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COMPARISON OF LAPLACIAN CENTRALITY WITH SOME NODE CENTRALITY MEASURES

A. BHARALI*[1], D. BARUAH[2]

Department of Mathematics Dibrugarh University, Assam-786004, India.

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ABSTRACT

The centrality measure of nodes plays a vital role in network analysis. Laplacian Centrality of a node is the relative change in the Laplacian Energy of the network due to removal of the node. The object of the paper is twofold. First, we discuss the Laplacian Centrality and its graph theoretic perspective. Second, we present a comparative study of Laplacian Centrality with some standard node centrality measures applying on classic synthetic and real network data sets.

Keywords*: Laplacian Centrality, Node Centrality, Barabási-Albert Network, Watts-Strogatz Network, Dolphins Network, American Football Network, ANI (Airport Network of India).*

2010 Mathematics Subject Classification*: 05C82, 90B06, 91B74.*

1. INTRODUCTION

Node centrality measures are proposed and used to analyze Social Networks. But soon these measures are well adopted and find their utility in almost all the network structures. The centrality of nodes, or the identification of nodes which are more "central" than others, has been a key issue in network analysis. The findings of some important nodes with high centralities to characterize the properties on the networks have significant uses in analyzing structure and dynamics of the network. These include the synchronization transition, epidemic spreading, and transmission of information. Various centrality measures like degree centrality, closeness centrality, betweenness centrality [5], eigenvector centrality [1] have been proposed for unweighted networks. Several attempts were made to generalize degree, betweenness and closeness centrality measures to weighted networks. Degree centrality was extended to weighted networks by *Barrat et al.* [3] and defined as the sum of the weights attached to the edges connected to a node. Some extensions of the closeness and betweenness centrality measures were proposed by *Newman* [12] and *Brandes* [4].

Degree Centrality of a node is the number of edges incident upon the node. *Closeness Centrality* of a node i was defined as the inverse sum of shortest distances to all other nodes from the node i, which is based on the mean shortest path between i and all other nodes reachable from it. *Betweenness Centrality* assesses the degree to which a node lies on the shortest paths between pairs of other nodes. *Eigenvector Centrality* of a node captures its number of connections recursively i.e., it will not only measure the number of edges incident upon a node but measures the number of connections that the connected nodes to the node possess. *Laplacian Centrality* [14] was proposed as a measure of node centrality capturing the effect of removal of the node on Laplacian Energy.

An efficient network would have small characteristic path length, high clustering coefficient, which are the properties of small-world network (*Watts and Strogatz*, 1998) [13]. A small-world network is a graph in which most nodes are not neighbors of one another, but most nodes can be reached from every other by a small number of hops, attributing to its small characteristics path length. Another common property of many large real networks is that, the degree distribution follows a scale-free power-law distribution (*Barabási and Albert*, 1999) [2]. These networks are popularly known as scale-free (SC) networks. In this paper, we present a comparative study of Laplacian Centrality (LC) with some other standard node centrality measures applying on classic synthetic and real network data sets.

Corresponding Author: A. Bharali*[1],
Department of Mathematics Dibrugarh University, Assam-786004, India.

The rest of this paper is organized as follows. In the next section, we discuss Laplacian Centrality and its structural description. In section 3, we compare LC with some standard node centrality measures in synthetic network environment like Barabási-Albert Network (BA-network), Watts-Strogatz Network (WS-network) and in real network datasets like Dolphins Network, American Football Network and Airport Network of India. In section 4 we present a consensus study to evaluate the performance of various measures in case of the three real networks. Conclusions are made in section 5.

2. L APLACIAN CENTRALITY AND ITS GRAPH THEORETIC PERSPECTIVE

Laplacian energy is another version of the graph invariant, known as graph energy [7]. Laplacian energy can be used to measure the importance (centrality) of a node by measuring the relative change of Laplacian energy in the network caused by the removal of the node from the network [14]. Let us start the discussion with the definition of Laplacian energy of a network and LC of a node.

