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# SPECIAL CLASSES OF IDEALS AND FILTERS OF PSEUDO-COMPLEMENTED ALMOST DISTRIBUTIVE LATTICES

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**ABSTRACT**

*In this paper we introduce the concepts of $\alpha-$ideal and $\beta-$filter of a weakly pseudo-complemented ADL in general and of a pseudo- complemented ADL in particular and discuss certain properties of these. Mainly, we prove that the sets of $\alpha-$ideals and $\beta-$filters of a pseudo-complemented ADL form algebraic lattices. Also, we characterize Stone ADLs and Almost Boolean algebras in terms of $\alpha-$ideals and $\beta-$filters.*

***Key words***: *Almost Distributive Lattice (ADL); weak pseudo-complementation; pseudo-complementation; $\alpha-$ideal; $\beta-$filter; minimal prime ideal; Almost Boolean algebra.*

***AMS Subject Classification (2000)***: *06D99, 06D15.*

## 1. INTRODUCTION

The concept of an Almost Distributive Lattice (ADL) was introduced by U. M. Swamy and G. C. Rao [3] as a common abstraction of several lattice theoretic and ring theoretic generalizations of Boolean algebra and Boolean ring. Further, U. M. Swamy, G. C. Rao and G. N. Rao [4] have introduced the notion of pseudo-complementation on an ADL and proved that the class of pseudo-complemented ADLs is equationally definable and they exhibited a one-to-one correspondence between maximal elements and pseudo-complementations on an ADL. Later, R. V. Babu, Ch. S. Sundar Raj and B. Venkateswarlu [7] have introduced the notion of weak pseudo-complementation on an ADL and proved several properties of this. In particular, they have proved that an ADL is pseudo-complemented if and only if it is weakly pseudo-complemented, even though a weak pseudo-complementation need not be a pseudo-complementation. In [1], Blyth defined the concepts of $*-$ideals and $*-$filters in pseudo-complemented semi lattices. Here, we extend these concepts to ADLs and we define these in the form of $\alpha-$ideals and $\beta-$filters of weakly pseudo-complemented ADLs in general and of pseudo-complemented ADLs in particular. Mainly, we prove that the $\alpha-$ideals ( $\beta-$filters) are independent of the weak pseudo (pseudo)-complementation. The main object of this paper is to study the classes of $\alpha-$ideals and $\beta-$filters of a pseudo-complemented ADL and prove that these classes form algebraic lattices. Also, in this paper we characterize minimal prime ideals in a weakly pseudo-complemented ADL. Mainly, we characterize Stone ADLs and Almost Boolean algebras in terms of their $\alpha-$ideals and $\beta-$filters.

We recall the notion of an Almost Distributive Lattice (abbreviated: ADL) and certain necessary results which will be used in the main text of this paper.

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## 2. PRELIMINARIES

**Definition 2.1 ([3]):** An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (ADL), if it satisfies the following conditions for all $a, b$ and $c$ in $A$.

(1) $0 \wedge a = 0$
(2) $a \vee 0 = a$
(3) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(4) $(a \vee b) \wedge c = (a \vee c) \vee (b \wedge c)$
(5) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(6) $(a \vee b) \wedge b = b$

An ADL is $(A, \wedge, \vee, 0)$ is said to be associative if the operation $\vee$ is associative. Throughout this paper by $A$ we mean an associative ADL $(A, \wedge, \vee, 0)$ unless otherwise mentioned.

For any $a, b \in A$, we say that $a$ is less than or equal to $b$ and we write $a \leq b$, if $a \wedge b = a$ (equivalently $a \vee b = b$). It can be easily proved that $\leq$ *is* a partial order on $A$.

**Lemma 1.2 ([3]):** The following hold for any $a$, $b$ and $c$ in an ADL $A$.

(1) $a \wedge 0 = 0$ and $a = 0 \vee a$
(2) $a \wedge a = a = a \vee a$
(3) $a \wedge b \leq b$ and $a \leq a \vee b$
(4) $a \wedge b = a \Leftrightarrow a \vee b = b$ and $a \wedge b = b \Leftrightarrow a \vee b = a$
(5) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (i.e., $\wedge$ is associative)
(6) $a \vee (b \vee a) = a \vee b$
(7) $(a \wedge b) \wedge c = (b \wedge a) \wedge c$
(8) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(9) $a \wedge b = b \wedge a \Leftrightarrow a \vee b = b \vee a$.

