P_n x - x\| \le C \|Q_n x - x\|$ where $Q_n x$ is the best approximation from X_n to x.

if $\{X_n, Y_n\}$ a regular pair is with a linear operator $\prod_n : X_n \to Y_n$ with $\prod_n X_n = Y_n$, then eq. (3) may be rewritten

$$\langle D_*^{\alpha} u_n - K u_n, \prod_n x_n \rangle = \langle f, \prod_n x_n \rangle \quad \text{for all } x_n \in X_n \quad (6)$$

Using the projection P_n defined earlier, eq. (3) is equivalent to

$$D_*^{\alpha} u_n - P_n K u_n = P_n f \quad (7)$$

eq. (7) can also be derived from the fact that P_nx = 0 for an $x \in X$ if and only if $\langle x, y_n \rangle = 0$ for all $y_n \in Y_n$. This equation indicates that the Petrov-Galerkin method is a projection method.

Now, assume $u_n \in X_n$ and $\{b_j\}_{j=1}^n$ is a basis for X_n (trial space) and $\{b_i^*\}_{i=1}^n$ (test space) is a basis for Y_n. Therefore the (PGM) on $[a, b]$ for Eq. (2) is:

$$\langle D_*^{\alpha} u_n - K u_n, b_i^* \rangle = \langle f, b_i^* \rangle, \quad i = 1,.. ,n \quad (8)$$

3. PETROV–GALERKIN METHOD (PGM) FOR APPROXIMATE SOLUTION OF LINEAR FRACTIONAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATION (LFVIDES) VIA ORTHONORMAL BERNSTEIN POLYNOMIALS BASIS

We recall the eq. (1) of the Linear fractional Volterra integro- differential equation (LFVIDEs) of the form:

$$D_*^{\alpha} u(x) = f(x) + \int_a^x k(x,t) u(t) dt, \quad u(0) = \beta, \quad x \in [0,1]$$

Our approach being by taking the fractional integration to both sides of eq. (1) we get

$$u(x) = u(0) + I^{\alpha} f(x) + I^{\alpha} \left(\int_0^x k(x,t) u(t) dt \right) \quad (9)$$

To approximate solution of eq. (1), we use the normalization polynomial basis on $[a, b]$ as:

$$u(x) = \sum_{i=0}^{n} a_i \, b_{i,n}(x) \tag{10}$$

Where ($a_i, i = 0,1, \ldots\ldots, n$) are unknown constants to be determined substituting eq. (10) in to eq. (9), we get

$$\sum_{i=0}^{n} a_i \, b_{i,n}(x) = u(0) + I^{\alpha} f(x) + I^{\alpha}\left(\int_0^x k(x,t) \sum_{i=0}^{n} a_i \, b_{i,n}(t)dt\right) \tag{11}$$

Hence

$$\sum_{i=0}^{n} a_i \, b_{i,n}(x) - I^{\alpha}\left(\int_0^x k(x,t) \sum_{i=0}^{n} a_i \, b_{i,n}(t)dt\right) = u(0) + I^{\alpha} f(x) \tag{12}$$

In the next step, apply Petrov-Galerkin method (PGM) for eq. (1) is a numerical method for finding $u(x) = \sum_{i=0}^{n} a_i \, b_{i,n}(x) \in X_n$, such that $\boldsymbol{a_i}$ is unknown and must be determined from eq. (12).

From eq. (8) it is clear that the eq.(12) can be written as :

$$\left\langle \sum_{i=0}^{n} \mathbf{a_i} \, \mathbf{b_{i,n}}(\mathbf{x}) - \mathbf{I^{\alpha}}\left(\int_0^x \mathbf{k}(\mathbf{x},\mathbf{t}) \sum_{i=0}^{n} \mathbf{a_i} \, \mathbf{b_{i,n}}(\mathbf{t})\mathbf{dt}\right), \mathbf{b}^*_{\mathbf{j,n}} \right\rangle = \left\langle \mathbf{u(0)} + \mathbf{I^{\alpha}\, f(x)}, \mathbf{b}^*_{\mathbf{j,n}} \right\rangle \tag{13}$$

Thus

$$\int_0^1 \{ \sum_{i=0}^{n} a_i \; b_{i,n}(x) - I^{\alpha}\left(\int_a^x k(x,t) \sum_{i=0}^{n} a_i \, b_{i,n}(t)dt\right)\} b^*_{j,n} = \int_0^1 \{u(0) - I^{\alpha} f(x)dx\} \, b^*_{j,n} \tag{14}$$

The Petrov-Galerkin using regular pairs $\{X_n, Y_n\}$ of piecewise polynomial spaces are called Petrov-Galerkin element.