Laplacian Energy [10]: Let G = (V, E, W) be a weighted network with n nodes, and $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of the laplacian of G. Then the Laplacian Energy can be defined as follows:

$$E(G) = \sum_{i=1}^{n} {\mu_i}^2$$

Laplacian Centrality [14]: Let G = (V, E, W) is a network with n nodes $\{v_1, v_2, \ldots, v_n\}$. Let us also consider that G_i be the network obtained by deleting v_i from G. The LC, $LC(v_i, G)$ of a node v_i is defined as follows

$$LC(v_i, G) = \frac{E(G) - E(G_i)}{E(G)} = \frac{(\Delta E)_i}{E}$$

Since $(\Delta E)_i = E(G) - E(G_i)$ is always a non-negative quantity, because of the interlacing property [8] of the eigenvalues of Laplacian matrix. Clearly, $0 \le LC(v_i, G) \le 1, \forall\, i$.

The graph theoretic description of LC can be understood using 2-walks. Let G = (V, E, W) be a weighted network of n nodes $\{v_1, v_2, \ldots, v_n\}$. A walk of length k or k-walk is a sequence of (not necessary different) $k+1$ nodes such that for each pair of consecutive nodes there is an edge. A walk is closed if the starting and the end nodes are same. A 2-walk is a walk of length 2. The 2-walks containing a node can be divided into three categories: first, Closed 2-walks containing the node v; say, $CW(v)$ denotes the number of such walks, second, Non-closed 2-walks containing the node v as one of the end-nodes; say, $NCW^E(v)$ denotes the number of such walks, and finally, Non-closed 2-walks containing the node v as the middle point; and the number of such 2-walks be $NCW^M(v)$. Using these notations we have the following theorem due to Xingqin Qi., *et. al.* [14] that helps to characterize LC as a graph measure.

Theorem: Let G = (V, E, W) is a weighted network of n nodes $\{v_1, v_2, \ldots, v_n\}$. Let G_i be the network obtained by deleting the node v_i from G, then the drop of Laplacian energy with respect to v_i is

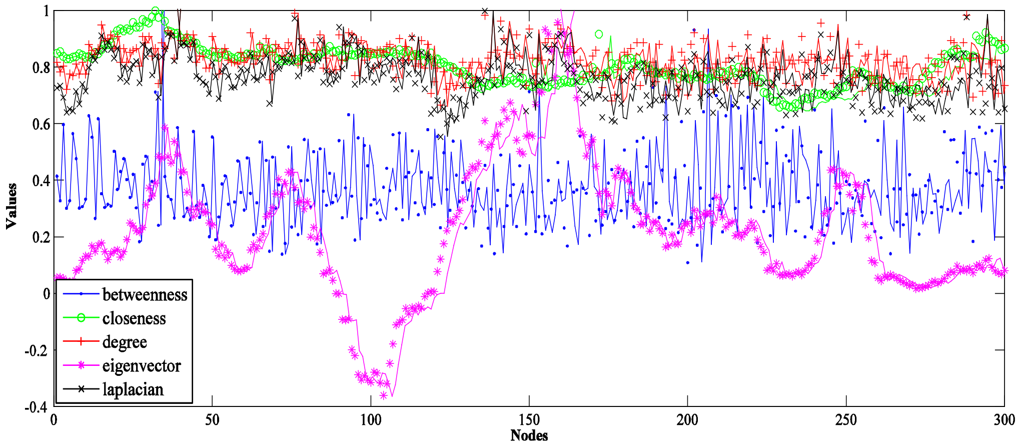
$$(\Delta E)_i = E(G) - E(G_i) = 4CW(v_i) + 2(NCW^E(v_i) + NCW^M(v_i)).$$

From this theorem we can observe that unlike Degree centrality, which considers only direct connections, LC of a node gathers information from the nodes which are reachable in two steps from the node. Also the weight on closed walks is 4 as compared to 2 on non-closed walks. This can be attributed to the fact that though the global environment may not be completely ignored but the local environment of a node is mainly responsible for the centrality value of the node, and the closed walks can capture this the most.