An element $m \in A$ is said to be maximal if $m \leq x$ implies $m = x$. It can be easily observed that $m$ is maximal if and only if $m \wedge x = x$ for all $x \in A$.

**Definition 1.3 ([3]):** A non-empty set $I$ of an ADL $A$ is said to be an ideal (filter) of $A$ if $a \vee b$ $(a \wedge b) \in I$ for all $a$ and $b \in I$ and $a \wedge x$ $(x \vee a) \in I$ for all $a \in I$ and $x \in A$.
It follows as a consequence that for any ideal (filter) $I$ of $A$, $x \wedge a$ $(a \vee x) \in I$ for all $a \in I$ and $x \in A$.
For any $X \subseteq A$, the smallest ideal (filter) of $A$ containing $X$ is called the ideal (filter) generated by $X$ and is denoted by $\langle X]$ $([X\rangle)$. If $X = \{x\}$, we simply write $\langle x]$ $([x\rangle)$ for $\langle\{x\}]$ $([\{x\}\rangle)$. we have the following for any $X \subseteq A$ and $x \in A$

$$\langle X] = \left\{\left(\bigvee_{i=1}^{n} x_i\right) \wedge a \;/\; n \geq 0,\ x_i \in X \text{ and } a \in A\right\}$$

$$[X\rangle = \left\{a \vee \left(\bigwedge_{i=1}^{n} x_i\right) \;/\; n \geq 0,\ x_i \in X \text{ and } a \in A\right\}$$

and $\langle x] = \{x \wedge a \mid a \in A\}$ *and* $[x\rangle = \{a \vee x \mid a \in A\}$.

$\langle x]$ $([x\rangle)$ is called the principal ideal (filter) generated by $x$.

For any subset $S$ of $A$, let $S^* = \{a \in A : a \wedge s = 0$ for all $s \in S\}$. Then $S^*$ is always an ideal of $A$ for all $S \subseteq A$. It can be proved that $S^* = \langle S]^*$ in particular for any $a \in A$,

$\langle a]^* = \{a\}^* = \{x \in A \mid a \wedge x = 0\}$.

**Definition 1.4 ([7]):** Let $A$ be an ADL. A mapping $a \mapsto a^*$ of $A$ into itself is called a weak pseudo-complementation on $A$ if $a \wedge b = 0 \Leftrightarrow a^* \wedge b = b$ for any $a$ and $b \in A$. An ADL $A$ is said to be weakly pseudo-complemented if there is a weak pseudo-complementation $a \mapsto a^*$ on $A$.

**Theorem 1.5 ([7]):** The following are equivalent to each other for any mapping $a \mapsto a^*$ of an ADL $A$ into itself.

(1) $a \mapsto a^*$ is a weak pseudo-complementation on $A$
(2) $\{a\}^* = \langle a^*]$ for any $a \in A$
(3) For any $a \in A$, $a \wedge a^* = 0$ and $a \wedge b = 0 \Rightarrow a^* \wedge b = b$ for any $b \in A$.

Let us recall from [8], that two elements $a$ and $b$ in an ADL $A$ are said to be associates to each other if $a \wedge b = b$ and $b \wedge a = a$ (equivalently $(a] = (b]$); in this case we write $a \sim b$. Also, in this case for any ideal (filter) $I$ of $A$, $a \in I \Leftrightarrow b \in I$.

**Theorem 1.6 ([7]):** Let $a \mapsto a^*$ and $a \mapsto a^+$ be two weak pseudo-complementations on an ADL $A$. Then the following hold for any $a$ and $b \in A$.

(1) $a^* \sim a^+$
(2) $a^{**} \sim a^{++}$
(3) $a^* \sim b^* \Leftrightarrow a^+ \sim a^+$
(4) $a^* = 0 \Leftrightarrow a^+ = 0$
(5) $a^* \wedge 0^+ \sim a^+$
(6) $a^* \vee a^{**} \sim 0^* \Leftrightarrow a^+ \vee a^{++} \sim 0^+$

**Theorem 1.7 ([7]):** Let $a \mapsto a^*$ be a weak pseudo-complementation on an ADL $A$. Then the following hold for any $a$ and $b \in A$.

