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ON DYNAMICAL SYSTEM AND SEMIGROUPS INDUCED BY COMPOSITE CONVOLUTION OPERATORS

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ABSTRACT

In this paper, we analyse Composite Convolution operators which are obtained by composing convolution operators with composition operators. We calculate the norm of composite convolution operators. The norm of trace of composite convolution operators has also been explored. In this paper, an attempt has been made to investigate semigroups of one-parameter family and two-parameter family of composite convolution operators. A dynamical system induced by composite convolution operator is also obtained.

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1. INTRODUCTION

Let (X, Ω, μ) be a σ-finite measure space. For $1\leq p < \infty$ and for each $f \in L^p(\mu)$, there exists a unique $\phi^{-1}(\Omega)$ measurable function E(f) such that

$$\int gf\, d\mu = \int g\, E(f) d\mu$$

for every $\phi^{-1}(\Omega)$ measurable function g for which left integral exists. The function E(f) is called conditional expectation of f with respect to the sub - algebra $\phi^{-1}(\Omega)$. For more details about expectation operators, one can refer to Parthasarthy [8]. Let ϕ: X → X be a non-singular measurable transformation, (i.e., $\mu(E) = 0 \Rightarrow \mu\, \phi^{-1}(E) = 0$). Then a composition transformation, for $1\leq p < \infty$, C_ϕ: $L^p(\mu) \to L^p(\mu)$ is defined by

$$C_\phi = f\, o \phi$$

for every $f \in L^p(\mu)$. If C_ϕ is continuous, then it is a composition operator induced by ϕ. It is well-known result of Singh [11] that C_ϕ is a bounded operator if and only if $f_d = \frac{d\mu\phi^{-1}}{d\mu}$, the Radon-Nikodym derivative of the measure $\mu\,\phi^{-1}$ with respect to the measure μ is essentially bounded. For more detail about composition operators, we refer to Singh and Manhas [11].

Given f, g $\in L^2(\mathbb{R})$, then convolution of f and g, $f * g$ is defined by

$$f* g(x) = \int g(x-y) f(y)\, d(y),$$

where g is fixed, k(x,y) = g(x-y) is a convolution kernel, and the integral operator defined by

$$I_k\, f(x) = \int k(x-y) f(y) d\mu(y)$$

is known as Convolution operator. The composite convolution operator is obtained by composing convolution operator I_k and a composition operator C_ϕ. Suppose ϕ: X → X is a measurable transformation, then

$$I_{k,\phi}\, f(x) = \int k(x-y) f\big(\phi(y)\big) d\mu(y) = \int k_\phi\, (x-y) f(y) d\mu(y)$$

is known as composite convolution operator (CCO) induced pair (k, ϕ) for every $f \in L^p(\mu)$,
where $k_\phi(x-y) = E\big(f_d\,(y) k(x-y) \phi^{-}(y)\big)$.

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The integral operators, in particular convolution operators have already been studied extensively over the last few decades. For more detail about composition operators, integral operators, convolution operators, composite integral operators and composite convolution operators we refer to Singh and Manhas [11], Halmos and Sunder ([5],[6]), Stepanov ([9], [10]), Gupta and Komal [1] and Gupta ([2], [3], [4]). Whitley [12] established the Lyubic's [7] conjecture and generalized it to Volterra composition operators on $L^p\,[0,1]$. This paper broadens the approach that was taken into account in the papers of Gupta ([2], [3]).

Here, I recall some basic notion in operator theory. Let H be a Hilbert space and B(H) be the algebra of all bounded linear operators acting on H. Let $L^2(\mu)$ consists of all measurable functions f: X → ℝ (or (C) such that $(\int |f(x)|^2 \, d\mu(x))^{1/2} < \infty$. The space $L^2(X, \Omega, \mu)$ is a Banach space under the norm defined by $\| f \| = (\int |f(x)|^2 \, d\mu(x))^{1/2}$. Also, $L^2(\mu)$ the space of square-integrable functions of complex numbers is a Hilbert space.

Let G be a group with identity e and X be a non-empty set. Let $\Pi : G\times X \to X$ be a mapping such that

$$\Pi(e, x) = x;\ \Pi(st,x) = \Pi(s, \Pi(t,x))$$

for all $x \in X$ and $s, t \in G$. Then Π is called an action of G. If G is a topological group, X is topological space and Π is continuous, then (G, X, Π) is known as a dynamical system.