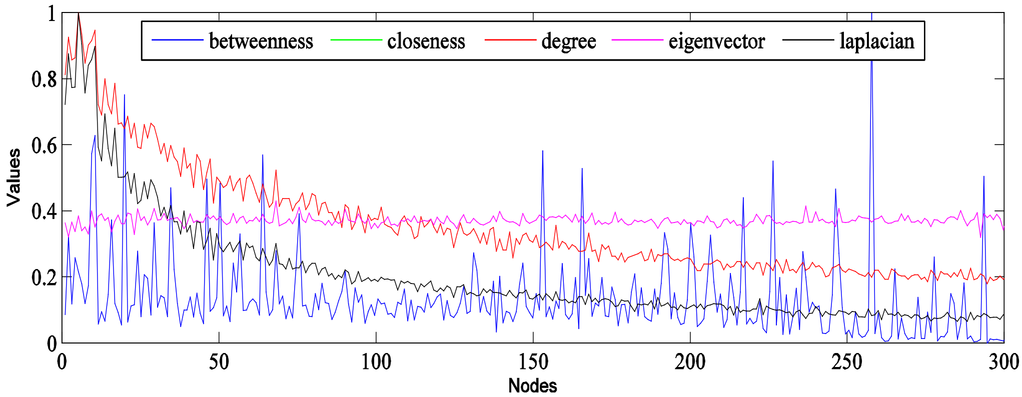
3. A COMPARATIVE STUDY

3.1 Synthetic Networks

In this section we apply the node centrality measures namely degree, betweenness, closeness, eigenvector centrality and LC on two synthetic networks, viz. WS-network and BA-network of different sizes, seeds and rewiring probabilities. We all know that WS-network and BA-network are probabilistic networks. So to minimize the effect of randomness, we consider the average result of ten networks generated for each of these models. And the values of the five centralities are also normalized for the sake of graphical comparison. We have also varied the size (number of nodes), seed size and rewiring probabilities to observe the effect of these parameters on these centrality measures. In general, we observe a positive correlation among the five centrality measures and LC is closely related to the degree centrality in case of WS-network. Also we can see from figure 1(a) that there is a high variation in the values of eigenvector centrality as compared to the other four centrality measures, this feature may be helpful for distinct ranking of the nodes. In case of BA-network, we observe a positive correlation among four out of the five measures, whereas eigenvector centrality is negatively correlated to LC as shown in figure 1(b). This may be attributed to the fact that unlike degree centrality, which is a local measure and eigenvector centrality, which is a global measure; LC is an intermediate measure between local and global characterization of a node's importance. Again if we increase the size of BA-network, keeping the seed size fixed or the size of WS-network with a fixed rewiring probability we observe that only degree centrality is positively correlated with LC. Similar results are obtained when we fix the size and vary the other parameters.



(a): WS-network with n=300, p=0.01.



(b): BA-network with n=300, $m = m_0 = 10$.

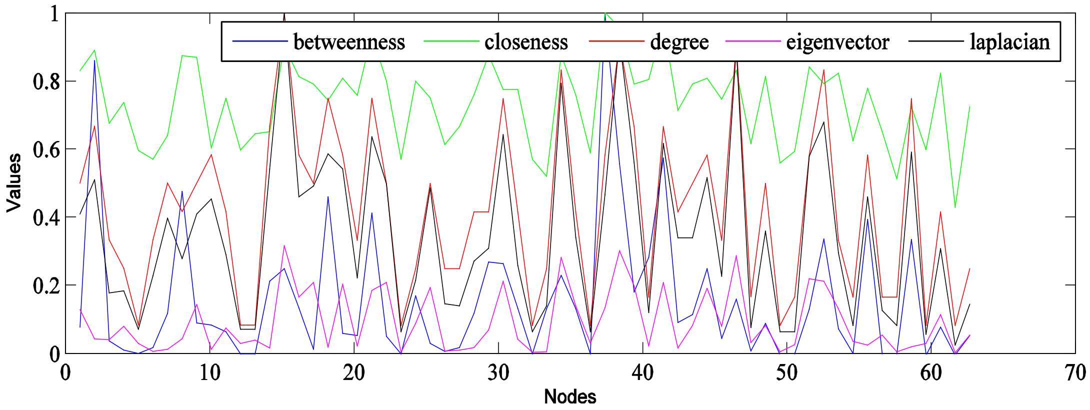
Figure-1: Five node centrality measures on two synthetic networks

