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U-COVERING SETS AND U-COVERING POLYNOMIALS OF CHAINS

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ABSTRACT

Let P be a finite poset. For a subset A of P, the upper cover set of A is defined as $U(A) = \{x \in P | x$ covers an $a \in A\}$. The upper closed neighbours of A is defined as $U[A]=U(A) \cup A$ and A is called an U – covering set of P if $U[A] = P$. The U – covering number $\vee(P)$ is the minimum cardinality of a U-covering set. Let U_n^i be the family of all U-covering sets of a chain P_n with cardinality i. Similarly we can define L – covering and N-covering sets of P_n with cordinality i. $\mathcal{u}(P_n,i) = |U_n^i|$, $\ell(P_n, i) = |L_n^i|$, $\mathcal{n}(P_n, i) = |N_n^i|$. In this paper, we construct U_n^i, and obtain a recursive formula for $U(P_n,i)$. Using this recursive formula we construct the polynomial $U(P_n,x) = \sum_{i=\lceil n/2 \rceil}^{n} \mathcal{u}(P_n,i)x^i$ called U-covering polynomial of P_n .

Keywords: *Poset, U-Covering set, U-Covering Polynomial.*

1. INTRODUCTION

A poset P is finite if it has finite number of elements. Let P be a finite poset. The open upper cover set of A is the set $U(A) = \{x \in P | x$ covers an $a \in A\}$. The closed upper cover set of A is the set $U[A] = U(A) \cup A$. We denote $U(\{x\})$ as $U(x)$. A set $A \subseteq P$ is a U-covering set of P if $U[A] = P$. The U-covering number $\vee(P)$ is the minimum cardinality of a U-covering set of P. A poset P is a chain if every pair of elements is comparable. Let P_n be the n element chain $x_1 < x_2 < \dots < x_n$. Let U_n^i be the family of U-covering sets of P_n with cardinality i and let $\mathcal{u}(P_n,i)= |U_n^i|$. The polynomial $U(P_n, x) = \sum_{i=\vee(P_n)}^{n} \mathcal{u}(P_n, i)x^i$ is called the U-covering polynomial of P_n.

2. U-COVERING SETS OF CHAINS

In this section we construct the family of U-covering sets of chains by a recursive method. We use $\lceil x \rceil$, for the smallest integer greater than or equal to x. Let U_n^i be the family of U-covering sets of P_n with cardinality i. The following lemma follows from observation.

Lemma 2.1: $\vee(P_n) = \lceil \frac{n}{2} \rceil$.

By the definition of U-covering set and by lemma 2.1, we have the following lemma

Lemma 2.2: $U_j^i = \varphi$ if and only if $i > j$ or $i < \lceil \frac{j}{2} \rceil$.

A chain connecting a and b where a < b is a simple chain if every element other than a and b in the chain has exactly one upper cover and lower cover.

The following lemma follows from observation.

Lemma 2.3: If a poset P contains a simple chain of length 2k-1, then every U-covering set of P must contain atleast k elements of the chain.

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To find a U-covering set of P_n with cardinality i, we do not need to consider U-covering sets of P_{n-3} with cardinality i-1. We show this in lemma 2.4. So, we only need to consider U_{n-1}^{i-1} and U_{n-2}^{i-1}.

Lemma 2.4: If $D \in U_{n-3}^{i-1}$ and if there exist $x \in P_n$ such that $D \cup \{x\} \in U_n^i$ then $D \in U_{n-2}^{i-1}$.

Proof: Suppose that $D \notin U_{n-2}^{i-1}$. Since $D \in U_{n-3}^{i-1}$, D contains x_{n-4} or x_{n-3}. If $x_{n-3} \in D$, then $D \in U_{n-2}^{i-1}$, a contradiction.

Hence $x_{n-4} \in D$. But in this case, $D \cup \{x\} \notin U_n^i$ for any $x \in P_n$, a contradiction.