The study of Composite Convolution operators has been initiated in the work of Gupta [2]. The present paper illustrates some properties of composite convolution operators. In this paper main focus is to obtain semigroups of composite convolution operators. We also obtain dynamical system induced by composite convolution operators (CCO).

2. BOUNDED AND HILBERT-SCHMIDT COMPOSITE CONVOLUTION OPERATORS (CCO)

In this section, it is shown that composite convolution operator $I_{k,\phi}$ is bounded and linear operator from $L^p(\mu)$ into $L^q((\mu)$. It has also been proved by Gupta [2] that CCO $I_{k,\phi}$ is bounded operator if and only if $k_\phi \in L^2(\mu \times \mu)$. We also obtain the norm of CCO and norm of trace of CCO.

Theorem 2.1: Let $\phi : X \to X$ be a non-singular measurable transformation. For $1\le p, q < \infty$, if $I_{k,\phi}$ maps $L^p(\mu)$ into $L^{(q)}(\mu)$, then $I_{k,\phi}$ is a bounded operator. Moreover, $I_{k,\phi}$ is a linear operator.

Proof: Let $\{f_n\}$ be a sequence in $L^p(\mu)$ such that $f_n \to f$ and $I_{k,\phi}\, f_n \to g$ for some $f \in L^p(\mu)$ and $g \in L^q(\mu)$ for μ almost all x∈ X. Then, we select a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k}(x) \to f(x)$ for μ-almost all x∈ X. Again, since $I_{k,\phi}\, f_n \to g$, we select a dominated subsequence $\{f_{n_{k_*}}\}$ of $\{f_{n_k}\}$ such that $I_{k,\phi}\, f_{n_{k_*}}(x) \to g(x)$.
That is,

$$\int k_\phi\,(x-y) f_{n_{k_*}}(y) d\mu(y) \ \to g(x).$$

For μ -almost all x ∈ X. Also, we have $|f_{n_{k_*}}| \le h$ for some $h \in L^q(\mu)$.

Again, we have,

$$k_\phi(x-y) f_{n_k}(y) \ \to\ k_\phi\, f(y) f(y)$$

for μ almost all x ∈ X, and

$$|\, k_\phi(x-y) f_{n_{k_*}} \,| \ \le \left|k_\phi(x-y) h(y)\right|.$$

But dominated subsequence $k_\phi\, f_{n_{k_*}}$ converges to $k_\phi\, f$ a. e.

Hence, by Lebesgue Dominated Convergence theorem, we conclude that

$$\int k_\phi(x-y) f_{n_{k_*}}(y) d\mu(y) \to \int k_\phi\,(x-y) f(y) d\mu(y)$$

for μ - almost all x ∈ X. Thus, we have

$$I_{k,\phi} f(x) = \int k_\phi(x-y) f(y) d\mu(y) = \ g(x),$$

which proves that graph of $I_{k,\phi}$ is closed. Hence, $I_{k,\phi}$ is a bounded operator by using closed graph theorem.

Again, if $f, g \in L^p(\mu)$, we have

$$\begin{aligned} I_{k,\phi}\,(\alpha f + \beta\, g)(x) &= \int k_\phi(x-y)\,(\alpha f + \beta g)(y) d\mu(y) \\ &= \alpha \int k_\phi\ (x-y) f(y)\, d\mu(y) + \beta \int k_\phi\ (x-y) g(y) d\mu(y) \\ &= \alpha\, I_{k,\phi}\, f\,(x) + \ \beta\, I_{k,\phi}\, g(x) \end{aligned}$$

Thus, we have

$$I_{k,\phi}(\ + \beta\, g) = \alpha\, I_{k,\phi}\, f + \ \beta\, I_{k,\phi}\, g$$

Hence, the result follows.

In the next result, the condition for CCO to be Hilbert-Schmidt is evaluated.

Theorem 2.2: Let $k_\phi \in L^2(\mu \times \mu)$. Then $I_{k,\phi}$ is a Hilbert-Schmidt operator.