(1) $0^*$ is a maximal element $in\ A$
(2) $m$ is maximal $in\ A \Rightarrow m^* = o$
(3) $0^{**} = 0$
(4) $a^* \wedge a = 0$
(5) $a^{**} \wedge a = a$
(6) $a \wedge b = 0 \Leftrightarrow a^{**} \wedge b = 0 \Leftrightarrow a \wedge b^{**} = 0 \Leftrightarrow a^{**} \wedge b^{**} = 0$
(7) $a^* \sim a^{***}$
(8) $a^* = 0 \Leftrightarrow a^{**}$ is maximal
(9) $a = 0 \Leftrightarrow a^{**} = 0$
(10) $(a \vee b)^* \sim a^* \wedge b^*$

**Theorem 1.8 ([7]):** Let $A$ be an ADL and $a \mapsto a^*$ be a weak pseudo-complementation on $A$. Then the following hold for any $a$ and $b \in A$.

(1) $a \sim b \Rightarrow a^* \sim b^*$
(2) $(a \wedge b)^* \sim (b \wedge a)^*$
(3) $(a \vee b)^* \sim (b \vee a)^*$
(4) $(a \wedge b)^{**} \sim a^{**} \wedge b^{**}$.

**Definition 1.9 ([4]):** Let $(A, \wedge, \vee, 0)$ be an ADL. Then a unary operation $a \mapsto a^*$ on $A$ is called a pseudo-complementation on $A$ if, for any $a, b \in A$, the following independent axioms are satisfied

(1) $a \wedge b = 0 \Rightarrow a^* \wedge b = b$
(2) $a \wedge a^* = 0$
(3) $(a \vee b)^* = a^* \wedge b^*$

## 3. $\alpha$ −IDEALS AND $\beta$ −FILTERS

In this section, we define an $\alpha-$ideal and a $\beta-$filter of a weakly pseudo-complemented ADL in general and of a pseudo-complemented ADL in particular and provide certain examples of these.

For any non-empty subset $X$ of an ADL $A$, let us denote $\alpha(X)$ by

$$\alpha(X) = \{y \in A : x \wedge y = 0 \text{ for some } x \in X\}.$$

Clearly $\alpha(X) = \bigcup_{x\in X} \{x\}^*$.

**Lemma 2.1:** Let $A$ be an ADL and $X$ a non-empty subset of $A$ such that $x \wedge y \in X$ for all $x, y \in X$. Then $\alpha(X)$ is an ideal of $A$.

**Corollary 2.2:** Let $A$ be an ADL and $F$ be a filter of $A$. Then $\alpha(F)$ is an ideal of $A$.

**Lemma 2.3:** Let $A$ be an ADL and $*$ a weak pseudo-complementation on $A$. Then for any filter $F$ of $A$,
$\alpha(F) = \{x \in A : x^* \in F\}$ and $\alpha(F)$ is independent of the weak pseudo-complementation $*$ on $A$.

**Proof:** Let $x \in \alpha(F)$. Then $x \wedge y = 0$ for some $y \in F$, therefore $x^* \wedge y = y$, and hence $x^* \vee y = x^*$, $y \in F$. Therefore, $x^* \in F$. On the other hand if $x^* \in F$, then $x \wedge x^* = 0$ and $x^* \in F$, it implies that $x \in \alpha(F)$. Let $+$ be a weak pseudo-complementation on $A$.

Then, $x^* \in F \Rightarrow x^+ \sim x^* \wedge 0^+ \in F \Rightarrow x^+ \in F$
and $x^+ \in F \Rightarrow x^* \sim x^+ \wedge 0^* \in F \Rightarrow x^* \in F$

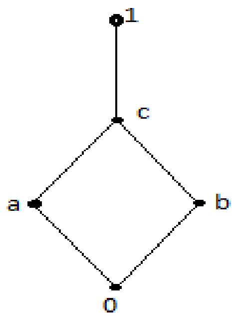
Since $0^*$ and $0^+$ are maximal and hence are in $F$. Therefore $\alpha(F)$ is independent of the weak pseudo complementation $*$ on $A$.

