

**NORMALIZATION BERNSTEIN  
FOR SOLVING FRACTIONAL LINEAR VOLTERRA-INTEGRO DIFFERENTIAL EQUATION**

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**ABSTRACT**

*In this paper, the Petrov-Galerkian method (PGM) is proposed to obtain approximate solutions of linear fractional Volterra integro-differential equation of the second kind (LFVIDE<sub>s</sub>) via the normalization Bernstein bases. The fractional derivatives are described in the Caputo sense. Some examples are given and their results shown in tables and figures, the Petrov-Galerkian method (PGM) is very effective and convenient and overcome the difficulty of traditional methods.*

**Key words:** Petrov-Galerkian method; Fractional Derivative; Caputo Sense.

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**1. INTRODUCTION**

Integro-differential equations are encountered in various fields of sciences. It plays an important role in many branches of linear and non-linear functional analysis and their applications are in the theory of sciences, engineering and social sciences. Many problems can be modeled by fractional integro-differential equations from various sciences and engineering applications. Finding the approximate or exact solutions of fractional integro-differential equations is an important task. Save in a limited number, there is difficulty in finding the analytical solutions of fractional integro-differential equations. Therefore, there have been attempts to develop new methods for obtaining analytical solutions which reasonably approximate the exact solutions.

Let us consider the linear fractional Volterra integro-differential equation of the second kind (LFVIDE<sub>s</sub>):

$$D_*^\alpha u(x) = f(x) + \int_a^x k(x, t)u(t)dt, \quad u(0) = \beta, \quad x \in [0, 1] \quad (1)$$

Where  $D_*^\alpha$  is Caputo fractional derivative,  $\alpha$  is a parameter describing the order of the fractional derivative,  $f(x), k(x, t)$  are given continuous functions and  $u(x)$  is the unknown function to be determined.

However, several numbers of algorithms for solving linear fractional Volterra integro-differential equation of the second kind (LFVIDE<sub>s</sub>) have been investigated. Zhao and Neville [1] use collocation methods are used for solving the fractional Volterra integro-differential equations with weakly singular kernels. Abdon and Necde [2] present Picard method to find existence and uniqueness of the solution Volterra fractional integral equation of the second kind. Saleh, Amer, Nagdy and Alngar [3] applied homotopy perturbation method (HPM) and variational iteration method (VIM) to approximate solutions for nonlinear Volterra fractional integro-differential equations with boundary conditions. Vedat and Shaher [4] obtain approximate analytical solutions to fractional Volterra integro-differential equations using the generalized differential transform method. Emran, Ezadkhah and Shateyi [5] adopt a computational approach for solving a class of nonlinear Volterra integro-differential equations of fractional order which is based on the Bernoulli polynomials approximation. Vanni and Aminatae [6] employed Operational Tau matrix method to approximation for a general class of fractional integro-differential equations.

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In this paper, we show how the approximately methods which are based on the Petrov-Galerkian method (PGM) can be used to solve (LFVIDE's) to obtain approximate solutions via the normalization Bernstein bases.

### 1.1. Basic Definitions of Fractional Derivatives

In this section some basic definitions and properties of fractional calculus theory which are necessary for the formulation of the problem are given

**Definition 1:** A real function  $f(t)$ ,  $t > 0$ , is said to be in the space  $C_\mu$ ,  $\mu \in \mathbb{R}$ , if there exists a real number  $p > \mu$ , such that  $f(t) = t^p h_1(t)$ ; where  $f_1(t) \in (0, \infty)$ , and it is said to be in space  $C_\mu^n$  if and only if  $f^{(n)} \in C_\mu$ ,  $n \in \mathbb{N}$ .

