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EXTENSION OF URYSOHN'S LEMMA ON MEASURE MANIFOLD

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ABSTRACT

In this paper, we study the necessary and sufficient condition for the existence of normal property on measure manifold by using the existence of measurable homeomorphism and measure invariant function on (M,τ,Σ,μ). We also observe that this property holds μ_1-a.e., on the measure manifold.

Keywords: *Measure manifold, Measurable normal, μ_1-a.e. on measure manifold.*

MSC (2010): *28-XX, 49Q15, 54-XX, 57NXX, 58-XX.*

1. INTRODUCTION

In [4] [7] it is shown that the subset A defined as the set of all points x in M such that the property P(x) is true μ-a.e., and μ(A) > 0 otherwise the property P(x) is not true, for all other points x ∈ U ⊂ M such that μ (A) = 0 and is identified as the dark region of the measure manifold. In [5] S. C. P. Halakatti has proved the measurable Hausdorff, regular and normal properties on (R^n, τ, Σ, μ) by showing the existence of measurable function $f : (R^n,\tau,\Sigma,\mu) \longrightarrow [0,1] \subset (R,\tau,\Sigma,\mu)$. Further it is shown that these properties remain invariant on the measure manifold with the help of measurable homeomorphism and measure invariant function $f \circ \phi: (M, \tau, \Sigma, \mu) \longrightarrow [0, 1] \subset (R,\tau,\Sigma,\mu)$ [3]. In this paper we study the necessary and sufficient condition for normal property on measure space and on measure manifold with additional conditions.

2. PRELIMINARIES

We use the following basic definitions and results to develop the study on measure manifold [1].

Definition 2.1: Measurable Normal
A measure space (R^n, τ, Σ, μ) is said to be measurable normal if for every pair of disjoint Borel closed sets E and F ∈ (R^n, τ, Σ, μ) ∃ Borel open sets A and B such that E ⊂ A, F ⊂ B and A ∩ B = ∅ with μ(A) > 0, μ(B) > 0 and μ (A ∩ B) = μ (∅) = 0.

We extend the study of above measurable topological property on measure manifold with the help of already developed concepts of measure manifold [2].

Definition 2.2: Measure Chart
A measure $\mu_{1/U}$ on a measurable chart $((U_1,\tau_{1/U}, \Sigma_{1/U}), \phi)$ is called a measure chart, denoted by $((U_1,\tau_{1/U}, \Sigma_{1/U}, \mu_{1/U}), \phi)$ satisfying the following conditions:

(*i*) ϕ is homeomorphism
(*ii*) ϕ is measurable and
(*iii*) ϕ is measure invariant
then, the structure $((U_1,\tau_{1/U}, \Sigma_{1/U}, \mu_{1/U}), \phi)$ is called as a measure chart.

Theorem 2.3: Let (M,τ) be a second countable, Hausdorff topological space which is modeled on a measure space (R^n,τ,Σ,μ) then ∃ a C^∞ measurable homeomorphism and measure invariant function $\phi: U \subset M \longrightarrow (R^n, \tau, \Sigma, \mu)$ generates a measure manifold of dimension n.

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Theorem 2.4: All Borel subsets in R^n are Lebesgue measurable.

Theorem 2.5: The Lebesgue measure of an interval is its length.

3. MAIN RESULTS

Let (R^n,τ, Σ, μ) be a measure space and let $\{f_n\}$ be a sequence of measurable functions converging to measure function f in (R^n, τ, Σ, μ). The point p∈ U ⊂ (R^n, τ, Σ, μ) carries additional information in terms of the ordered pair ($\{f_n\}$, f) which induces a measurable set A_n such that the measure of such set is always positive on measure space [4][7].

This means that if $\{f_n\}$ is a sequence of measurable functions on (R^n, τ, Σ, μ) then the ordered pair ($\{f_n\}$, f) induces a Borel subset A_n satisfying the following conditions:
A_n = {x ∈ (R^n, τ, Σ, μ): $|f_n(x)$ - $f(x)| <\ \varepsilon$}, ∀ n ∈ N, where

i) $\mu(A_n) > 0$, if $|f_n(x)$ - $f(x)| <\ \varepsilon$ ∀ n ∈ N
ii) $\mu(A_n) = 0$, if $|f_n(x)$ - $f(x)| \geq\ \varepsilon$ ∀ n ∈ N that is, $\mu(A_n)$ = 0 as n → ∞

Lemma 3.1: A measure space (R^n, τ, Σ, μ) is measurable normal if for each pair of disjoint Borel closed sets E and F carrying additional information ($\{f_n\}$, f), if ∃ a measurable function f : (R^n, τ, Σ, μ) ⟶ [0,1] ⊂(R, τ, Σ, μ) such that $f(E)$ = 0 and $f(F)$ = 1.

Proof: Suppose (R^n, τ, Σ, μ) is measurable normal (e-normal).

Let E, F be a pair of disjoint Borel closed sets in (R^n,τ, Σ, μ) that is, E ∩ F = ∅
⇒ E ⊂ (R^n - F)

E is a Borel closed set and (R^n - F) is Borel open set.