For any non-empty subset $X$ of an ADL $A$, let us define $\beta(X)$ by

$$\beta(X) = \{x \in A : \{x\}^* \subseteq X\}.$$

**Remark 2.4:** $\beta(I)$ need not be a filter even though $I$ is an ideal of $A$. For, consider the following example.

**Example 2.5:** Let $L$ be a lattice represented by the Hasse diagram given below,




Then $I = \{0, a, b, c\}$ is an ideal of $L$. Clearly, $\{0\}^* = L$, $\{a\}^* = \{0, b\}, \{b\}^* = \{0, a\}$, $\{c\}^* = \{0\}$ and $\{1\}^* = \{0\}$. Therefore $\beta(I) = \{a, b, c, 1\}$ which is not a filter of $L$, since $a \wedge b = 0 \notin \beta(I)$.

**Lemma 2.6:** Let $A$ be an ADL and $*$ a weak pseudo-complementation on $A$. Then for any ideal $I$ of $A$, $\beta(I) = \{x \in A : x^* \in I\}$ and $\beta(I)$ is independent of the weak pseudo-complementation $*$ on $A$.

**Lemma 2.7:** For any ideal $I$ of a weakly pseudo-complemented ADL $A$, $\alpha(\beta(I)) \subseteq I$.

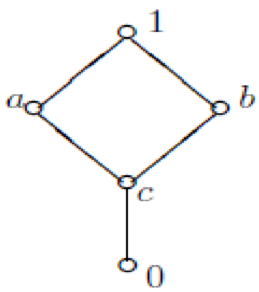
**Proof:** Let $*$ be a weak pseudo-complementation on $A$ and $x \in \alpha(\beta(I))$. Then $x \wedge y = 0 = y \wedge x$, for some $y \in \beta(I)$. This implies, $x = y^* \wedge x$ and $y^* \in I$, therefore $x \in I$.

**Lemma 2.8:** For any filter $F$ of a weakly pseudo-complemented ADL $A$, $F \subseteq \beta(\alpha(F))$.

**Proof:** Let $*$ be a weak pseudo-complementation on $A$ and $x \in F$. Then, $x^{**} = x^{**} \vee x \in F$ (since $x^* \wedge x = 0$ and hence $x^{**} \wedge x = x$) and hence $x^* \in \alpha(F)$, it follows that $x \in \beta(\alpha(F))$.

**Remark 2.9:** The following examples shows that the equality may not hold in 2.7 and 2.8.

**Example 2.9(1):** Let $L$ be the lattice represented by the Hasse diagram given below




Define the map $*$ on $L$ by $0^* = 1$ and $a^* = b^* = c^* = 1^* = 0$. Then $*$ is a pseudo-complementation on $L$ and $I = \{0, c, a\}$ is an ideal of $L$. Now $\beta(I) = \{x \in L : x^* \in I\} = \{c, a, b, 1\}$ and
$\alpha(\beta(I)) = \{x \in L : x^* \in \beta(I)\} = \{0\}$. This shows $\alpha(\beta(I)) \neq I$.

**Example 2.9(2):** Let $A = \{a,\ b,\ c,\ 0\}$. Define ∨ and ∧ on $A$ as follows

| ∨ | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | a | a | a | a |
| b | b | b | b | b |
| c | c | a | b | c |

| ∧ | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | b | c |
| b | 0 | a | b | c |
| c | 0 | c | c | c |

Then $(A,\ \wedge,\ \vee,\ 0)$ is an ADL. Define the map $*$ on $A$ by $x^* = 0$ for all $x \neq 0$ and $0^* = a$. Then $*$ is a pseudo - complementation on $A$. Let $F = \{a,\ b\}$, then $F$ is a filter of $A$ and $\alpha(F) = \{0\}$ and $\beta(\alpha(F)) = \{a,\ b,\ c\}$. This shows, $F \neq \beta(\alpha(F))$.

**Theorem 2.10:** Let $*$ be a weak pseudo-complementation on an ADL $A$ and $I$ an ideal of $A$. Then the following are equivalent to each other.
(1) $x \in I \Rightarrow x^{**} \in I$
(2) $I \subseteq \alpha(\beta(I))$
(3) $I = \alpha(\beta(I))$
(4) $I = \alpha(F)$ for some filter $F$ of $A$.