**Definition 2:** The Riemann-Liouville fractional integral operator of order  $\alpha$  for a function in  $C_\mu$ , where  $\mu \geq -1$ , is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad \alpha > 0$$

$$J^\alpha f(x) = f(x).$$

**Definition 3:** Let  $f \in C_{-1}^m$ ,  $m \in \mathbb{N} \cup \{0\}$ . Then the Caputo fractional derivatives of  $f(x)$  is defined as:

$$D^\alpha f(x) = \begin{cases} J^{m-\alpha} f^{(m)}(x), & m-1 < \alpha \leq m, m \in \mathbb{N} \\ \frac{d^m f(x)}{dx^m}, & \alpha = m \end{cases}$$

Hence, we have following properties

1.  $J^\alpha J^\nu f = J^{\alpha+\nu} f$ ,  $\alpha, \nu > 0, f \in C_\mu, \mu > 0$
2.  $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \alpha > 0, \gamma > -1, x > 0$
3.  $J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, x > 0, m-1 < \alpha \leq m$
4.  $J^\alpha D^\alpha f(x) = f(x), x > 0, m-1 < \alpha \leq m$
5.  $D^\alpha C = 0$ ,  $C$  is constant
6.  $D^\alpha x^\beta = \begin{cases} 0 & \beta \in N_0, \beta < [\alpha] \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} & \beta \in N_0, \beta \geq [\alpha] \end{cases}$

where  $[\alpha]$  denoted the smallest integer greater than or equal to  $\alpha$  and  $N_0 = \{0, 1, 2, \dots\}$ .

### 1.2. The Derivative for Orthonormal Bernstein Polynomials

The Bernstein polynomials of  $n$ th degree are defined on the interval  $[0, 1]$  as [7].

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad \binom{n}{i} = \frac{n!}{i!(n-i)!} \quad \text{for } i = 0, 1, 2, \dots, n$$

The representation of the orthonormal Bernstein Polynomials, denoted by  $b_{i,n}(x)$  here, was discovered by analyzing the resulting orthonormal polynomials after applying the Gram-Schmidt process on sets of Bernstein polynomials of degree  $B_{i,n}(x)$ , [8]. Then the following sets of orthonormal polynomials  $b_{i,n}(x)$ ,  $0 \leq i \leq n$ . For  $n = 3$ , the four orthonormal Bernstein polynomials are given as:

$$b_{0,3}(x) = \sqrt{7} (1-x)^3,$$

$$b_{1,3}(x) = \sqrt{20} \left( 3x(x-1)^2 + \frac{1}{2}(x-1)^3 \right),$$

$$b_{2,3}(x) = \frac{10}{\sqrt{3}} \left( -3x(x-1)^2 - 3x^2(x-1) - \frac{3}{10}(x-1)^3 \right)$$

$$b_{3,3}(x) = 4 \left( 3x(x-1)^2 + \frac{9}{2}x^2(x-1) + \frac{1}{4}(x-1)^3 + x^3 \right)$$

## 2. CONVERGENCE OF THE PETROV-GALERKIN METHOD

In this section we introduce the (PGM) for Eq. (2). For the proof of all results in this section we can use the same manner used in [9], but for Eq. (2). Let  $X$  be a Banach space with the norm  $\|\cdot\|$  and let  $X^*$  denote its dual space. Assume  $K: X \rightarrow X$  is a compact linear operator. We rewrite this eq. (1) in operator form as:

$$D_*^\alpha u - Ku = f, \quad f \in X \tag{2}$$

where  $u \in X$  is the unknown to be determined. The Petrov-Galerkin method (PGM) used for the numerical solutions of eq. (2). The Petrov-Galerkin methods (PGM) interpolate between the Galerkin method and the collocation method. For this purpose for each positive integer  $n$ , we assume that  $X_n \subset X$ ,  $Y_n \subset X^*$ , and  $X_n, Y_n$  are finite dimensional vector spaces with  $\dim X_n = \dim Y_n$ , then  $X_n, Y_n$  satisfy condition (H): for each  $x \in X$  and  $y \in X^*$ , there exists  $x_n \in X_n$  and

$y_n \in Y_n$  such that  $\|x_n - x\| \rightarrow 0$  as  $n \rightarrow \infty$ . when The Petrov-Galerkin method (PGM) for Eq.(2) is a numerical method for finding  $u_n \in X$  such that

$$\langle D_*^\alpha u_n - K u_n, y_n \rangle = \langle f, y_n \rangle \text{ for all } y_n \in Y_n \quad (3)$$

