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PERFORMANCE MEASURES OF VARIANT WORKING VACATIONS ON BATCH ARRIVAL QUEUE WITH RENEGING

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ABSTRACT

This paper analyzes a batch arrival infinite-buffer single server queueing system with variant working vacations wherein customers arrive according to a Poisson process. As soon as the system becomes empty, the server takes working vacation. The service times during regular busy period, working vacation period and vacation times are assumed to be exponentially distributed and are mutually independent. During working vacations the customer may renege and the reneging time follows exponential distribution. We derive the probability generating function of the steady-state probabilities and obtain the closed form expressions of the system size when the server is in different states. In addition, we obtain some other performance measures and discuss their monotonicity. A cost model is formulated to determine the optimal service rate during working vacation and regular busy period.

***Keywords**: Queue, Batch arrival, Geometric distribution, Reneging, Variant working vacations, Probability generating function.*

1. INTRODUCTION

Queueing systems with server vacation have been investigated extensively due to their wide applications in several areas including computer communication systems, manufacturing and production systems. Vacation models are useful in systems where the server wants to utilize the idle time for different purposes. For more detail on this topic the reader may refer to the surveys of Doshi [6], Takagi [19] and Tian and Zhang [20].

In classical vacation queues, the server completely stops service during the vacation period. However, there are numerous situations where the server remains active during the vacation period which is called working vacation (WV). Servi and Finn [18] introduced this class of semi-vacation policy. They studied an $M/M/1$ queue with multiple working vacations $(MWVs)$. Baba [2] analyzed a $GI/M/1$ queue with MWV. Wu and Takagi [24] generalized Servi and Finn's [18] $M/M/1/WV$ queue to an $M/G/1/WV$ queue. Banik *et al.* [4] studied a $GI/M/1/N$ WV queue with limited waiting space. Liu *et al.* [13] derived the stochastic decomposition results in an $M/M/1$ queue with WV. In computer networks and communication systems the units arrive in batches, thus bulk input queue models have extensive applications. For the batch arrival queues, Xu *et al.* [25] investigated a bulk input $M^X/M/1$ queue with single working vacation. The probability generating function (PGF) of the stationary system length distribution is derived using quasi upper triangular transition matrix of two-dimensional Markov chain and matrix analytic method. The stochastic decomposition structure of system length has been evaluated which indicates the relationship with that of the $M^X/M/1$ queue without vacation. A similar analysis has been carried out in Baba [3] for $M^X/M/1$ queue with MWV and Liu and Song [14] for $M^X/M/1$ queue with working breakdown. A Steady state analysis and computation of the $GI^X/M^b/1/L$ queue with multiple working vacation and Partial batch rejection presented by Yu *et al.* [27] and Goswami and Vijaya Laxmi [8] analyzed the $GI^X/M/1/N$ queue with single working vacation and partial batch rejection. Recently, a retrial queue with working vacation for the batch arrival $Geo^X/Geo/1$ queue has been analyzed by Upadhyaya [21] by considering the general early arrival system.

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In real-life, many queueing situations arise where customers tend to be discouraged by a long queue. As a result, customers after joining the queue depart(renege) without getting service. Palm's [15] work seems to be the first to analyze the act of impatient customers in an $M/M/c$ queueing system, where the customers have independent exponential distribution sojourn times. Altman and Yechiali [1] presented the analysis for impatient customers in $M/M/1, M/G/1$ and $M/M/c$ queueing models with server vacations. An $M/M/c$ queue with impatient customers has been studied by Yechiali [26]. Perel and Yechiali [16] considered a 2-phase (fast and slow) Markovian random environment with impatient customers. Yue *et al.* [28] analyzed the $M/M/1$ queueing system with impatient customers and the variant of multiple vacation policy and obtained the closed-form expressions of the mean system sizes when the server is in different states using probability generating functions. Kim and Kim [12] analyzed a multi-server batch arrival $M^X/M/c$ queue with impatient customers.

The concept of variant multiple vacation policy is relatively a new one where the server is allowed to take a certain fixed number of consecutive vacations, if the system remains empty at the end of a vacation. This kind of vacation schedule is investigated by Zhang and Tian [29] for the $Geo/G/1$ queue with multiple adaptive vacations. Ke [9] analyzed the operating characteristic of $M^{[X]}/G/1$ system with a variant vacation policy with balking. Banik [5] studied the infinite-buffer single server queue with variant of multiple vacation policy and batch Markovian arrival process by using matrix analytic method. The literature related to this kind of vacation can be identified in papers by Ke and Chang [10], Ke *et al.* [11] and Wang *et al.* [23]. In case of WV, Zhang and Hou [30] analyzed a steady state renewal input $GI/M/1/N$ queue with a variant of multiple working vacation by using matrix analytic method. Gao and Yao [7] develop the variant working vacation (VWV) policy for the $M^X/G/1$ queueing system, where the server operates a randomized vacation policy and takes at most J working vacations when the system becomes empty. A finite buffer $M/M/1$ queue with VWV and balking and reneging has been analyzed by Vijaya Laxmi and Jyothsna [22]. They obtained the steady state probabilities using matrix form solutions.