Proof: Firstly, we show that $I_{k,\phi} \in B(L^2(\mu)$. For $f \in L^2[0,1]$, we have

$$\begin{aligned}\| I_{k,\phi}\ f \|^2 &= \int_0^1 | I_{k,\phi}\ f(x)|^2 \, d\,\mu(x)\\ &= \int_0^1 | \int_0^1 k_\phi\ (x-y)\ f(y)\ d\,\mu(y)|^2\ d\mu(x)\\ &\leq \int_0^1 \left[\int_0^1 | k_\phi\ (x-y)\ |^2\ d\mu(y) \int_0^1 | f(y)|^2 d\mu(y)\right] d\,\mu(x)\\ &= \int_0^1 \left[\int_0^1 | k_\phi\ (x-y)\ |^2 d\mu(y)\ d\mu(x) \int_0^1 | f(y)|^2\ d\mu(y)\right]\\ &= C^2 \| f \|_2^2\\ &< \infty,\end{aligned}$$

where $C^2 = \int_0^1 \left[\int_0^1 | k_\phi\ (x-y)\ |^2 d\mu(y)\ d\mu(x)\right.$

Thus, $I_{k,\phi} \in B(L^2(\mu)$.

Again,

$$\begin{aligned}\| k_\phi \|_2^2 \| &= \int_0^1 \left[\int_0^1 | k_\phi\ (x-y)\ |^2 d\mu(y)\ d\mu(x)\right.\\ &< \infty,.\end{aligned}$$

Indeed,

$$\begin{aligned}||I_{k,\phi}\ f\ || &= \int_0^1 [\ |\ \int_0^1 k_\phi(x-y) f(y) d\mu(y)|^2\ d\mu(x)]^{1/2}\\ &\leq ||k_\phi\ ||_2^2\ \int_0^1 |f(y)|^2 d\mu(y)\\ &= ||k_\phi\ ||_2^2\ \ ||f\ ||_2^2\end{aligned}$$

This implies that, $\| I_{k,\phi} \| \leq \| k_\phi \|_2$.

Therefore, $I_{k,\phi}$ is a Hilbert-Schmidt operator.

In the next result, we have estimated the norm of CCO minus identity operator I. We consider $f \in L^1[0,1]$ such that $f = f^+ - f^-$ and $| f | = f^+ + f^-$. Suppose kernel $k_\phi(x-y) = \delta\ (x-y)$ and $\delta\ (x-y) = 1$, if x = y and $\delta\ (x-y) = 0$, if $x \neq y$.

Theorem 2.3: Let $I_{k,\phi} \in B(L^1[0,1])$. Then

$$\| I_{k,\phi} - I \|_1 \leq 2\|f\|_1,$$

where I is an identity operator.

Proof: Let $f \in L^1[0, 1]$.

$$\begin{aligned}\|I_{k,\phi}\ (f) - I(f)\ \|_1 &= \int | I_{k,\phi} f(x) - If(x)| d\mu(x)\\ &\leq \int_0^1 (\ | I_{k,\phi} f(x)|\ d\mu(x)\ +\ \int_0^1 | f(x)|\ )\ d\mu(x) \qquad (1)\end{aligned}$$

Also, we have

$$\begin{aligned}| I_{k,\phi}\ f\ | &= | I_{k,\phi}\,(\ f^+ -\ f^-\ )\ |\\ &\leq \left| I_{k,\phi}\ f^+\ \right| + | I_{k,\phi}\, f^-|\\ &= I_{k,\phi}\, f^+ +\ I_{k,\phi}\ f^-\\ &= I_{k,\phi}\,(f^+ +\ f^-\ )\\ &= I_{k,\phi}\, | f | \qquad (2)\end{aligned}$$

Now using equations (1) and (2), we have

$$\begin{aligned}||I_{k,\phi}\ f - I\ ||_1 &\leq (\int_0^1 I_{k,\phi}\, | f |(x)\ \ +\ \ |\ If(x)|\ )\ d\mu(x)\\ &= \int_0^1 (\ I_{k,\phi}\ [\ f^+(x) -\ f^-(x)] +\ \ (f^+(x) -\ f^-(x)\ )\ ) d\mu(x)\\ &= \int_0^1 (\int_0^1 k_\phi(x-y)\ (f^+ +\ f^-)(y) d\mu(y))\ d\mu(x)\ +\ \int_0^1 [f^+(x)\ +\ f^-(x)]\ d\mu(x)\\ &= 2\int_0^1 [\ f^+(x) +\ f^-(x)]\ d\mu(x),\end{aligned}$$

by using given condition, $k_\phi(x-y) = \delta\ (x-y)$, we conclude that

$$||I_{k,\phi}\ f - I\ ||_1 \ \leq\ 2\ ||\ f\ ||_1\ .$$

Hence, the result follows.