**Proof:**
$(\mathbf{1}) \Rightarrow (\mathbf{2})$: Assume (1). Let $x \in I$. Then $x^{**} \in I$ and hence $x^* \in \beta(I)$. Since $x \wedge x^* = 0$ and $x^* \in \beta(I)$, we get $x \in \alpha(\beta(I))$. Therefore $I \subseteq \alpha(\beta(I))$.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$ is clear by the lemma 2.7.

$(\mathbf{3}) \Rightarrow (\mathbf{4})$: Assume that $I = \alpha(\beta(I))$. It suffices to prove that $\beta(I)$ is a filter of $A$. Let $x, y \in \beta(I)$.
Then $x, y^* \in I$ and hence $x^* \vee y^* \in I$. Therefore $x^* \vee y^* \in \alpha(\beta(I))$, implies $(x^* \vee y^*) \wedge a = 0$ for some $a \in \beta(I)$ and hence $(x^* \vee y^*)^{**} \wedge a = 0,\ a \in \beta(I)$. This shows that $(x^* \vee y^*)^{**} \in \alpha(\beta(I))$.
Now, $(x \wedge y)^* \sim (x \wedge y)^{***} \sim (x^{**} \wedge y^{**})^* \sim (x^* \vee y^*)^{**} \in I$. Therefore $(x \wedge y)^* \in I$ and hence $x \wedge y \in \beta(I)$. Further, let $x \in \beta(I)$ and $a \in A$. Then $x^* \in I$ and $(a \vee x)^* \sim (x \vee a)^* \sim x^* \wedge a^*\, I$. This implies $(a \vee x)^* \in I$ and hence $a \vee x \in \beta(I)$. Therefore $\beta(I)$ is a filter of $A$.

$(\mathbf{5}) \Rightarrow (\mathbf{1})$: Assume that $I = \alpha(F)$ for some filter $F$ of $A$. Then,

$$\begin{aligned} x \in I \Rightarrow x \in \alpha(F) &\Rightarrow x^* \in F \\ &\Rightarrow x^{***} \sim x^* \in F \\ &\Rightarrow x^{***} \in F \\ &\Rightarrow x^{**} \in \alpha(F) \\ &\Rightarrow x^{**} \in I. \end{aligned}$$

**Theorem 2.11:** Let $*$ be a weak pseudo-complementation on an ADL $A$ and $F$ a filter of $A$. Then the following are equivalent to each other.
(1) $x^{**} \in F \Rightarrow x \in F$
(2) $\beta(\alpha(F)) \subseteq F$
(3) $F = \beta(\alpha(F))$
(4) $F = \beta(I)$ for some ideal $I$ of $A$

Now, we introduce $\alpha-$ideals and $\beta-$filters in weakly pseudo-complemented ADLs.

**Definition 2.12:** Let $A$ be an ADL and $*$ a weak pseudo-complementation on $A$. Then
(1) an ideal $I$ of $A$ is said to be an $\alpha$ −ideal of $A$ if any one (and hence all) of the conditions in theorem 2.10 holds
(2) a filter $F$ of $A$ is said to be a $\beta$ −filter of $A$ if any one (and hence all) of the conditions in theorem 2.11 holds.

**Example 2.12:** Let $A = \{0, a\}$ and $B = \{0, b_1, b_2\}$ be two discrete ADLs.
Write $L = A \times B = \{(0,0), (a,0), (0,b_1), (0,b_2), (a,b_1), (a,b_2)\}$. Then $(L,\ \wedge,\ \vee,\ 0)$ is an ADL under point-wise operations. Consider the ideals $I_1 = \{(0,0), (a,0)\}$ and $I_2 = \{(0,0), (0,b_1), (0,b_2)\}$.
Then $I_1$ and $I_2$ are $\alpha$ −ideals since $I_1 = \alpha(F_1)$, $I_2 = \alpha(F_2)$
where $F_1 = \{(0,b_1),\ (0,b_2),\ (a,b_1),\ (a,b_2)\}$ and $F_2 = \{(a,0),\ (a,b_1),\ (a,b_2)\}$ are filters. Also, $F_1$ and $F_2$ are $\beta$ −filters, since $F_1 = \beta(I_1)$ and $F_2 = \beta(I_2)$.