It is clear that the Petrov-Galerkin method (PGM) is closely related to a generalized best approximation from  $X_n$  to  $x \in X$  with respect to  $Y_n$ . Given  $x \in X$ , an element  $P_n x \in X_n$  is called a generalized best approximation from  $X_n$  to  $x$  with respect to  $Y_n$  if it satisfies the equation

$$\langle x - P_n x, y_n \rangle = 0 \text{ for all } y_n \in Y_n \quad (4)$$

Similarly, given  $y \in X^*$ , an element  $p'_n y \in Y_n$  is called best approximation from  $Y_n$  to  $y$  with respect to  $Y_n$  if it satisfies the equation

$$\langle x_n, y - p'_n y \rangle = 0 \text{ for all } x_n \in X_n.$$

Now will demonstrate for each  $x \in X$  has a unique generalized best approximation

**Proposition 1:** For each  $x \in X$ , the generalized best approximation from  $X_n$  to  $x$  with respect to  $Y_n$  exists uniquely if and only if

$$Y_n \cap X_n^\perp = \{0\} \quad (5)$$

Under this condition,  $P_n$  is a projection; i.e.,  $P_n^2 = P_n$

Assume that, for each  $n$ , there is a linear operator  $\Pi_n: X_n \rightarrow Y_n$  with  $\Pi_n X_n = Y_n$ , and satisfying the condition

$$(H-1) \|x_n\| \leq C_1 \langle x_n, \Pi_n x_n \rangle^{\frac{1}{2}} \text{ for all } x_n \in X_n,$$

$$(H-2) \|\Pi_n x_n\| \leq C_2 \|x_n\| \text{ for all } x_n \in X_n,$$

Where  $C_1$  and  $C_2$  are positive constants independent of  $n$ . if a pair of sequence  $\{X_n\}$  and  $\{Y_n\}$  satisfy (H-1) and (H-2), we call  $\{X_n, Y_n\}$  a regular pair.

For each  $x \in X$ , let  $Q_n x$  be a best approximation from  $X_n$  to  $x$ , that is,  $Q_n x \in X_n$  satisfies the equation

$$\|x - Q_n x\| = \inf_{x_n \in X_n} \|x - x_n\|.$$

If a regular Pair  $\{X_n, Y_n\}$  satisfies  $\dim X_n = \dim Y_n$  and condition (H), then the corresponding generalized projection  $P_n$  satisfies:

$$(1) \text{ for all } x \in X, \|P_n x - x\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$(2) \text{ there is a constant } C > 0 \text{ such that, } \|P_n\| < C, \quad n = 1, 2, \dots$$

$$(2) \text{ for some constant } C > 0 \text{ independent of } n, \|P_n x - x\| \leq C \|Q_n x - x\| \text{ where } Q_n x \text{ is the best approximation from } X_n \text{ to } x.$$

if  $\{X_n, Y_n\}$  a regular pair is with a linear operator  $\Pi_n: X_n \rightarrow Y_n$  with  $\Pi_n X_n = Y_n$ , then eq. (3) may be rewritten

$$\langle D_*^\alpha u_n - K u_n, \Pi_n x_n \rangle = \langle f, \Pi_n x_n \rangle \text{ for all } x_n \in X_n \quad (6)$$

Using the projection  $P_n$  defined earlier, eq. (3) is equivalent to

$$D_*^\alpha u_n - P_n K u_n = P_n f \quad (7)$$

eq. (7) can also be derived from the fact that  $P_n x = 0$  for an  $x \in X$  if and only if  $\langle x, y_n \rangle = 0$  for all  $y_n \in Y_n$ . This equation indicates that the Petrov-Galerkin method is a projection method.