Then, Eq. (14) is equivalent to linear system can be formed as follows:

$$\left.\begin{aligned} R(x, a_i) &= \int_0^1 [\sum_{i=0}^{n} a_i \, b_{i,n}(x) - I^{\alpha} \left(\int_0^x k(x,t) \sum_{i=0}^{n} a_i b_{i,n} \; (t)dt\right)] \\ h_j &= \int_0^1 [u(0) + I^{\alpha} f(x)] b^*_{j,n} \end{aligned}\right] \tag{15}$$

We can represent the system eq.(15) as a matrix form:

$$RA = H \tag{16}$$

where

$$R = \begin{bmatrix} \int_0^1 R(x,a_0) b^*_{0,n} \, dt & \cdots & \int_0^1 R(x,a_n) b^*_{0,n} dt \\ \vdots & \ddots & \vdots \\ \int_0^1 R(x,a_0) b^*_{n,n} \, dt & \cdots & \int_0^1 R(x,a_n) b^*_{n,n} \, dt \end{bmatrix}, A = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}, H = \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_n \end{bmatrix}$$

Then we are solving the system to calculate the value a_i

6. NUMERICAL EXAMPLES

Example 1: consider th e following linear volterra fractional linear integro-differential equation:

$D_x^{\alpha} \, y(x) = 1 - \frac{x^2}{2} + \int_0^x y(t)dt$, y(0)= 0, $0 < \alpha \leq 1$ with the exact solution $y(x) = x$

Table-1: represent a comparison between the exact solution and approximate solution with different value $\alpha = 0.25, 0.5, 1$

X	**Exact solution**	**Approximate solution**		
		$\boldsymbol{\alpha = 1}$	$\boldsymbol{\alpha = 0.5}$	$\boldsymbol{\alpha = 0.25}$
0	0	0.0001		0.44006
0.1	0.1	0.1006	0.36432	0.64414
0.2	0.2	0.20209	0.52486	0.79142
0.3	0.3	0.30393	0.65311	0.90025
0.4	0.4	0.40548	0.76384	0.98899
0.5	0.5	0.50611	0.87186	1.0760
0.6	0.6	0.60516	0.99198	1.1796
0.7	0.7	0.70200	1.1390	1.3182
0.8	0.8	0.79599	1.3277	1.5101
0.9	0.9	0.88649	1.5728	1.7737
1	1	0.97286	1.8893	2.1273

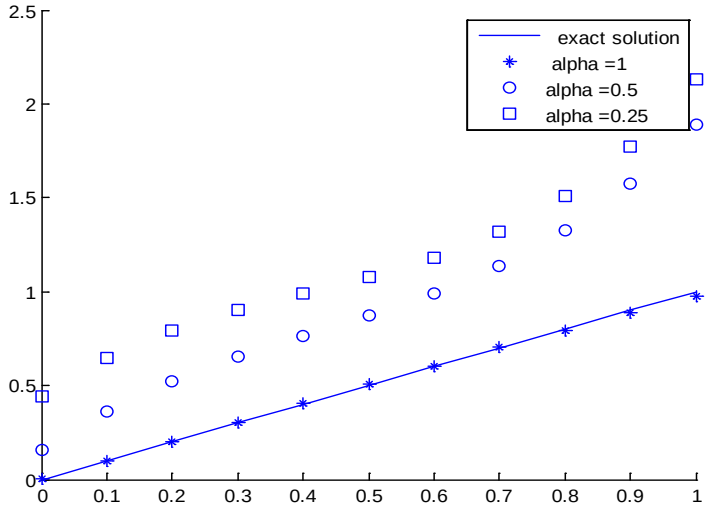

Figure-1: comparison between the approximate solution and exact solution

Example 2: consider the following linear volterra fractional integro-differential equation:

$D_x^{\alpha} y(x) = 2\cos x - 1 + \int_0^x y(t)dt$, y(0) = 0, $0 < \alpha \leq 1$

With the exact solution y(x) = $\sin x$

Table-2: represent a comparison between the exact solution and approximate solution with different value $\alpha = 0.25,0.5,1$

X	Exact solution	Approximate solution		
		$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.25$
0	0	0	0.15616	0.43945
0.1	0.09983	0.10041	0.36302	0.64068
0.2	0.19867	0.20063	0.5182	0.77708
0.3	0.29552	0.29909	0.63544	0.86648
0.4	0.38942	0.39417	0.72849	0.92672
0.5	0.47943	0.48422	0.81108	0.97564
0.6	0.56464	0.56764	0.89695	1.0311
0.7	0.64422	0.64277	0.99984	1.1109
0.8	0.71736	0.7080	1.1335	1.2329
0.9	0.78333	0.76168	1.3117	1.4149
1	0.84147	0.8022	1.5481	1.6748