Lemma 2.5:

(i) If $U_{n-1}^{i-1} = U_{n-3}^{i-1} = \varphi$ then $U_{n-2}^{i-1} = \varphi$.
(ii) If $U_{n-1}^{i-1} \neq \varphi$ and $U_{n-3}^{i-1} \neq \varphi$ then $U_{n-2}^{i-1} \neq \varphi$.
(iii) If $U_{n-1}^{i-1} = U_{n-2}^{i-1} = \varphi$ then $U_n^i = \varphi$.

Proof:

(i) Since $U_{n-1}^{i-1} = U_{n-3}^{i-1} = \varphi$ by lemma 2.2, i-1 > n-1 or $i-1 < \lceil \frac{(n-3)}{2} \rceil$.
$\therefore$ i-1 > n-2 or $i-1 < \lceil \frac{(n-2)}{2} \rceil$ and hence $U_{n-2}^{i-1} = \varphi$

(ii) Suppose that $U_{n-2}^{i-1} = \varphi$, then by lemma 2.2 i-1 > n-2 then $i-1 < \lceil \frac{(n-2)}{2} \rceil$.
If i-1 > n-2 or i-1 > n-3 and hence $U_{n-3}^{i-1} = \varphi$, a contradiction.
Hence $i-1 < \lceil \frac{(n-2)}{2} \rceil < \lceil \frac{(n-1)}{2} \rceil$ and hence $U_{n-1}^{i-1} = \varphi$, a contradiction.

(iii) Suppose that $U_n^i \neq \varphi$. Let $D \in U_n^i$. Then x_n or x_{n-1} is in D. If $x_n \in D$, then by lemma 2.3, atleast one of x_{n-1} or x_{n-2} is in D. If $x_{n-1} \in D$ or $x_{n-2} \in D$ then $D-\{x_n\} \in U_{n-1}^{i-1}$, a contradiction. If $x_{n-1} \in D$, then by lemma 2.3 atleast one of x_{n-2} or $x_{n-3} \in D$. If $x_{n-2} \in D$ or $x_{n-3} \in D$ then $D-\{x_{n-1}\} \in U_{n-2}^{i-1}$, a contradiction.

Lemma 2.6: If $U_n^i \neq \varphi$, then

(i) $U_{n-1}^{i-1} = \varphi$ and $U_{n-2}^{i-1} \neq \varphi$ if and only if n=2k and i=k for some $k \in \mathbb{N}$.
(ii) $U_{n-1}^{i-1} \neq \varphi$ and $U_{n-2}^{i-1} = \varphi$ if and only if i=n.
(iii) $U_{n-1}^{i-1} \neq \varphi$, and $U_{n-2}^{i-1} \neq \varphi$ if and only if $\lceil \frac{(n-1)}{2} \rceil + 1 \leq i \leq n-1$.

Proof:

(i) ($\Longrightarrow$) since $U_{n-1}^{i-1} \neq \varphi$, by lemma 2.2, i-1>n-1 or $i-1 < \lceil \frac{(n-1)}{2} \rceil$. If i-1 > n-1, then i>n and hence by lemma 2.2 $U_n^i = \varphi$, a contradiction. Therefore, $i-1 < \lceil \frac{(n-1)}{2} \rceil$ and since $U_n^i \neq \varphi$ $\lceil \frac{n}{2} \rceil \leq i < \lceil \frac{(n-1)}{2} \rceil + 1$. This gives us n=2k and i=k for some $k \in \mathbb{N}$.
($\Longleftarrow$) If n=2k and i=k for some $k \in \mathbb{N}$, then $i < \lceil \frac{(n-1)}{2} \rceil + 1$ and hence $i-1 < \lceil \frac{(n-1)}{2} \rceil$. Therefore by lemma 2.2, $U_{n-1}^{i-1} = \varphi$