Now, assume  $u_n \in X_n$  and  $\{b_j\}_{j=1}^n$  is a basis for  $X_n$  (trial space) and  $\{b_i^*\}_{i=1}^n$  (test space) is a basis for  $Y_n$ . Therefore the (PGM) on  $[a, b]$  for Eq. (2) is:

$$\langle D_*^\alpha u_n - K u_n, b_i^* \rangle = \langle f, b_i^* \rangle, \quad i = 1, \dots, n \quad (8)$$

### 3. PETROV-GALERKIN METHOD (PGM) FOR APPROXIMATE SOLUTION OF LINEAR FRACTIONAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATION (LFVIDES) VIA ORTHONORMAL BERNSTEIN POLYNOMIALS BASIS

We recall the eq. (1) of the Linear fractional Volterra integro- differential equation (LFVIDEs) of the form:

$$D_*^\alpha u(x) = f(x) + \int_a^x k(x, t) u(t) dt, \quad u(0) = \beta, \quad x \in [0, 1]$$

Our approach being by taking the fractional integration to both sides of eq. (1) we get

$$u(x) = u(0) + I^\alpha f(x) + I^\alpha \left( \int_0^x k(x, t) u(t) dt \right) \quad (9)$$

To approximate solution of eq. (1), we use the normalization polynomial basis on  $[a, b]$  as:

$$u(x) = \sum_{i=0}^n a_i b_{i,n}(x) \quad (10)$$

Where  $(a_i, i = 0, 1, \dots, n)$  are unknown constants to be determined substituting eq. (10) in to eq. (9), we get

$$\sum_{i=0}^n a_i b_{i,n}(x) = u(0) + I^\alpha f(x) + I^\alpha \left( \int_0^x k(x, t) \sum_{i=0}^n a_i b_{i,n}(t) dt \right) \quad (11)$$

Hence

$$\sum_{i=0}^n a_i b_{i,n}(x) - I^\alpha \left( \int_0^x k(x, t) \sum_{i=0}^n a_i b_{i,n}(t) dt \right) = u(0) + I^\alpha f(x) \quad (12)$$

In the next step, apply Petrov-Galerkin method (PGM) for eq. (1) is a numerical method for finding  $u(x) = \sum_{i=0}^n a_i b_{i,n}(x) \in X_n$ , such that  $a_i$  is unknown and must be determined from eq. (12).

From eq. (8) it is clear that the eq.(12) can be written as :

$$\langle \sum_{i=0}^n a_i b_{i,n}(x) - I^\alpha \left( \int_0^x k(x, t) \sum_{i=0}^n a_i b_{i,n}(t) dt \right), b_{j,n}^* \rangle = \langle u(0) + I^\alpha f(x), b_{j,n}^* \rangle \quad (13)$$

Thus

$$\int_0^1 \left\{ \sum_{i=0}^n a_i b_{i,n}(x) - I^\alpha \left( \int_0^x k(x, t) \sum_{i=0}^n a_i b_{i,n}(t) dt \right) \right\} b_{j,n}^* dx = \int_0^1 \{u(0) + I^\alpha f(x)\} b_{j,n}^* dx \quad (14)$$

The Petrov-Galerkin using regular pairs  $\{X_n, Y_n\}$  of piecewise polynomial spaces are called Petrov-Galerkin element.

Then, Eq. (14) is equivalent to linear system can be formed as follows:

$$\left[ \begin{array}{c} R(x, a_i) = \int_0^1 \left[ \sum_{i=0}^n a_i b_{i,n}(x) - I^\alpha \left( \int_0^x k(x, t) \sum_{i=0}^n a_i b_{i,n}(t) dt \right) \right] b_{j,n}^* dx \\ h_j = \int_0^1 \{u(0) + I^\alpha f(x)\} b_{j,n}^* dx \end{array} \right] \quad (15)$$

We can represent the system eq.(15) as a matrix form:

$$RA = H \quad (16)$$

where

$$R = \begin{bmatrix} \int_0^1 R(x, a_0) b_{0,n}^* dx & \cdots & \int_0^1 R(x, a_n) b_{0,n}^* dx \\ \vdots & \ddots & \vdots \\ \int_0^1 R(x, a_0) b_{n,n}^* dx & \cdots & \int_0^1 R(x, a_n) b_{n,n}^* dx \end{bmatrix}, A = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}, H = \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_n \end{bmatrix}$$