## 3. PROPERTIES OF $\alpha$ −IDEALS AND $\beta$ –FILTERS

Let us recall from [5, 4] that, an element $a$ of an ADL $A$ is said to be dense if $\{a\}^* = \{0\}$. It can be verified that $a$ is dense if and only if $a^* = 0$ where $*$ is a weak pseudo (pseudo)-complementation on $A$. Also, the set $D(A)$ of dense elements of $A$ is a filter of $A$.

**Proposition 3.1:** Let $A$ be a weakly pseudo-complemented ADL. Then $A, \{0\}$ are $\alpha$ −ideals and $A, D(A)$ are $\beta$ −filters.

**Proof:** It is by the fact that $A = \alpha(A)$, $\{0\} = \alpha(D(A))$ and $A = \beta(A)$, $D(A) = \beta(\{0\})$. It can be easily observed that, for any subset $S$ of $A$, $S^*$ is an $\alpha$ −ideal of $A$.

**Lemma 3.2:** Let $A$ be a weakly pseudo-complemented ADL and $F$ be a filter of $A$. Then $F$ is a $\beta$ −filter of $A$ if and only if $D(A) \subseteq F$.

**Proof:** Let $*$ be a weak pseudo-complementation on $A$. Suppose that $F$ is a $\beta$ −filter of $A$ and $x \in D(A)$. Then $x^* = 0$ and hence $x^{**} = 0^*$ which is maximal. Therefore $x^{**} \in F$ and hence $x \in F$. Thus $D(A) \subseteq F$. Conversely, let $x^{**} \in F$ then $x \vee x^* \in F$, since $x \vee x^*$ is dense. Therefore $x^{**} \wedge (x \vee x^*) \in F$, implies $(x^{**} \wedge x) \vee (x^{**} \wedge x^*) \in F$ which implies $x \vee 0 \in F$ and hence $x \in F$. Therefore $F$ is a $\beta$ −filter of $A$.

**Corollary 3.3:** $D(A)$ is the smallest $\beta$ −filter of $A$.

Now, we shall discuss certain properties of $\alpha$ −ideals and $\beta$ −filters in pseudo-complemented ADLs.

In general the lattice $\mathcal{I}(A)$ of ideals of an ADL $A$ is a complete lattice since it is closed under arbitrary intersections. However, the lattice $\mathcal{F}(A)$ of filters of $A$ is not necessary complete; infact $\mathcal{F}(A)$ is complete if and only if $A$ has a maximal element. If $*$ is a pseudo-complementation on $A$, then $0^*$ is necessarily a maximal element in $A$ and hence $0^* \in F$ for all $F \in \mathcal{F}(A)$. Therefore, every class $\{F_i\}_{i\in\Delta}$ of filters of a pseudo complemented ADL $A$ has infimum $\bigcap_{i\in\Delta} F_i$ and supremum $\bigvee_{i\in\Delta} F_i = \left[\bigcup_{i\in\Delta} F_i\right\rangle$ in $\mathcal{F}(A)$.

**Theorem 3.4:** Let $A$ be a pseudo-complemented ADL. Then the set $\mathcal{I}_\alpha(A)$ of $\alpha$ −ideals of $A$ is a complete distributive lattice ordered by set inclusion, in which the lattice operations are as follows:
If $\{I_i\}_{i\in\Delta} \subseteq \mathcal{I}_\alpha(A)$ then

$$glb\{I_i : i \in \Delta\} = \cap_{i\in\Delta} I_i \text{ and } lub\{I_i : i \in \Delta\} = \alpha\left(\bigvee_{i\in\Delta} F_i\right)$$

where to each $i \in \Delta$, $I_i = \alpha(F_i)$, for some filter $F_i$ of $A$.

**Proof:** Straight forward.

**Theorem 3.5:** Let $A$ be a pseudo-complemented ADL. Then the lattice $\mathcal{I}_\alpha(A)$ of $\alpha$ −ideals of $A$ is an algebraic lattice.