(ii) ($\Longrightarrow$) since $U_{n-2}^{i-1} = \varphi$, by lemma 2.2, i-1> n-2 or $i-1 < \lceil \frac{(n-2)}{2} \rceil$. If $i-1 < \lceil \frac{(n-2)}{2} \rceil$ then $i-1 < \lceil \frac{(n-1)}{2} \rceil$ and hence $U_{n-1}^{i-1} = \varphi$, a contradiction. Therefore, i-1 > n-2 and so i>n-1. Also, since $U_n^i \neq \varphi$, $i \leq n$ and hence i = n.
($\Longleftarrow$) If i=n, then by lemma 2.2, $U_{n-1}^{i-1} \neq \varphi$, and $U_{n-2}^{i-1} = \varphi$

(iii) ($\Longrightarrow$) since $U_{n-1}^{i-1} \neq \varphi$ and $U_{n-2}^{i-1} \neq \varphi$, $\lceil \frac{(n-1)}{2} \rceil \leq i-1 \leq n-2$ and hence $\lceil \frac{(n-1)}{2} \rceil + 1 \leq i \leq n-1$.
($\Longleftarrow$) If $\lceil \frac{(n-1)}{2} \rceil + 1 \leq i \leq n-1$, then the result follows from lemma 2.2

Theorem 2.7: For every $n \geq 3$ and $i \geq \lceil \frac{n}{2} \rceil$

(i) If $U_{n-1}^{i-1} = \varphi$ and $U_{n-2}^{i-1} \neq \varphi$, then $U_n^i = \{\{x_1, x_3, x_5, \ldots, x_{n-1}\}\}$
(ii) If $U_{n-1}^{i-1} \neq \varphi$ and $U_{n-2}^{i-1} = \varphi$, then $U_n^i = \{\{x_1, x_2, x_3, \ldots, x_n\}\}$
(iii) If $U_{n-1}^{i-1} \neq \varphi$ and $U_{n-2}^{i-1} \neq \varphi$, then
$U_n^i = \{\{x_n\} \cup X | X \in U_{n-1}^{i-1}\} \cup \{\{x_{n-1}\} \cup X | X \in U_{n-2}^{i-1} \setminus U_{n-1}^{i-1}\} \cup \{\{x_{n-1}\} \cup X | X \in U_{n-2}^{i-1} \cap U_{n-1}^{i-1}\}$

Proof:

(i) $U_{n-1}^{i-1} = \varphi$ and $U_{n-2}^{i-1} \neq \varphi$. So, by lemma 2.6 (i), n=2k and i=k for some $k \in N$.
Therefore, $U_n^i = U_n^{\frac{n}{2}} = \{\{x_1, x_3, x_5, \ldots, x_{n-3}, x_{n-1}\}\}$

(ii) $U_{n-1}^{i-1} \neq \varphi$ and $U_{n-2}^{i-1} = \varphi$. So, by lemma 2.6 (ii), i=n.
Therefore, $U_n^i = U_n^n = \{\{x_1, x_2, x_3, \ldots, x_{n-1}, x_n\}\}$

(iii) $U_{n-1}^{i-1} \neq \varphi$ and $U_{n-2}^{i-1} \neq \varphi$. Let $X_1 \in U_{n-1}^{i-1}$. Then $x_{n-2} \in X_1$ or $x_{n-1} \in X_1$. In both cases, $X_1 \cup \{x_n\} \in U_n^i$.
Let $X_2 \in U_{n-2}^{i-1} \setminus U_{n-1}^{i-1}$. Then $X_2 \in U_{n-2}^{i-1}$ but $X_2 \notin U_{n-1}^{i-1}$. $X_2 \in U_{n-1}^{i-1}$ implies that x_{n-2} or x_{n-3} is in X_2.
Since $X_2 \notin U_{n-1}^{i-1}$, $x_{n-2} \notin X_2$ and hence $x_{n-3} \in X_2$. Therefore, $\{x_{n-1}\} \cup X_2 \in U_n^i$. Let $X_3 \in U_{n-2}^{i-1} \cap U_{n-1}^{i-1}$.