In this paper, we extend the work of Yue *et al.* [28] by considering batch arrival and the concept of VWV with reneging. The customers become impatient when the server is on vacation. Variant working vacation policy refers to the phenomena of taking a maximum number K of consecutive WV s, if the system remains empty at the end of a WV. The VWV generates MWV when $K \to \infty$ and SWV when K is equal to 1. However, after the end of the K^{th} vacation, the server switches to normal busy period and stays idle or busy depending on the availability of the customers in the system. We have obtain the explicit expressions of the steady-state probabilities by using probability generating functions, Various performance measures and the monotonicity on some performance measures with respect to K are discussed and a cost optimization problem through quadratic fit search method $(QFSM)$ has been considered. $QFSM$ is an optimization technique which can be used when the objective function is highly complex and obtaining the derivative is a difficult task. Given a 3-point pattern, one can fit a quadratic function through corresponding functional values that has a unique minimum for given objective function. For details of $QFSM$, one may refer Rardin [17].

The rest of the paper is organized as follows. In section 2, model description is given. In Section 3, we have obtained the probability generating functions of the stationary analysis of the system . In section 4, the closed-form expression of the average system size when the server is in different states are derived. The closed-form expressions of the performance measures, monotonicity with respect K and cost model are presented in Section 5. Some numerical results are presented in the form of table and graphs in Section 6. Finally, Section 7 concludes the paper.

2. DESCRIPTION OF THE MODEL

We consider an infinite buffer queueing system wherein customers arrive in batches according to a Poisson process with rate λ. The arrival batch size X is a random variable with probability mass function $P(X = l) = b_l,\ l = 1,2,\cdots$.The service is provided by a single server with exponential service rate μ. At the end of a service, if there is no customer in the system, the server begins a vacation of random length which is exponentially distributed with parameter ϕ. If the server finds customer at a vacation completion instant, it returns to regular busy period; otherwise, the server takes K vacations sequentially. After the K vacations the server switches to busy period and depending on the arriving batch of customers, he stays idle or busy with the next arrivals. During these K vacations, the server keeps on serving the customers, generally with a slower rate. The service time during vacations are assumed to be exponentially distributed with parameter η. This type of vacation policy is referred to as VWV. One may note that the results of the corresponding queueing model with MWV $(K \to \infty)$ and single $WV(K=1)$ can be obtained from the results of our study.

During WV customers become impatient. That is, whenever a batch of customers arrives during WV, an " impatience timer" T is activated, which is exponentially distributed with parameter α. During WV, if the service does not commence before the time T expires, the customer abandons the queue and never returns. Since the arrival and departure of an impatient customer without service are independent, the average reneging rate of a customer is given by $n\alpha$, where n denotes the number of customers in the system. If the server is available in WV before the time T expires, the customer is served with rate η. If WV finishes before the impatient timer expires, server switches to normal working period and the customer is served with rate μ. We further assume that the switching time between the normal working and working vacations is negligible.

3. ANALYSIS OF THE MODEL

In the next subsection, we develop the difference equations for the $PGFs$ of the steady-state probabilities and solution of the differential equations.

3.1 STEADY STATE EQUATIONS

At time t, let $L(t)$ be the number of customers in the system and $J(t)$ denote the status of the server, which is defined as follows:

$$J(t) = \begin{cases} j, \text{ the server is on } (j+1)^{th} \text{ working vacation at time } t \text{ for } j = 0,1,\ldots,K-1, \\ K, \text{ the server is idle or busy at time } t. \end{cases}$$

The process $\{(L(t),J(t)), t \geq 0\}$ defines a continuous-time Markov process with state space

$$\Omega = \{(n, j) : n \geq 0, j = 0,1,\ldots,K\}.$$

Let $P_{n,j} = \lim_{t\to\infty} P\{L(t) = n, J(t) = j\}, n \geq 0, j = 0,1,\ldots,K,$ denote the steady-state probabilities of the process $\{(L(t), J(t)), t \geq 0\}$. Using Markov theory, the set of balance equations are given below:

$$(\lambda+\phi)P_{0,0} = (\eta+\alpha)P_{1,0} + \mu P_{1,K}, \tag{1}$$

$$(\lambda+\phi+\eta+\alpha)P_{1,0} = \lambda b_1 P_{0,0} + (\eta+2\alpha)P_{2,0}, \tag{2}$$

$$(\lambda+\phi+\eta+n\alpha)P_{n,0} = \lambda\sum_{m=1}^{n} b_m P_{n-m,0} + (\eta+(n+1)\alpha)P_{n+1,0}, \ \ n \geq 2, \tag{3}$$

$$(\lambda+\phi)P_{0,j} = (\eta+\alpha)P_{1,j} + \phi P_{0,j-1}, 1 \leq j \leq K-1, \tag{4}$$

$$(\lambda+\phi+\eta+\alpha)P_{1,j} = \lambda b_1 P_{0,j} + (\eta+2\alpha)P_{2,j}, 1 \leq j \leq K-1, \tag{5}$$

$$(\lambda+\phi+\eta+n\alpha)P_{n,j} = \lambda\sum_{m=1}^{n} b_m P_{n-m,j} + (\eta+(n+1)\alpha)P_{n+1,j}, 1 \leq j \leq K-1, n \geq 2, \tag{6}$$

$$\lambda P_{0,K} = \phi P_{0,K-1}, \tag{7}$$

$$(\lambda+\mu)P_{1,K} = \lambda b_1 P_{0,K} + \mu P_{2,K} + \phi\sum_{j=0}^{K-1} P_{1,j}, \tag{8}$$