In this result, we calculate the norm of CCO on $L^2[0,1]$ space.

Theorem 2.4: Let $I_{k,\phi} \in B(L^2[0,1])$ and k_ϕ be kernel function on $[0,1]\times[0,1]$ with $k_\phi \in L^2(\mu \times \mu)$. Then $||I_{k,\phi}f||^2 < ||k_\phi||^2$.

Proof: Let $f \in L^2[0,1]$ be such that $f = f^+ - f^-$. Then, we have

$$\begin{aligned} ||I_{k,\phi}\ f\,||^2 &= |I_{k,\phi}\,(f^+ - f^-)||^2 \\ &= ||I_{k,\phi}\ f^+ - I_{k,\phi}f^-\,||^2 \\ &= ||I_{k,\phi}\ \ f^+||^2 + ||I_{k,\phi}f^-||^2 - 2I_{k,\phi}f^+I_{k,\phi}f^- \end{aligned}$$

since $\int f^+(x)\ f^-(x)dx = 0.$ Now, we have

$$\begin{aligned} ||I_{k,\phi}\,f\,||^2 &= ||I_{k,\phi}\ \ f^+(x)||^2 + ||\ I_{k,\phi}\ \ f^-(x)\,||^2 \\ &\le ||\,k_\phi\,||^2[\,||f^+||^2 + ||\,f^-\,||^2] \\ &< |k_\phi\,||^2\,. \end{aligned}$$

Here, we have make use of Theorem 2.3. Hence, the result is proved.

In the next result, we explore the norm of trace of CCO. We define the trace of K, denoted by tr(K) = $\int_0^1 k(x,x)d\mu(x)$.

In the proof of this theorem we use the result of Gupta [3], that is, product of two CCO operators is again a CCO.

Theorem 2.5: Let $K_{k,\phi}$ and $J_{j,\phi}$ be two composite convolution operators induced by kernel functions k_ϕ and j_ϕ respectively. Then

$$|tr(K_{k,\phi}\,J_{j,\phi})|^2 \le ||k_\phi\,||_2\,||j_\phi\,||_2.$$

Moreover,

$$||\,K_{k,\phi}\,J_{j,\phi}\,||_2^2 \le ||k_\phi||_2^2\,||j_\phi||_2^2\,.$$

Proof: For any $f \in L^2[0, 1]$, we have

$$\begin{aligned} (K_{k,\phi}\ \ J_{j,\phi}\ f)(x) &= K_{k,\phi}\,(J_{j,\phi}\,f(x)) \\ &= \int_0^1[\int_0^1 k_\phi\,(x-y)\,j_\phi\,(y-z)\,d\mu(y)]\,f(z)\,d\mu(z) \\ &= \int_0^1 t_\phi\ \,(x-z)f(z)d\mu(z). \end{aligned}$$

where t_ϕ is the kernel of $K_{k,\phi}\ \ J_{j,\phi}$ and $t_\phi\,(x-z) = \int_0^1 k_\phi(x-y)j_\phi\,(y-z)\,d\mu(y)$.