Since (R^n,τ, Σ, μ) is e-normal ∃ Borel open set say, $G_{1/2}$ such that

$$E \subset G_{1/2} \subset \overline{G_{1/2}} \subset (R^n - F)$$

Again, E ⊂ $G_{1/2}$ where E is Borel closed set and $G_{1/2}$ is Borel open set and $G_{1/2}$ ⊂ (R^n - F)
where $G_{1/2}$ is Borel closed set and $R^n - F$ is Borel open set.

By normality of (R^n,τ, Σ, μ) ∃ Borel open sets say $G_{1/4}$ and $G_{3/4}$ such that

$$E \subset G_{1/4} \subset \overline{G_{1/4}} \subset G_{1/2} \subset \overline{G_{1/2}} \subset G_{3/4} \subset \overline{G_{3/4}} \subset (R^n - F)$$

We continue this process,
∀ $t \in D$ is the set of all dyadic rationals in [0, 1], a Borel open set G_t satisfying the following two conditions,

i) E ⊂ G_t ⊂ $\overline{G_t}$ ⊂ (R^n - F), ∀ t ∈ D
ii) if r, s ∈ D with r < s then $\overline{G_r}$ ⊂ G_s

Define a measurable function f: (R^n, τ, Σ, μ) ⟶ [0, 1] ⊂ (R, τ, Σ, μ) as follows:

$$f(x) = inf\{\ t \in D:\ x \in G_t\},\ \text{if } x \in (R^n - F)$$
$$= 1,\ \text{if } x \in F$$

Therefore, f maps (R^n, τ, Σ, μ) into [0, 1].

Let x ∈ E
implies, x ∈ (R^n - F)
and $f(x) = inf\{$ t ∈ D: $x \in G_t\}$
$= inf\{$ t: t ∈ D$\}$
= 0

Therefore, $f(t)$ = {0}

Also, by definition of f, $f(F)$= {1}

Therefore, we need to show that f is measurable, that is, f : (R^n, τ, Σ, μ) ⟶ [0, 1] is measurable since, (b, 1] and [0, a) are Borel open subsets in the measure subspace [0, 1] of (R^n,τ, Σ, μ) are Lebesgue measurable. that is,

$$\mathcal{L}\ (b, 1] = \text{Length of } (b, 1] \subset (R^n, \tau, \Sigma, \mu)$$
$$= 1 - b\ \text{(for any arbitrary b)}$$

and

$$\mathcal{L}\,[0, a) = \text{Length of } [0, a) \subset (R^n, \tau, \Sigma, \mu)$$
$$= a \text{ (for any arbitrary a)}$$

To show that f is measurable we need to show f^{-1}((b, 1]) and f^{-1} ([0, a)) are Borel open subsets in (R^n, τ, Σ, μ), for all 0 < a, b < 1.

We prove that

$$f^{-1}\,([0, a)) = \cup\{\, G_t : t < a\} \quad (1)$$

$$f^{-1}\,((b\ 1]) = \cup\{R^n - \overline{G_t} : t > b\} \quad (2)$$

Clearly each is arbitrary union of Borel open subset.
proof of (1): Let x ∈ f^{-1} ([0, a))
implies, $f(x)$ ∈ [0, a) and $0 \leq f(x) < a$

Now D is dense in [0, 1] implies, $\overline{D}$ = [0, 1] and $f(x) \in \overline{D}$ this implies, each Borel open set containing $f(x)$ meets D, that is, D ∩ E ≠ ∅ that is, [$f(x)$, a) ∩ D ≠ ∅ ∃ t_x in D such that $t_x \in$ [f(x), a) that is, $f(x) = inf\{\text{ t : } x \in G_t\} \leq t_x < a$

Suppose, $inf\{\text{ t : } x \in G_t\}$ = r then $x \in G_r$

Also, r ≤ $t_x \Rightarrow G_r \subset \overline{G_r} \subset G_{t_x}$ implies, $x \in G_{t_x}$

Therefore, $x \in G_{t_x}$, where t_x < a

Therefore, $x \in \cup\{G_t : t < a\}$

Hence, f^{-1} ([0, a)) ⊂ $\cup\{G_t : t < a\}$... ... ... (i)

On the other hand,

Let $y \in \cup\{G_t : t < a\}$
then ∃ t_y ∈ D such that y ∈ G_{t_y} and t_y < a

Therefore, $f(y) = inf\{\text{t : } y \in G_t\} \leq t_y < a$
implies, $0 \leq f(y) \leq t_y < a$
and $0 \leq f(y) < a$, $f(y)$ ∈ [0,a)
implies, $y \in f^{-1}$ ([0, a))

Therefore, $\cup\{G_t : t < a\} \subset f^{-1}$ ([0, a)) ... ... (ii)

Therefore from (i) and (ii), equality (1) follows,

Now, we prove (2):