$$(\lambda+\mu)P_{n,K} = \mu P_{n+1,K} + \lambda\sum_{m=1}^{n} b_m P_{n-m,K} + \phi\sum_{j=0}^{K-1} P_{n,j}, \ \ n \geq 2, \tag{9}$$

and the normalizing condition is

$$\sum_{n=0}^{\infty}\sum_{j=0}^{K} P_{n,j} = 1. \tag{10}$$

The state probabilities are obtained by solving the equations (1) to (9) using $PGFs$. Let us define the PGF of $P_{n,j}$ as

$$G_j(z) = \sum_{n=0}^{\infty} P_{n,j} z^n, 0 \leq z \leq 1, j = 0,1,..., K.$$

Define $G_j'(z) = \frac{d}{dz} G_j(z) = \sum_{n=1}^{\infty} n z^{n-1} P_{n,j}, \ j = 0,1,..., K$, and PGF of the arrival batch size X is $G(z) = \sum_{l=1}^{\infty} b_l z^l$, $|z| \leq 1$ with $G(1) = \sum_{l=1}^{\infty} b_l = 1.$

We assume that the arrival batch size X follows a geometric distribution with parameter q, that is, $P(X=l)=(1-q)^{l-1}q, 0<q<1 (\ l=1,2,\cdots)$. It is easy to observe that

$$G(z)=\frac{qz}{1-(1-q)z}. \tag{11}$$

Now, multiplying equations (1), (2) and (3) by z^n, summing over all possible values of n and re-arranging the terms, we get

$$\alpha z(1-z)G_0'(z)+(\lambda z(G(z)-1)-(\phi+\eta)z+\eta)G_0(z)=\eta(1-z)P_{0,0}-\mu zP_{1,K}. \tag{12}$$

Similarly, from equations (4), (5) and (6) and (7), (8) and (9), respectively, we get

$$\alpha z(1-z)G_j'(z)+(\lambda z(G(z)-1)-(\phi+\eta)z+\eta)G_j(z)=\eta(1-z)P_{0,j}-\phi zP_{0,j-1}, j=1,2,\cdots,K-1. \tag{13}$$

$$(\lambda z(G(z)-1)+(1-z)\mu)G_K(z)+\phi z\sum_{j=0}^{K-1}G_j(z)-\mu(1-z)P_{0,K}=z\left[\mu P_{1,K}+\phi\sum_{j=0}^{K-2}P_{0,j}\right]. \tag{14}$$

By taking $z=1$ in equations (12) and (13), we obtain

$$\phi G_0(1)=\mu P_{1,K} \tag{15}$$

and

$$G_j(1)=P_{0,j-1}. \tag{16}$$

3.2 SOLUTION OF THE DIFFERENTIAL EQUATION

We solve the differential equations (12) and (13) by following the method used in Altmana and Yechiali [1] and Yue et al. [28]. Equation (12) can be written as

$$G_0'(z)+\left[\frac{\lambda(G(z)-1)}{\alpha(1-z)}-\frac{\phi+\eta}{\alpha(1-z)}+\frac{\eta}{\alpha z(1-z)}\right]G_0(z)=\frac{\eta}{\alpha z}P_{0,0}-\frac{\mu}{\alpha(1-z)}P_{1,K}. \tag{17}$$

The above equation is a linear differential equation whose solution is given by

$$G_0(z)=\frac{(1-(1-q)z)^{\frac{-\lambda}{\alpha(1-q)}}}{(1-z)^{\frac{\phi}{\alpha}}z^{\frac{\eta}{\alpha}}}\left[\frac{\eta}{\alpha}I_1(z)P_{0,0}-\frac{\mu}{\alpha}I_2(z)P_{1,K}\right], \tag{18}$$

where

$$\left.\begin{aligned} I_1(z)&=\int_0^z(1-(1-q)x)^{\frac{\lambda}{\alpha(1-q)}}(1-x)^{\frac{\phi}{\alpha}}x^{\frac{\eta}{\alpha}-1}dx,\\ I_2(z)&=\int_0^z(1-(1-q)x)^{\frac{\lambda}{\alpha(1-q)}}(1-x)^{\frac{\phi}{\alpha}-1}x^{\frac{\eta}{\alpha}}dx.\end{aligned}\right\} \tag{19}$$

Proceeding similarly, equation (13) gives

$$G_j(z)=\frac{(1-(1-q)z)^{\frac{-\lambda}{\alpha(1-q)}}}{(1-z)^{\frac{\phi}{\alpha}}z^{\frac{\eta}{\alpha}}}\left[\frac{\eta}{\alpha}I_1(z)P_{0,j}-\frac{\phi}{\alpha}I_2(z)P_{0,j-1}\right], 1\le j\le K-1. \tag{20}$$

To get $P_{1,K}$ and $P_{0,j}$ in terms of $P_{0,0}$, $z=1$ and $z=0$ are the roots of the denominator of the right hand side of equations (18) and (20), we have $z=1$ and $z=0$ must be the roots of the numerator of the right hand side of those equations. Taking $z=1$ in equations (18) and (20), we get

$$P_{1,K}=\frac{\eta}{\mu}\frac{I_1(1)}{I_2(1)}P_{0,0} \tag{21}$$

and

$$P_{0,j}=\frac{\phi}{\eta}\frac{I_2(1)}{I_1(1)}P_{0,j-1}, 1\le j\le K-1. \tag{22}$$