Then, we have

$$\begin{aligned} |\ tr\ (\,K_{k,\phi}\ \ J_{j,\phi}\,)|^2 &= |\int_0^1 t_\phi\ \,(x-z)f(z)d\mu(z)\ | \\ &= |\int_0^1 t_\phi\ \,(x-x)\ d\mu(x)\ | \\ &= |\int_0^1\int_0^1 k_\phi\ \,(x-y)j_\phi\ \,(y-x)\ d\mu(y)\ d\mu(x)| \\ &\le \int_0^1\ (\int_0^1|\,k_\phi\,(x-y)|^2d\mu(y)\,)^{1/2}\left(\int_0^1|\,j_\phi\ \,(y-x)|^2d\mu(y))^{1/2}\right)d\mu(x) \\ &= \|\,k_\phi||_2^2\ \ \|\,j_\phi\ \,||_2^2\,. \end{aligned}$$

Also, we have

$$\begin{aligned} \|\,K_{k,\phi}\ \ J_{j,\phi}\ \ f(x)\,||_2^2 &= \int_0^1|\ \int_0^1 t_\phi\,(x-z)f(z)\ d\mu(z)|^2\ d\mu(x) \\ &\le \int_0^1\ (\int_0^1|\,t_\phi(x-z)|^2\ d\mu(z)\ \int_0^1|\,f(z)\,|^2\ d\mu(z))\ d\mu(x) \\ &= \int_0^1\ (\int_0^1|\,t_\phi(x-z)|^2\ d\mu(z)d\mu(x)\ \int_0^1|\,f(z)\,|^2\ d\mu(z)) \\ &= \|\,t_\phi\,||_2^2\ \ ||\,f\,||_2^2\,, \end{aligned}$$

which proves that

$$||\,K_{k,\phi}\ \ J_{j,\phi}\,)\,||_2^2 \le \|\,k_\phi\,||_2^2\ \ \|\,j_\phi\ \,||_2^2\,.$$

3. SEMIGROUP OF COMPOSITE CONVOLUTION OPERATORS (CCO)

For each real, $t \ge 0$, a collection {T(t)} of bounded linear operators on Banach space X is said to be a semigroup of one-parameter if

$$T(0) = 1 \text{ and } T(t_1 + t_2) = T(t_1)\,T(t_2) \quad \forall\, t_1,\ t_2\ \ge 0.$$

A collection $A \subset B(X)$ is said to be two parameter semigroup of operators of

$$A = \{\,T(t,s): (t,s) \in [0,\infty)\ \times [0,\infty),$$

where T(t, s) is a bounded linear operator on X if the collection A satisfies

$$T(0, 0) = 1 \text{ and } T(t_1 + t_2, s_1 + s_2) = T(t_1, s_1)\, T(t_2, s_2).$$

Suppose $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, define translation map $\phi_t : \mathbb{N}_0 \longrightarrow \mathbb{N}_0$ as $\phi_t(x) = x + t$. In this section, we explore the conditions for CCO to be semigroup of one-parameter and two-parameter family. The kronecker delta function is defined as

$$\delta(x-y) = 1, \text{ if } x = y \text{ and } \delta(x-y) = 0, \text{ if } x \neq y.$$

Theorem 3.1: Let $I_{k,\phi_t} \in B(L^2[0,1])$. Then $I_{k,\phi_t} : t \in \mathbb{N}_0$ is a semigroup of one-parameter family if and only if $k_\phi(x-y) = \delta(x-y)$, where convolution kernel is equal to kroneker delta function.

Proof: Firstly, suppose that the condition is true.

For $f \in L^2[0,1]$, we have

$$I_{k,\phi_t} f(x) = \int k(x-y) f(\phi_t(y)) d\mu(y).$$

Then,

$$\begin{aligned} I_{k,\phi_0} f(x) &= \int_0^1 k(x-y)\, f(\phi_0(y)) d\mu(y) \\ &= \int_0^1 k(x-y)\, f(y) d\mu(y) \\ &= f(x) = If(x). \end{aligned}$$

Hence,

$$I_{k,\phi_0} = I.$$

Now,

$$\begin{aligned} I_{k,\phi_{s+t}} f(x) &= \int_0^1 k(x-y) f(\phi_{s+t}(y)) d\mu(y) \\ &= \int_0^1 k(x-y) f(y+s+t) d\mu(y) \\ &= f(s+t+x). \end{aligned}$$

Again,

$$\begin{aligned} I_{k,\phi_s}\ I_{k,\phi_t} f(x) &= \int_0^1 k(x-y)\, (I_{k,\phi_t} f)(\phi_s(y)) d\mu(y) \\ &= \int_0^1 k(x-x)\, (I_{k,\phi_t} f)(s+x)\, d\mu(x) \\ &= \int_0^1 I_{k,\phi_t}\ f(s+x) d\mu(x) \\ &= \int_0^1 k(s+x-y) f(\phi_t(y)) d\mu(y) \\ &= \int_0^1 k(s+x-y) f(t+y) d\mu(y) \\ &= f(s+t+x). \end{aligned}$$

Therefore, we have,

$$I_{k,\phi_{s+t}} = I_{k,\phi_s}\ I_{k,\phi_t}.$$

This shows that $I_{k,\phi_t} : t \in \mathbb{N}_0$ is a semigroup of one-parameter family.