Then we are solving the system to calculate the value  $a_i$

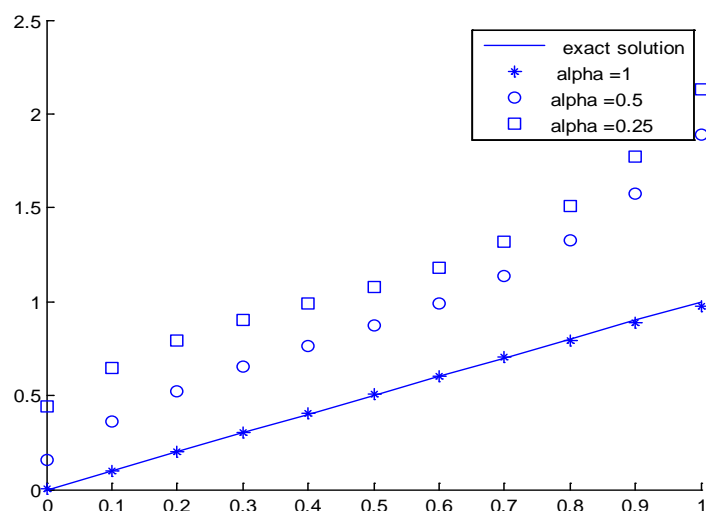
## 6. NUMERICAL EXAMPLES

**Example 1:** consider the following linear volterra fractional linear integro-differential equation:

$$D_x^\alpha y(x) = 1 - \frac{x^2}{2} + \int_0^x y(t) dt, \quad y(0) = 0, \quad 0 < \alpha \leq 1 \text{ with the exact solution } y(x) = x$$

**Table-1:** represent a comparison between the exact solution and approximate solution with different value  $\alpha = 0.25, 0.5, 1$

X	Exact solution	Approximate solution		
		$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.25$
0	0	0.0001		0.44006
0.1	0.1	0.1006	0.36432	0.64414
0.2	0.2	0.20209	0.52486	0.79142
0.3	0.3	0.30393	0.65311	0.90025
0.4	0.4	0.40548	0.76384	0.98899
0.5	0.5	0.50611	0.87186	1.0760
0.6	0.6	0.60516	0.99198	1.1796
0.7	0.7	0.70200	1.1390	1.3182
0.8	0.8	0.79599	1.3277	1.5101
0.9	0.9	0.88649	1.5728	1.7737
1	1	0.97286	1.8893	2.1273



**Figure-1:** comparison between the approximate solution and exact solution

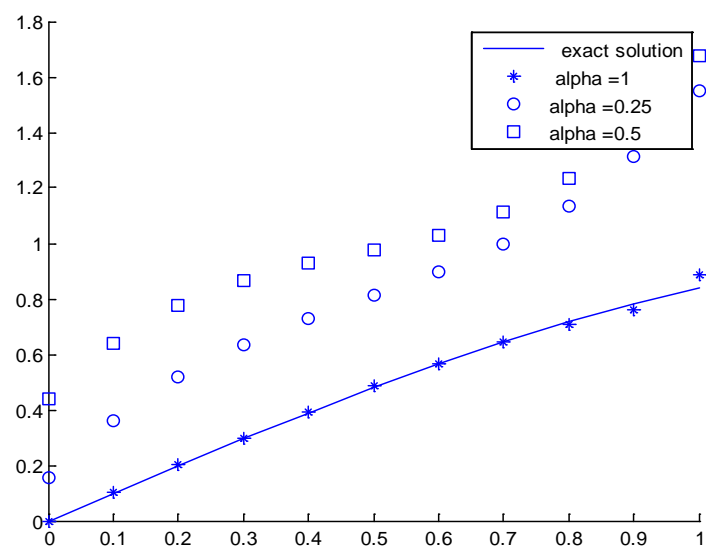
**Example 2:** consider the following linear volterra fractional integro-differential equation:

$$D_x^\alpha y(x) = 2 \cos x - 1 + \int_0^x y(t)dt, y(0) = 0, 0 < \alpha \leq 1$$