Equation (22) can be written as

$$P_{0,j} = C^j P_{0,0}, 1 \le j \le K-1, \tag{23}$$

where $C = \frac{\phi}{\eta}\frac{I_2(1)}{I_1(1)}$. It is easy to observe that $0 < C < 1$ and C^j decreases with j for $j = 1,\ldots,K-1$. Using equations (7) and (23), we obtain

$$P_{0,K} = \frac{\phi}{\lambda} C^{K-1} P_{0,0}. \tag{24}$$

Using equations (21) and (23) in equations (18) and (20) respectively, we get

$$G_0(z) = \frac{(1-(1-q)z)^{\frac{-\lambda}{\alpha(1-q)}}}{(1-z)^{\frac{\phi}{\alpha}} z^{\frac{\eta}{\alpha}}}\left[I_1(z) - I_2(z)\frac{I_1(1)}{I_2(1)}\right]\frac{\eta}{\alpha} P_{0,0}, \tag{25}$$

$$G_j(z) = \frac{(1-(1-q)z)^{\frac{-\lambda}{\alpha(1-q)}}}{(1-z)^{\frac{\phi}{\alpha}} z^{\frac{\eta}{\alpha}}}\left[\frac{I_2(1)}{I_1(1)} I_1(z) - I_2(z)\right]\frac{\phi}{\alpha} C^{j-1} P_{0,0}, 1 \le j \le K-1. \tag{26}$$

From equations (21), (23) and (24) show that $P_{1,K}, P_{0,j}, j = 0,1,\ldots,K$ are expressed in terms of $P_{0,0}$. Equations (25) and (26) $G_j(z)$ for $j = 0,1,\ldots,K-1,$ is expressed in terms of $P_{0,0}$. Hence, from equation (14), $G_K(z)$ can also be expressed in terms of $P_{0,0}$. Therefore, once $P_{0,0}$ is calculated, $G_j(z)$ for $j = 0,1,\ldots,K,$ are completely determined.

In the next section, we derive the probability $P_{0,0}, P_{0,K}$ and the mean system size when the server is in different states.

4. MEAN SYSTEM SIZE

Let L_j be the system size when the server is in the state j, and the mean system size at j is given by

$$E[L_j] = G_j'(1) = \sum_{n=1}^{\infty} n P_{n,j},\ j = 0,1,\ldots,K.$$

Here, $E[L_j]$ for $j = 0,1,\ldots,K-1,$ represents the mean system size when the server is taking the $(j+1)^{th}$ WV and $E[L_K]$ represents the mean system size when the server is busy or idle. We first derive $E[L_j]$ for $j = 0,1,\ldots,K-1.$

From equations (12) and (15), we have for $z = 1$ and using L'Hospital rule

$$(\lambda+\phi)G_0'(1) = (\lambda G'(1) - \eta)G_0(1) + \eta P_{0,0}. \tag{27}$$

Similarly, from equations (13) and (16), we get

$$(\lambda+\phi)G_j'(1) = (\lambda G'(1) - \eta)G_j(1) + \eta P_{0,j},\ j = 1,2,\ldots,K-1. \tag{28}$$

Equations (27) and (28) imply

$$E[L_j] = G_j'(1) = \frac{(\lambda G'(1) - \eta)}{\alpha+\phi} G_j(1) + \frac{\eta}{\alpha+\phi} P_{0,j},\ j = 0,1,\ldots,K-1. \tag{29}$$

From equations (25) and (26), using L'Hospital rule, we get

$$G_j(1) = C^{j-1} P_{0,0},\ j = 0,1,\ldots,K-1. \tag{30}$$

Using equations (30) and (23), equation (29) can be written as

$$E[L_j] = G_j'(1) = \frac{(\lambda G'(1) - \eta)C^{j-1} + \eta C^j}{\alpha+\phi} P_{0,0},\ j = 0,1,\ldots,K-1. \tag{31}$$

Therefore, the mean system size when the server is on WV, denoted by $E[L_{WV}]$, is obtained as

$$E[L_{WV}] = \sum_{j=0}^{K-1} E[L_j] = \left[\frac{\lambda G'(1) - \eta(1-C)}{\alpha+\phi}\right]\frac{(1-C^K)}{C(1-C)} P_{0,0}. \tag{32}$$

Next, we derive $G_K(1)$ and $P_{0,0}$. By using equations (15) and (16), we obtain

$$\mu P_{1,K} + \phi\sum_{j=0}^{K-2} P_{0,j} = \phi\sum_{j=0}^{K-1} G_j(1). \tag{33}$$

Substituting the above equation in (14), we get

$$G_K(z) = \frac{\mu(1-z)P_{0,K} - \phi z\sum_{j=0}^{K-1}\left[G_j(z) - G_j(1)\right]}{\lambda z(G(z)-1) + (1-z)\mu}. \tag{34}$$

Taking $z = 1$ and applying L'Hospital rule, we get

$$G_K(1) = \frac{\phi E[L_{WV}] + \mu P_{0,K}}{(\mu - \lambda G'(1))}. \tag{35}$$

Substituting equations(24) and (32) in the above equation, we get

$$G_K(1) = \frac{\phi}{(\mu - \lambda G'(1))}\left[\frac{(\lambda G'(1) - \eta(1-C))}{\alpha+\phi}\frac{(1-C^K)}{C(1-C)} + \frac{\mu}{\lambda}C^{K-1}\right]P_{0,0}. \tag{36}$$