3.2 Real networks

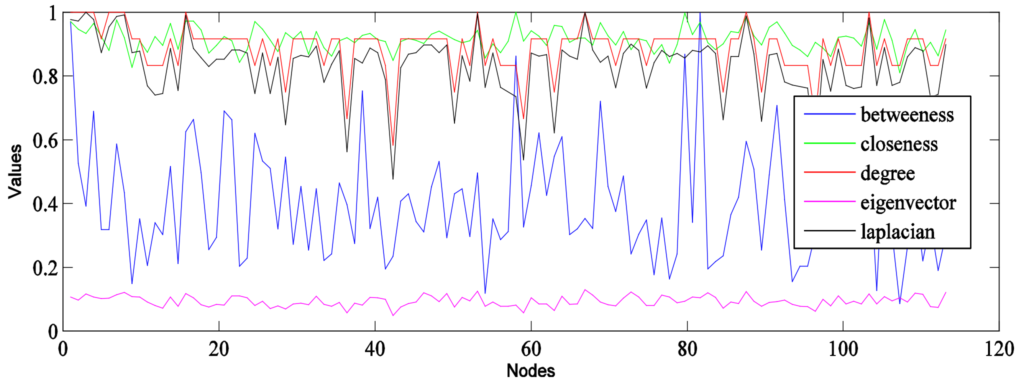
We have applied the aforesaid centrality measures on three real networks. One of them is the ANI and the other two are the bench-mark data sets in Network analysis, namely, Dolphins Network [11] and American Football Network [6]. These three real networks are selected from three different categories namely, Infrastructural, Animal Social and Human Social respectively. The results obtained are as shown in the figure 2. From the figure 2(b) and figure 2(c), we can conclude that closeness centrality is not synchronized with the other measures for Football Network and ANI. Whereas, eigenvector centrality is the measure that does not conform with the other measures in case of Dolphins Network. The top 10 scores of ANI for different centrality measures are listed in table 1. There we can observe that the ranks of airport according to LC are mostly in consensus with the other measures except few, e.g., Rank 4 is different for LC and betweenness. The following consensus study will clarify the observation.

Rank	Degree	Closeness	Betweenness	Eigenvector	Laplacian
1	Delhi	Delhi	Delhi	Delhi	Delhi
2	Mumbai	Mumbai	Mumbai	Mumbai	Mumbai
3	Kolkata	Bangalore	Bangalore	Bangalore	Bangalore
4	Bangalore	Kolkata	Kolkata	Kolkata	Kolkata
5	Madras	Madras	Madras	Hyderabad	Madras
6	Hyderabad	Hyderabad	Varanasi	Madras	Hyderabad
7	Guwahati	Guwahati	Hyderabad	Pune	Ahmedabad
8	Ahmedabad	Ahmedabad	Guwahati	Ahmedabad	Pune
9	Pune	Pune	Kochi	Goa	Guwahati
10	Goa	Goa	Port Blair	Jaipur	Goa