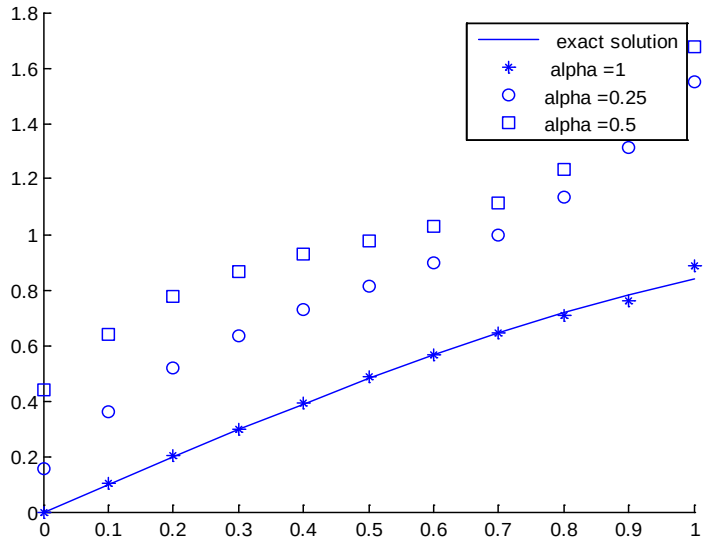

Figure-2: comparison between the approximate solution and exact solution

Example 3: consider the following linear volterra fractional integro-differential equation:

$D_x^{\alpha} y(x) = e^x - xe^x + x + \int_0^x xy(t)dt$, y(0)=1, $0 < \alpha \leq 1$

With the exact solution y(x) = e^x

Table-3: represent a comparison between the exact solution and approximate solution with different value $\alpha = 0.25,0.5,1$

X	Exact solution	Approximate solution		
		$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.25$
0	1.000	1.000	1.1541	1.4379
0.1	1.1052	1.1049	1.3804	1.6703
0.2	1.2214	1.2191	1.5706	1.857
0.3	1.3499	1.3418	1.738	2.0148
0.4	1.4918	1.4722	1.8958	2.1604
0.5	1.6487	1.6093	2.057	2.3105
0.6	1.8221	1.7523	2.235	2.4818
0.7	2.0138	1.9002	2.4429	2.691
0.8	2.2255	2.0521	2.6938	2.9547
0.9	2.4596	2.2072	3.0009	3.2898
1	2.7183	2.3644	3.3775	3.7129

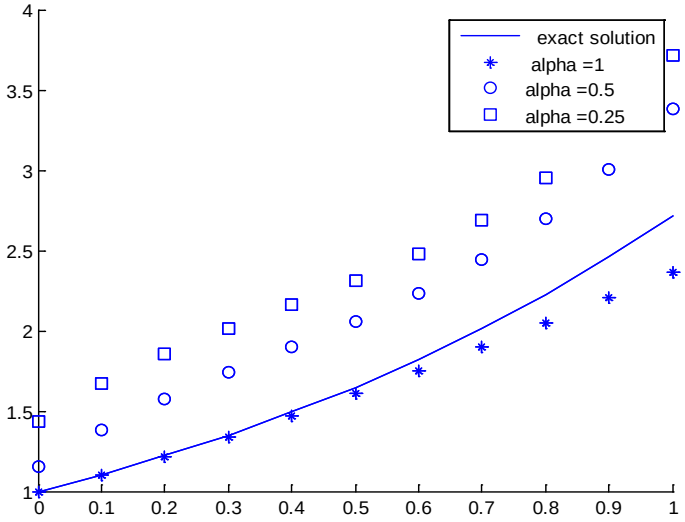

Figure-3: comparison between the approximate solution and exact solution

7. CONCLUSION

In this paper, Petrov-Galerkin method (PGM) has been successfully applied to finding the approximate solution of linear fractional Volterra integro-differential equation of the second kind ($LFVIDE_s$) via the normalization Bernstein basis. This method is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear fractional Volterra integro-differential equation of the second kind ($LFVIDE_s$), for the special case α = 1 is shown in Figure 1 Figure 2 and Figure3. It can be seen from this figures that the solution obtained by the present method is identical with the exact solution. In our paper, we use the Matlab language to calculate the Petrov-Galerkin method by using normalization Bernstein basis.

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