Using (30) and (36) in normalization condition $\sum_{j=0}^{K-1} G_j(1) + G_K(1) = 1$, we get

$$P_{0,0} = \frac{\lambda(\mu - \lambda G'(1))(\alpha+\phi)h(K)}{\lambda\alpha(\mu - \lambda G'(1)) + \phi[\lambda(\mu - \eta(1-C)) + \mu(\alpha+\phi)H(K)]}, \tag{37}$$

where $h(K) = \dfrac{C(1-C)}{(1-C^K)}$ and $H(K) = \dfrac{C^K(1-C)}{(1-C^K)}$ and which decreases with K. Substituting equation (37) in equation (32), we obtain the mean system size when the server is on WV as follows:

$$E[L_{WV}] = \frac{\lambda(\mu - \lambda G'(1))(\lambda G'(1) - \eta(1-C))}{\lambda\alpha(\mu - \lambda G'(1)) + \phi[\lambda(\mu - \eta(1-C)) + \mu(\alpha+\phi)H(K)]}. \tag{38}$$

Now, we derive $E[L_K]$. From equation (34), using L'Hospital rule, we obtain

$$E[L_K] = G_K'(1) = \frac{\phi}{2(\mu - \lambda G'(1))}\sum_{j=0}^{K-1} G_j''(1) + \frac{\phi(2\mu + \lambda G''(1))}{2(\mu - \lambda G'(1))^2}\sum_{j=0}^{K-1} G_j'(1) + \frac{\lambda\mu\left[G''(1) + 2G'(1)\right]}{2(\mu - \lambda G'(1))^2}P_{0,K}, \tag{39}$$

where $G_j''(1)$ is obtained by differentiating twice $G_j(z)$ at $z=1$ for $j = 0,1,\ldots, K-1$. Differentiating twice (12) and (13) and taking $z = 1$, we get

$$G_j''(1) = \frac{1}{\phi + 2\alpha}\left[(\lambda G'(1) - (\phi+\eta+\alpha))2G_j'(1) + \lambda(2G'(1) + G''(1))G_j(1)\right], j = 0,1,\ldots,K-1. \tag{40}$$

Substituting equation (40) in (39), we obtain

$$\begin{aligned} E[L_K] = {} & \frac{\phi}{(\mu - \lambda G'(1))}\left[\left(\frac{\lambda G'(1) - (\phi+\eta+\alpha)}{(\phi+2\alpha)} + \frac{2\mu + \lambda G''(1)}{2(\mu - \lambda G'(1))}\right)E[L_{WV}]\right] \\ & + \frac{\lambda(2G'(1) + G''(1))}{2(\mu - \lambda G'(1))}\left[\frac{\phi}{(\phi+2\alpha)}\frac{1-C^K}{C(1-C)}P_{0,0} + \frac{\mu}{(\mu - \lambda G'(1))}P_{0,K}\right]. \end{aligned} \tag{41}$$

From equation (11), we obtain that $G'(1) = g = 1/q$, $G''(1) = 2(1-q)/q^2$ and $2G'(1) + G''(1) = 2/q^2$ and $\rho = (\lambda g)/\mu$. Therefore, equations (38) and (41) can be written as

$$E[L_{WV}] = \frac{\lambda\mu(1-\rho)(\mu\rho - \eta(1-C))}{\lambda\alpha\mu(1-\rho) + \phi[\lambda(\mu - \eta(1-C)) + \mu(\alpha+\phi)H(K)]} \tag{42}$$

and

$$\begin{aligned} E[L_K] = {} & \frac{\phi}{\mu(1-\rho)}\left[\left(\frac{\mu\rho - (\phi+\eta+\alpha)}{(\phi+2\alpha)} + \frac{\mu q^2 + \lambda(1-q)}{\mu q^2(1-\rho)}\right)E[L_{WV}]\right] \\ & + \frac{\lambda}{\mu q^2(1-\rho)}\left[\frac{\phi}{(\phi+2\alpha)}\frac{1-C^K}{C(1-C)}P_{0,0} + \frac{1}{(1-\rho)}P_{0,K}\right], \end{aligned} \tag{43}$$

where $E[L_{WV}]$ is calculated by equation (42) and the probability when the server is idle $(P_{0,K})$, is calculated by using equations (24) and (37) as follows:

$$P_{0,K} = \frac{\phi\mu(1-\rho)(\alpha+\phi)H(K)}{\lambda\alpha\mu(1-\rho)+\phi[\lambda(\mu-\eta(1-C))+\mu(\alpha+\phi)H(K)]}. \tag{44}$$

Let L be the number of customers in the system. The mean system size $E[L] = E[L_{WV}] + E[L_K]$ can be calculated from equations (42) and (43).

5. PERFORMANCE MEASURES AND COST MODEL

In this section, some other performance measures and their monotonicity with respect to K are presented, and we develop a cost model to determine the optimal service rate during WV.