Then $X_3 \in \cup_{n-2}^{i-1}$ and $X_3 \in \cup_{n-1}^{i-1}$. $X_3 \in \cup_{n-2}^{i-1}$ implies that $x_{n-3} \in X_3$ or $x_{n-2} \in X_3$.
Since $X_3 \in \cup_{n-1}^{i-1}$, $x_{n-2} \in X_3$. Therefore, $\{x_{n-1}\} \cup X_3 \in \cup_n^i$. Hence, we have

$$\{\{x_n\} \cup X | X \in \cup_{n-1}^{i-1}\} \cup \{\{x_{n-1}\} \cup X | X \in \cup_{n-2}^{i-1} \setminus \cup_{n-1}^{i-1}\} \cup \{\{x_{n-1}\} \cup X | X \in \cup_{n-2}^{i-1} \cap \cup_{n-1}^{i-1}\} \subseteq \cup_n^i \quad (1)$$

Conversely, let $Y \in \cup_n^i$. Then $x_n \in Y$ or $x_{n-1} \in Y$. If $x_n \in Y$, then by lemma 2.3, atleast one of x_{n-1} or $x_{n-2} \in Y$.

Therefore, $Y = X \cup \{x_n\}$ for some $X \in \cup_{n-1}^{i-1}$. If $x_{n-1} \in Y$ and $x_n \notin Y$, then By lemma 2.3, atleast one of x_{n-2} or $x_{n-3} \in Y$.

If $x_{n-2} \notin Y$ and $x_{n-3} \in Y$ then $Y = X \cup \{x_{n-1}\}$ for some $X \in \cup_{n-2}^{i-1} \mid \cup_{n-1}^{i-1}$. If $x_{n-2} \in Y$, then $Y = X \cup \{x_{n-1}\}$ where $X \in \cup_{n-2}^{i-1} \cap \cup_{n-2}^{i-1}$.

$$\text{Therefore } \cup_n^i \subseteq \{\{x_n\} \cup X | X \in \cup_{n-1}^{i-1}\} \cup \{\{x_{n-1}\} \cup X | X \in \cup_{n-2}^{i-1} \setminus \cup_{n-1}^{i-1}\} \cup \{\{x_{n-1}\} \cup X | X \in \cup_{n-2}^{i-1} \cap \cup_{n-1}^{i-1}\} \quad (2)$$

From (1) and (2), we get (iii).

Table-1: $\mathcal{u}(P_n,j)$ the number of U-Covering sets of P_n with cardinality j.

j	1	2	3	4	5	6	7	8	9	10
n										
1	1									
2	1	1								
3	0	2	1							
4	0	1	3	1						
5	0	0	3	4	1					
6	0	0	1	6	5	1				
7	0	0	0	4	10	6	1			
8	0	0	0	1	10	15	7	1		
9	0	0	0	0	5	20	21	8	1	
10	0	0	0	0	1	15	35	28	9	1

3. U-COVERING POLYNOMIAL OF A CHAIN

Let $U(P_n, x) = \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \mathcal{u}(P_n, i)\, x^i$ be the U-covering polynomial of a chain P_n. In this section we study this polynomial.

Theorem 3.1:
(i) If $\cup_n^i$ is the family of U-covering sets with cardinality I of P_n, then $|\cup_n^i| = |\cup_{n-1}^{i-1}| + |\cup_{n-2}^{i-1}|$
(ii) For every $n \geq 3$, $U(P_n, x) = x\,[U(P_{n-1}, x) + U(P_{n-2}, x)]$ with initial values $U(P_1,x)=x$ and $U(P_2, x)=x^2+x$.

Proof:
(i) It follows from Theorem 2.7
(ii) It follows from part (i) and the definition of the U-Covering Polynomial.

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