Converse follows trivially, if we take f = e_n, where $e_n(m) = \delta(n-m)$.

Hence, the result is proved.

Theorem 3.2: Let $I_{k,\phi_t} \in B(L^2[0,1])$. Then $I_{k,\phi_t} : t \in \mathbb{N}_0 \times \mathbb{N}_0$ is a semigroup of two-parameter family if and only if $k_\phi(x-y) = \delta(x-y)$, where translation map $\phi_t : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$ defined as $\phi_t(a)$ = a +t, where t = (t_1, t_2), a = (a_1, a_2).

Proof: First, we suppose that the condition is true. That is, $k_\phi(x-y) = \delta(x-y)$.

For $f \in L^2[0,1]$, we have

$$I_{k,\phi_t} f(x) = \int_0^1 k(x-y) f(\phi_t(t))\, d\mu(y)$$

Then, we have

$$\begin{aligned} I_{k,\phi_{(0,0)}} f(x) &= \int_0^1 k(x-y) f(\phi_{(o,o)}(y))\, d\mu(y). \\ &= \int_0^1 k(x-y) f((0,0)+y)\, d\mu(y). \\ &= \int_0^1 k(x-y) f(y) d\mu(y). \\ &= f(x) = If(x), \end{aligned}$$

for $x = (x_1, x_2), y = (y_1, y_2)$.

Again, if $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{N}_0 \times \mathbb{N}_0$, then

$$\begin{aligned} I_{k,\phi_{(a+b)}} f(x) &= I_{k,\phi_{(a_1,a_2)+(b_1,b_2)}} f(x) \\ &= \int_0^1 k(x-y) f(\phi_{(a_1,a_2)+(b_1,b_2)}(y))\, d\mu(y). \\ &= \int_0^1 k(x-y) f(a_1 + b_1 + y_1,\ a_2 + b_2 + y_2)\, d\mu(y). \\ &= f(a_1 + b_1 + x_1,\ a_2 + b_2 + x_2) \end{aligned}$$

For $x = (x_1, x_2),\ y = (y_1, y_2)$, we have

$$\begin{aligned} I_{k,\phi_a} I_{k,\phi_b} f(x) &= \int_0^1 k(x-y)\, I_{k,\phi_{(b_1,b_2)}}\ f(\phi_{(a_1,a_2)}(y) d\mu(y) \\ &= \int_0^1 k(x-x)\, (I_{k,\phi_{(b_1,b_2)}}\ f((a_1, a_2) + x))\, d\mu(x) \\ &= (I_{k,\phi_{(b_1,b_2)}}\ f((a_1, a_2) + (x_1, x_2)) \\ &= (I_{k,\phi_{(b_1,b_2)}}\ f(a_1 + x_1,\ a_2 + x_2) \\ &= \int_0^1 k\,((a_1 + x_1,\ a_2 + x_2) - (y_1 + y_2))\, f(\phi_{(b_1,b_2)})(y)\, d\mu(y) \\ &= \int_0^1 k\,((a_1 + x_1,\ a_2 + x_2) - (y_1 + y_2))\, f((b_1, b_2) + (y_1 + y_2))\, d\mu(y) \\ &= f(a_1 + b_1 + x_1,\ a_2 + b_2 + x_2) \end{aligned}$$

This shows that $I_{k,\phi_t} : t \in \mathbb{N}_0 \times \mathbb{N}_0$ is a semigroup of two-parameter family.

Here, converse follows trivially.