5.1 PERFORMANCE MEASURES

- When the system is in state (n, j), $n \geq 0, j = 0,1,\cdots,K-1$, the rate of abandonment of a customer due to impatience is $n\alpha$. Thus, the average rate of abandonment due to impatience is given by

$$R_a = \sum_{j=0}^{K-1}\sum_{n=1}^{\infty} n\alpha P_{n,j} = \alpha E[L_{WV}]. \tag{45}$$

- From equation (44), the probability that the server is idle is given by

$$P_{0,K} = \frac{\phi\mu(1-\rho)(\alpha+\phi)H(K)}{\lambda\alpha\mu(1-\rho)+\phi[\lambda(\mu-\eta(1-C))+\mu(\alpha+\phi)H(K)]}.$$

  If we consider K as a continuous variable and take the derivative of $P_{0,K}$ with respect to K, we have $\frac{dP_{0,K}}{dK} < 0$ and the inequality follows from the fact that $H(K)$ decreases with K. Therefore, $P_{0,K}$ is a decreasing function of K.

- Let P_{WV} be the probability when the server is on WV. From equation (30), we have

$$P_{WV} = \sum_{j=0}^{K-1} G_j(1) = \frac{1-C^K}{C(1-C)} P_{0,0}. \tag{46}$$

  Substituting equation (37) in (46), we get

$$P_{WV} = \frac{\lambda\mu(1-\rho)(\alpha+\phi)}{\lambda\alpha\mu(1-\rho)+\phi[\lambda(\mu-\eta(1-C))+\mu(\alpha+\phi)H(K)]}. \tag{47}$$

  We see that P_{WV} increases with K because of the decrease of $P_{0,K}$ with respect to K.

- The probability of busy server is given by

$$P_b = \sum_{n=1}^{\infty} P_{n,K} = 1 - P_{0,K} - P_{WV}. \tag{48}$$

  Substituting equations (44) and (47) in the above equation, we obtain

$$P_b = \frac{\lambda\phi[(1/q)(\lambda+(\alpha+\phi)H(K)) - \eta(1-C)]}{\lambda\alpha\mu(1-\rho)+\phi[\lambda(\mu-\eta(1-C))+\mu(\alpha+\phi)H(K)]}. \tag{49}$$

  From the fact that $P_{0,K}$ decreases, P_{WV} increases with K, we find that P_b decreases with K.

- The probability when the system is empty and the server is on WV is given by

$$P_e = \sum_{j=0}^{K-1} P_{0,j} = \frac{(1-C^K)}{(1-C)} P_{0,0}. \tag{50}$$

  Using equation (46) in (50), we get

$$P_e = CP_{WV}. \tag{51}$$

- The average number of customers in the queue $E[L_q]$ is given by

$$E[L_q] = \sum_{j=0}^{K}\sum_{n=1}^{\infty} (n-1) P_{n,j}, \tag{52}$$

  and it can be written as

$$E[L_q] = E[L] - \left[1 - \sum_{j=0}^{K} P_{0,j}\right]. \tag{53}$$

5.2 COST MODEL

In this subsection, we formulate an expected cost function in which mean service rate η during WV is the control variable. Let us define

$C_1 \equiv$ service cost per unit time when the server is busy,
$C_2 \equiv$ service cost per unit time when the server is on WV,
$C_3 \equiv$ fixed cost per unit time when the server is on WV,
$C_4 \equiv$ cost per unit time when the server is idle,
$C_5 \equiv$ cost per unit time when the server is idle during WV,
$C_6 \equiv$ cost per unit time of every customer in the queue and waiting for service.

Using the definition of each cost element and its corresponding system characteristics, the total expected cost function per unit time is given by

$$F = C_1\mu P_b + (C_2\eta + C_3\phi)P_{WV} + C_4 P_{0,K} + C_5 P_e + C_6 E[L_q], \quad (54)$$

where $P_{0,K}, P_{WV}, P_b, P_e$ and $E[L_q]$ are given in equations (44), (47), (49), (51), and (53), respectively. Our objective is to determine the optimal mean service rate η^* during WV and μ^* during regular busy period that minimizes the cost function F. We employ the $QFSM$ to solve the above optimization problem, as the computation of derivatives of the above expected cost function is a nontrivial task.

$QFSM$ is a 3-point pattern, we can fit a quadratic function through corresponding functional values that has a unique minimum, x^q, for the given objective function $F(x)$. Quadratic fit uses this approximation to improve the current 3-point pattern by replacing one of its points with approximate optimum x^q. The unique optimum x^q of the quadratic function agreeing with $F(x)$ at 3-point operation (x^l, x^m, x^h) is given by

$$x^q \cong \frac{1}{2}\left[\frac{F(x^l)(s^m - s^h) + F(x^m)(s^h - s^l) + F(x^h)(s^l - s^m)}{F(x^l)(x^m - x^h) + F(x^m)(x^h - x^l) + F(x^h)(x^l - x^m)}\right],$$

where $s^l = (x^l)^2, s^m = (x^m)^2, s^h = (x^h)^2$.

6. NUMERICAL ANALYSIS

To study the parameter impact on the system performance, numerical computations are carried out and a few of those are presented in this section in the form of graphs. We have consider the cost parameters as: $C_1 = 50,\ C_2 = 25,\ C_3 = 35,\ C_4 = 20,\ C_5 = 18$ and $C_6 = 30$ and the other parameters as $\lambda = 1.9, \mu = 4.0, K = 6, \alpha = 0.2, \eta = 1.2, \phi = 2.0$ and $q = 0.7$ for all the figures, unless they are considered as variables or their values are mentioned in the respective figures. The performance measures $P_{O,K}$ and P_b are decreasing and P_{WV}, P_e and R_a are increasing function of K presented numerically in Table 1.