With the exact solution  $y(x) = \sin x$

**Table-2:** represent a comparison between the exact solution and approximate solution with different value  $\alpha = 0.25, 0.5, 1$

X	Exact solution	Approximate solution		
		$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.25$
0	0	0	0.15616	0.43945
0.1	0.09983	0.10041	0.36302	0.64068
0.2	0.19867	0.20063	0.5182	0.77708
0.3	0.29552	0.29909	0.63544	0.86648
0.4	0.38942	0.39417	0.72849	0.92672
0.5	0.47943	0.48422	0.81108	0.97564
0.6	0.56464	0.56764	0.89695	1.0311
0.7	0.64422	0.64277	0.99984	1.1109
0.8	0.71736	0.7080	1.1335	1.2329
0.9	0.78333	0.76168	1.3117	1.4149
1	0.84147	0.8022	1.5481	1.6748



**Figure-2:** comparison between the approximate solution and exact solution

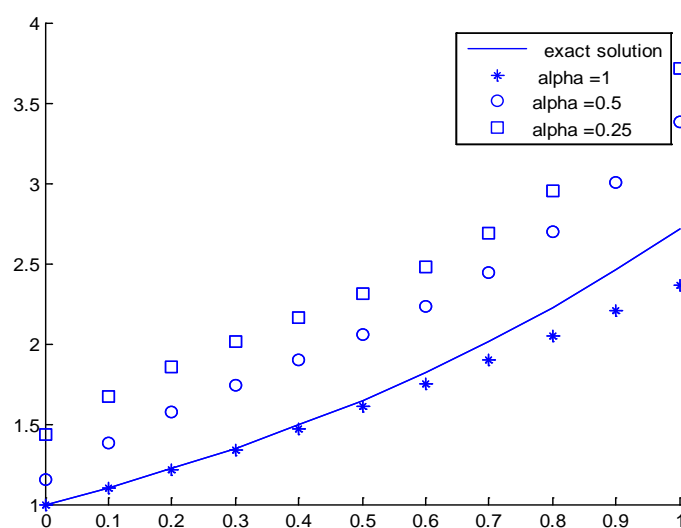
**Example 3:** consider the following linear volterra fractional integro-differential equation:

$$D_x^\alpha y(x) = e^x - xe^x + x + \int_0^x xy(t)dt, y(0)=1, 0 < \alpha \leq 1$$

With the exact solution  $y(x) = e^x$

**Table-3:** represent a comparison between the exact solution and approximate solution with different value  $\alpha = 0.25, 0.5, 1$

X	Exact solution	Approximate solution		
		$\alpha = 1$	$\alpha = 0.5$	$\alpha = 0.25$
0	1.000	1.000	1.1541	1.4379
0.1	1.1052	1.1049	1.3804	1.6703
0.2	1.2214	1.2191	1.5706	1.857
0.3	1.3499	1.3418	1.738	2.0148
0.4	1.4918	1.4722	1.8958	2.1604
0.5	1.6487	1.6093	2.057	2.3105
0.6	1.8221	1.7523	2.235	2.4818
0.7	2.0138	1.9002	2.4429	2.691
0.8	2.2255	2.0521	2.6938	2.9547
0.9	2.4596	2.2072	3.0009	3.2898
1	2.7183	2.3644	3.3775	3.7129



**Figure-3:** comparison between the approximate solution and exact solution

## 7. CONCLUSION

In this paper, Petrov-Galerkin method (PGM) has been successfully applied to finding the approximate solution of linear fractional Volterra integro-differential equation of the second kind (LFVIDE<sub>s</sub>) via the normalization Bernstein basis. This method is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear fractional Volterra integro-differential equation of the second kind (LFVIDE<sub>s</sub>), for the special case  $\alpha = 1$  is shown in Figure 1 Figure 2 and Figure3. It can be seen from this figures that the solution obtained by the present method is identical with the exact solution. In our paper, we use the Matlab language to calculate the Petrov-Galerkin method by using normalization Bernstein basis.

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