4. A DYNAMICAL SYSTEM INDUCED BY COMPOSITE CONVOLUTION OPERATORS (CCO))

In this section, we obtain a dynamical system induced by CCO I_{k,ϕ_a} where we have consider translation mapping Π_a, $\forall a \in \mathbb{R}$, $\Pi_a : \mathbb{R} \to \mathbb{R}$ defined as $\Pi_a(x) = a + x$, $\forall x \in \mathbb{R}$.

Theorem 4.1: Suppose $1 \leq p, q < \infty$ and k is a convolution kernel such that $k(x-y) = \delta(x-y)$. If $\Pi : \mathbb{R} \times L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is a mapping defined by $\Pi(a, f) = I_{k,\Pi_a} f$, $\forall a \in \mathbb{R}$ and $f \in L^p(\mathbb{R})$. Then $(\mathbb{R}, L^p(\mathbb{R}), \Pi)$ is a dynamical system.

Proof: For $a \in \mathbb{R}$ and $f \in L^p(\mathbb{R})$, we have

$$\begin{aligned} \Pi(0, f) &= I_{k,\Pi_0} f \\ &= \int_0^1 k(x-y) f(\Pi_0(y))\, d(y) \\ &= \int_0^1 k(x-y) f(y)\, d(y) \\ &= f(x) = I f(x) \end{aligned}$$

Now, for $a, b \in \mathbb{R}$, we have

$$\begin{aligned} \Pi(a+b, f) &= I_{k,\Pi_{a+b}} f(x) \\ &= \int_0^1 k(x-y) f\big(\Pi_{a+b}(y)\big) d(y) \\ &= \int_0^1 k(x-y) f(a+b+y)\, d(y) \\ &= f(a+b+x) \end{aligned}$$

Also, we have

$$\begin{aligned} \Pi(a, \Pi(b, f)) &= I_{k,\Pi_a} \Pi(b, f) \\ &= \int_0^1 k(x-y)\, \Pi(b, f)\, \big(\Pi_a(y)\big) d(y) \\ &= \int_0^1 k(x-y)\, \Pi(b, f)(a+y)\, d(y) \\ &= \Pi(b, f)(a+x) \\ &= I_{k,\Pi_b} f(a+x) \\ &= \int_0^1 k(a+x-y)\ f\big(\Pi_b(y)\big) d(y) \\ &= \int_0^1 k(a+x-y) f(b+y)\, d(y) \\ &= f(a+b+x) \end{aligned}$$

Hence, we have

$$\Pi(a+b, f) = \Pi(a, \Pi(b, f)).$$

This shows that Π is a motion. Next, to show that $(\mathbb{R}, L^p(\mathbb{R}), \Pi)$ is a dynamical system, we have to show that Π is separately continuous. For, let $\{a_n\}$ be a sequence in $\mathbb{R}$ such that $a_n \to a$.

$$\| \Pi(a_n, f) - \Pi(a, f) \|_p = \lim_{a_n \to \infty} \| I_{k,\Pi_{a_n}} f - I_{k,\Pi_a} f \|_p$$
$$= \lim_{a_n \to \infty} \int (| I_{k,\Pi_{a_n}} f(x) - I_{k,\Pi_a} f(x)|^p dx)^{1/p}$$
$$= \lim_{a_n \to \infty} \int | \int k(x-y) [f(a_n + y) - f(a+y)] \, dy|^p \, dx)^{1/p}$$
$$\longrightarrow 0 \text{ as } n \to \infty.$$

Again, we have to prove the continuity of Π in the second co-ordinate, for $\{f_n\} \in L^p(\mathbb{R})$ such that $f_n \to f$, we have

$$\| \Pi(a, f_n) - \Pi(a, f) \|_p = \lim_{a_n \to \infty} \| I_{k,\Pi_a} f_n - I_{k,\Pi_a} f \|_p$$
$$= \lim_{a_n \to \infty} \int (| I_{k,\Pi_a} f_n(x) - I_{k,\Pi_a} f(x)|^p dx)^{1/p}$$
$$= \lim_{a_n \to \infty} \int | \int k(x-y) [f_n(a + y) - f(a+y)] \, dy|^p \, dx)^{1/p}$$
$$\longrightarrow 0 \text{ as } n \to \infty.$$

Thus, $(\mathbb{R}, L^p(\mathbb{R}), \Pi)$ is a dynamical system. Hence, the theorem is proved.

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