Table-1: Effect of K on performance measures

K	$P_{0,K}$	P_b	P_{WV}	P_e	R_a
1	0.136383828	0.638687628	0.224928542	0.129564637	0.045098513
2	0.068211513	0.623994041	0.307794444	0.177297532	0.061713252
3	0.036520349	0.617163455	0.346316194	0.199487053	0.069436921
4	0.020215369	0.613649145	0.366135484	0.210903474	0.073410718
5	0.011388462	0.611746629	0.376864907	0.217083898	0.075561984
6	0.006477977	0.610688243	0.382833778	0.220522122	0.076758752
7	0.003704783	0.610090520	0.386204695	0.222463857	0.077434626

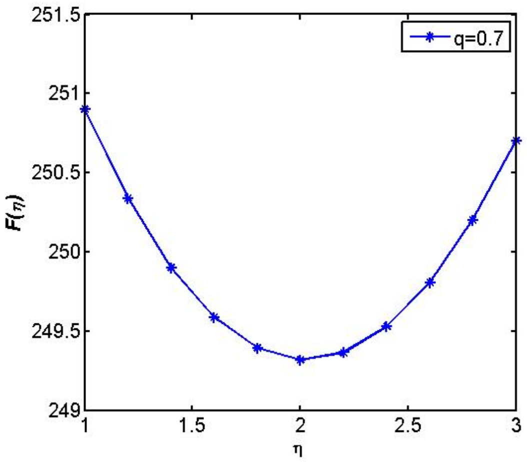

Figure-1: Effect of η on $F[\eta]$

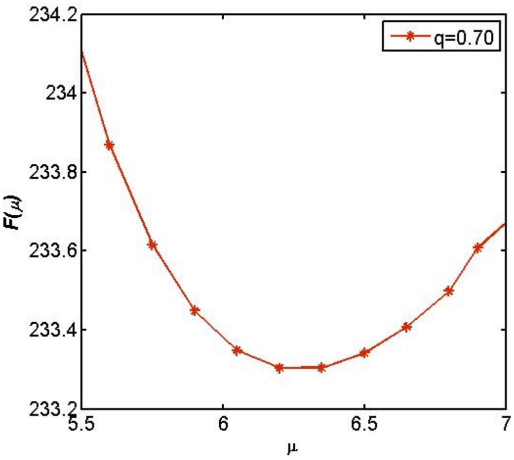

Figure-2: Effect of μ on $F[\mu]$

The effect of η and μ on the total expected cost function (F) for fixed mean batch size $(g = 1/q)$ is shown in " Fig. 1" and "Fig. 2", with constant vacation rates. With the information of " Fig. 1" and " Fig. 2", the convexity of the curves show that there exists a certain value of η and μ that minimizes the total expected cost function for the chosen set of model parameters. We have adopted $QFSM$ by choosing the initial 3-point pattern as $(\eta^l, \eta^m, \eta^h) = (1.8, 2.0, 2.2)$ and $(\mu^l, \mu^m, \mu^h) = (6.05, 6.2, 6.35)$ and the stopping tolerance $\varepsilon = 10^{-6}$. After finite iterations, Table 2 and Table 3 shows that the minimum expected operating cost per unit time converges to the solution $F = 249.3179157$ at $\eta^* = 2.0219858$ and $F = 233.2993391$ at $\mu^* = 6.2719852$.

Table-2: Search for optimum service rate during working vacation period (η^*)

η^l	η^m	η^h	$F(\eta^l)$	$F(\eta^m)$	$F(\eta^h)$	η^q	$F(\eta^q)$
1.8	2.0000000	2.2000000	249.3917453	249.3186366	249.3649432	2.0224444	249.3179160
2.00	2.0224444	2.2000000	249.3186366	249.3179160	249.3649432	2.0220343	249.3179157
2.00	2.0220343	2.0224444	249.3186366	249.3179157	249.3179160	2.0219859	249.3179157
2.00	2.0219859	2.0220343	249.3186366	249.3179157	249.3179157	2.0219858	249.3179157
2.00	2.0219858	2.0219859	249.3186366	249.3179157	249.3179157	2.0219858	249.3179157
2.00	2.0219858	2.0219858	249.3186366	249.3179157	249.3179157	2.0219858	249.3179157

Table-3: Search for optimum service rate during regular busy period (μ^*)

μ^l	μ^m	μ^h	$F(\mu^l)$	$F(\mu^m)$	$F(\mu^h)$	μ^q	$F(\mu^q)$
6.05	6.2000000	6.3500000	233.3485275	233.3041510	233.3046107	6.2734620	233.2993410
6.20	6.2734620	6.3500000	233.3041510	233.2993410	233.3046107	6.2732888	233.2993406
6.20	6.2732888	6.2734620	233.3041510	233.2993406	233.2993410	6.2720310	233.2993391
6.20	6.2720310	6.2732888	233.3041510	233.2993391	233.2993406	6.2720076	233.2993391
6.20	6.2720076	6.2720310	233.3041510	233.2993391	233.2993391	6.2719864	233.2993391
6.20	6.2719864	6.2720076	233.3041510	233.2993391	233.2993391	6.2719857	233.2993391
6.20	6.2719857	6.2719864	233.3041510	233.2993391	233.2993391	6.2719853	233.2993391
6.20	6.2719853	6.2719857	233.3041510	233.2993391	233.2993391	6.2719852	233.2993391