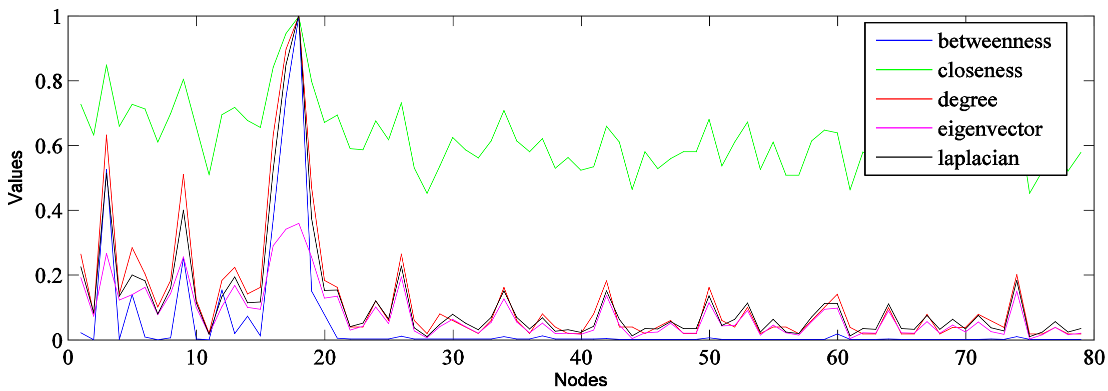
Table-1: Top 10 airports of ANI according to different centrality measures



(a): Dolphins Network (n=62, m=159)



(b): Football Network (n=115, m=613)



(c): Airport Network of India (n=79, m=248)

Figure-2: Five node centrality measures on three real networks

4. CONSENSUS ANALYSIS

Consensus analysis of outputs is a widely used method in bio-informatics [9]. We adopt a similar consensus method to evaluate the performance of various measures in the three real networks. A measure is considered to be more robust or more efficient, if its outputs are closer to consensus result. The nodes are ranked based on their centrality values for different centrality measures. Then the consensus ranks are calculated for each node as the mean of the five ranks obtained for the same node using the five centrality measures. For example, if a node gets ranks (2 3 3 4 1) from the five centrality measures, then the consensus rank of the node will be 2.6. To further analyze, we use the following simple “consensus deviation” to evaluate the output based on the five centrality measures from the consensus output.

$$\sigma(\alpha) = \frac{1}{n}\sum_{i=1}^{n}(rank_\alpha(i) - consensus(i))^2 ,$$

where n is the number of nodes, $rank_\alpha(i)$ is the rank of the i^{th} node based on centrality measurement α and $consensus(i)$ is the consensus rank of the i^{th} node. Clearly, smaller distance means better output. A method with smaller distance is considered as the one, which is best suited for the network. The different distance values for the three networks are presented in the Table 2. There, we can see that the consensus deviation of LC is the lowest for all the real networks under consideration.

	ANI	Dolphins Network	American Football Network
Degree	0.6778	1.1758	2.4804
Closeness	0.7743	1.0077	2.2750
Betweenness	1.7205	0.8345	2.4412
Eigenvector	0.7007	0.8345	1.9790
Laplacian	**0.6771**	**0.7636**	**1.5918**

Table-2: Consensus deviation values of different centrality measures

5. CONCLUSIONS

In this paper, we have discussed the graph theoretic aspect of Laplacian centrality and a comparative study is also presented with other standard centrality measures. In the study we observe that Laplacian centrality is a potential node centrality measure, which is based on the 2-walks of the network. It can capture both local and global environment about a node in a network. The graph theoretic aspect of the measure suggests that weight on the closed 2-walks is more than that on non-closed 2-walks, so we can say that it has given priority to the local environment than global, which is an important feature of centrality. But the Laplacian centrality does not ignore the global effect, which makes it more informative and realistic measure as compared to measures like degree centrality. In comparison with global measure like eigenvector centrality, it has a less computational cost and if the direction of ties can be ignored, or the pathway of connection is not important then Laplacian centrality can be the best centrality measure in a network. From the consensus study of the measure with other four centrality measures, we can conclude that Laplacian centrality is more robust or more efficient as its outputs are closer to the consensus output.

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