Let $x \in f^{-1}$ ((b, 1]) implies, $f(x)$ ∈ (b, 1] and b $< f(x) \leq$ a

Now D is dense in [0, 1] implies, $\overline{D}$ = [0, 1] and $f(x) \in \overline{D}$

Therefore each Borel subset containing f(x) meets D

Hence, ∃ t_1, t_2 ∈ D such that b < $t_1 < t_2 < f(x)$

Therefore, $f(x) = \cup\{\text{ t : } x \in G_t\} > t_2$

suppose, $inf\{\text{ t : } x \in G_t\} = r$ then $r > t_2$

Therefore, $G_{t_2} \subset G_r$

Therefore, $x \notin G_{t_2}$, also $t_1 < t_2$ and $G_{t_1} \subset G_{t_2}$ implies, $x \notin \overline{G_{t_1}}$

Therefore, $x \in R^n$ -$\overline{G_{t_1}}$, where t_1> b implies, $x \in \cup$ {R^n -$\overline{G_t}$: t > b}

Therefore, f^{-1} ((b, 1]) ⊂ ∪ {R^n -$\overline{G_t}$: t > b}... ... ... (iii)

On the other hand,

Let y ∈ ∪ {R^n -$\overline{G_t}$: t > b} then ∃, t_y ∈ D such that y ∈ R^n -$\overline{G_{t_y}}$ and t_y > b that is, $y \notin \overline{G_{t_y}}$

But ∀ $t < t_y$ we have $G_t \subset G_{t_y} \subset \overline{G_{t_y}}$

Therefore, $y \notin \overline{G_{t_y}}$, ∀ $t < t_y$

Therefore, $f(y) = inf$ { t : $y \in G_t$}≥ t_y < b that is, b < $t_y \leq f(y) \leq 1$

Therefore, $f(y) \in$ [0,a) implies, $y \in f^{-1}$ (b, 1]

Therefore, ∪ {R^n -$\overline{G_t}$: t > b}⊂ f^{-1} (b, 1] ... ... (iv)

From (iii) and (iv) equality (2) holds.

The right hand side of (1) and (2) being union of Borel open subsets in R^n are Borel open in R^n.

Therefore, f^{-1} ([0, a)) and f^{-1} ((b, 1]) are Borel open subsets in R^n are Lebesgue measurable.

Therefore f is measurable.

Hence ∃ a measurable function f: (R^n, τ, Σ, μ) ⟶ [0,1] such that $f(A)$ = {0} and $f(B)$ = {1}.

Conversely, suppose for each pair of disjoint Borel closed sets $E, F \in R^n$ ∃ a measurable function f : (R^n, τ, Σ, μ) ⟶ [0, 1] ⊂ (R, τ, Σ, μ) such that $f(E)$ = {0} and $f(F)$ = {1}.

To prove: (R^n, τ, Σ, μ) is e-normal.

proof follows from theorem 3.7 [5].

Using Theorem 3.8 [5], we show that the measurable normal property on (R^n, τ, Σ, μ) is invariant under measurable homeomorphism and measure invariant function ϕ.

Theorem 3.2: A measure manifold $(M,\ \tau_1,\ \Sigma_1, \mu_1)$ is measurable normal if and only if for each pair of disjoint Borel closed sets E and F carrying additional information ({$f_n \circ \phi$}, $f \circ \phi$), if ∃ a measurable function $f \circ \phi$: $(M, \tau_1, \Sigma_1, \mu_1)$ ⟶ [0,1] ⊂ (R, τ, Σ, μ) such that $f \circ \phi(E)$ = 0 and $f \circ \phi(F)$ = 1.

Proof: Let [0, a) and (b, 1] are Borel open subsets in the measure subspace [0, 1].

Let $(U_1, \phi_1) = f^{-1}$ ([0, a)) and $(U_2, \phi_2) = f^{-1}$ ((b, 1]).

Since ($f \circ \phi$) is measurable, (U_1, ϕ_1) and (U_2, ϕ_2) are measure charts in $(M, \tau_1, \Sigma_1, \mu_1)$.

Therefore all measure charts in $(M, \tau_1, \Sigma_1, \mu_1)$ being measurable homeomorphic to (R^n,τ,Σ,μ) are Lebesgue measurable.

Hence (U_1, ϕ_1) =$(f \circ \phi)^{-1}$ [0, 1/2) and (U_2, ϕ_2) =$(f \circ \phi)^{-1}$(1/2,1] are Lebesgue measurable in $(M, \tau_1, \Sigma_1, \mu_1)$.

According to the above Lemma and theorem 3.8[5] it follows that,
$(M, \tau_1, \Sigma_1, \mu_1)$ is measurable normal if and only if for each pair of disjoint Borel closed sets E and F carrying additional information ({$f_n \circ \phi$}, $f \circ \phi$) ∃ a measurable function $f \circ \phi$: $(M, \tau_1, \Sigma_1, \mu_1)$ ⟶ [0,1] ⊂ (R,τ,Σ,μ) such that $f \circ \phi(E)$ = 0 and $f \circ \phi(F)$ = 1.

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