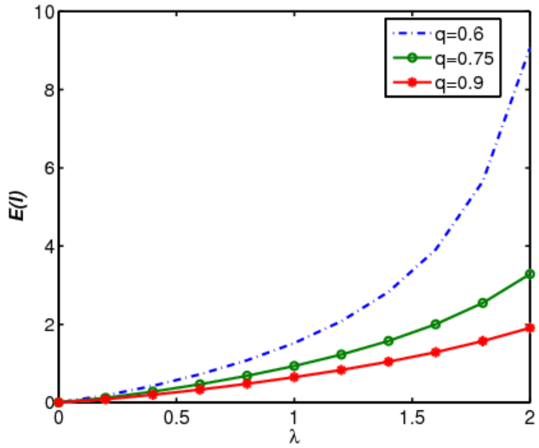

Figure-3: Effect of λ on $E[L]$

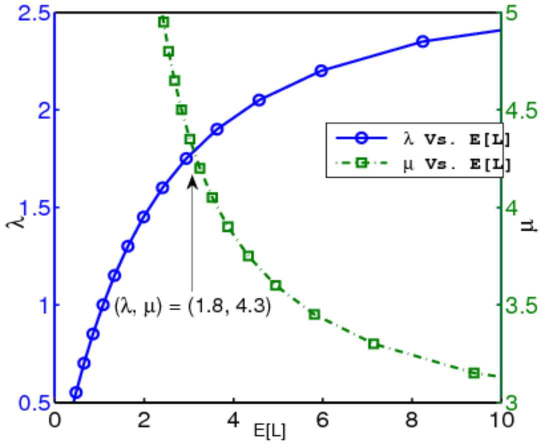

Figure-4: Effect of λ and μ on $E[L]$

The impact of arrival rate on $E[L]$ is shown in " Fig. 3", for different mean batch sizes $(g = 1/q)$. It can be observed that as λ increases, $E[L]$ increases for fixed q. For fixed λ, $E[L]$ increases with g as it should be. The effect of λ and μ on $E[L]$ is depicted in " Fig. 4". As expected, one may observe that $E[L]$ increases as λ increases whereas it decrease with μ. Further, the intersection point of the two curves at $(\lambda,\mu) = (1.8,4.3)$ gives us the optimum value of $E[L]$. " Fig. 5" shows the impact of reneging rate (α) on $E[L_{WV}]$ for different K values. It can be observed that as α increases $E[L_{WV}]$ decreases. We can also observe that single WV has better performance than MWV.

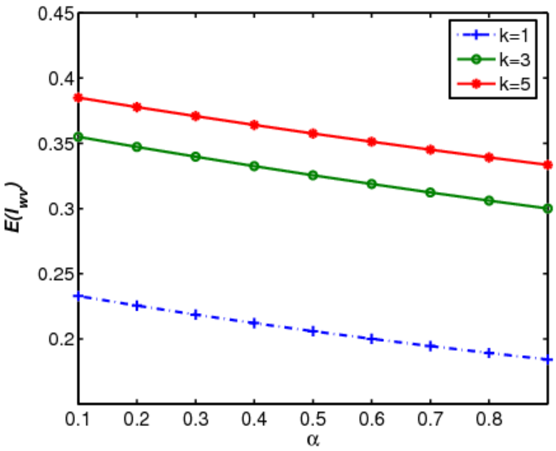

Figure-5: Effect of α on $E[L_{WV}]$

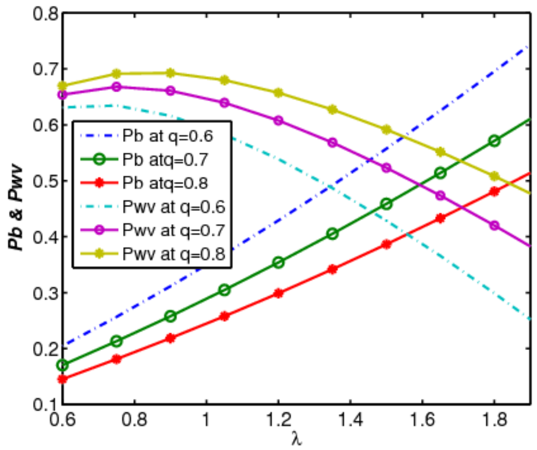

Figure-6: Effect of λ on P_b and P_{WV}

Finally, " Fig. 6" depicts the impact of probabilities during regular busy period (P_b) and WV (P_{WV}) depends on λ for different q. We observed that P_b increases as increases of λ, where as P_{WV} gradually increases and then decreases. For fixed λ, P_b decreases along with the increases of q, otherhand P_{WV} increases along with the increases of q. And the intersect points enumerates both P_b and P_{WV} are equal at particular λ corresponding to the q.

7. CONCLUSION

In this paper, we have studied an $M^X/M/1$ queueing system with variant WVs and customer's impatience. We have derived $PGFs$ of the number of customers in the system and the corresponding mean system sizes when the server is in different states. We have derived closed-form expressions for some other performance measures, the rate of abandonment due to impatience and also formulated a cost model to determine the optimal value of η. The effects of some parameters on the performance measures of the system have been investigated and the results are presented in the form of graphs. The technique adopted in this paper can be applied to analyze models like $M/M^X/1$ queue with variant working vacations, inpatient customer $M^X/M/c$ queue with variant working vacations, etc.

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