**Proof:** Let $*$ be a pseudo-complementation on $A$ and $I \in \mathcal{I}_\alpha(A)$. Then

$$I = \alpha(F) = \bigcup_{a\in F}\{a\}^* = sup\{(a^*] : a \in F\}$$

for some filter $F$ of $A$. Further, it can be verified that, for any $a \in A$, $(a^*]$ is a compact element in $\mathcal{I}_\alpha(A)$ and it follows that $\mathcal{I}_\alpha(A)$ is an algebraic lattice.

**Theorem 3.6:** Let $A$ be a weakly pseudo (pseudo)-complemented ADL and $I$ be an $\alpha$ −ideal of $A$. Then $\beta(I)$ is a filter of $A$ (and hence $\beta$ −filter).

**Proof:** Straight forward.
Form the theorem 3.6, $\beta$ and $\alpha$ induce isotone mappings $\hat{\beta} : \mathcal{I}_\alpha(A) \to \mathcal{F}_\beta(A)$ and $\hat{\alpha} : \mathcal{F}_\beta(A) \to \mathcal{I}_\alpha(A)$ where $\mathcal{F}_\beta(A)$ denote the set of $\beta$ −filters of a pseudo-complemented ADL $A$. Also, from 2.10 and 2.11, $\hat{\alpha}$ and $\hat{\beta}$ are isomorphisms which are inverses to each other. Therefore we have the following.

**Theorem 3.7:** $\mathcal{I}_{\alpha}(A) \cong \mathcal{F}_{\beta}(A)$.

The following are immediate consequences.

**Theorem 3.8:** The set $\mathcal{F}_{\beta}(A)$ of $\beta-$filters of a pseudo-complemented ADL $A$, ordered by set inclusion, is a complete distributive lattice in which the lattice operations are as follows: If $\{F_i\}_{i\in\Delta} \subseteq \mathcal{F}_{\beta}(A))$, then

$$inf\{F_i : i \in \Delta\} = \bigcap_{i\in\Delta} F_i \text{ and}$$

$$sup\{F_i : i \in \Delta\} = \left\{x \in A : x^{**} \in \bigvee_{i\in\Delta} F_i\right\} = \beta\left(\alpha\left(\bigvee_{i\in\Delta} F_i\right)\right)$$

**Theorem 3.9:** Let $A$ be a pseudo-complemented ADL. Then the lattice $\mathcal{F}_{\beta}(A)$ is an algebraic lattice.
Recall from [5], that an ADL $A$ with a pseudo-complementation $*$ is said to be a Stone ADL, if $x^* \vee x^{**} = 0^*$ for all $x \in A$ or equivalently $(x \wedge y)^* = x^* \vee y^*$ for all $x, y \in A$.

Here we characterize Stone ADLs interms of $\alpha-$ideals and $\beta-$filters.

**Theorem 3.10:** Let $A$ be an ADL and $*$ a pseudo-complementation on $A$. Then $A$ is a Stone ADL if and only if $\mathcal{I}_{\alpha}(A)$ is a sublattice of $\mathcal{I}(A)$.

**Proof:** Suppose that $A$ is a Stone ADL. Let $I, J \in \mathcal{I}_{\alpha}(A)$ and $x \in I \vee J$. Then $x = a \vee b$ for some $a \in I$ and $b \in J$ and hence $a^{**} \in I$ and $b^{**} \in J$.
Now, $x^{**} = (a \vee b)^{**} = (a^* \wedge b^*)^* = a^{**} \vee b^{**} \in I \vee J$. Therefore $I \vee J$ is an $\alpha-$ideal and hence $I \vee J \in \mathcal{I}_{\alpha}(A)$ and clearly $I \cap J \in \mathcal{I}_{\alpha}(A)$. Thus $\mathcal{I}_{\alpha}(A)$ is a sublattice of $\mathcal{I}(A)$.
Conversely, we suppose that $\mathcal{I}_{\alpha}(A)$ is a sublattice $of$ $\mathcal{I}(A)$. Let $x \in A$. Then, $(x^*]$ and $(x^{**}]$ are $\alpha-$ideals of $A$, by assumption $< x^*] \vee (x^{**}] = < x^* \vee x^{**}]$ is an $\alpha-$ideal of $A$. This implies $0^* = (x^{**} \wedge x^*)^* = (x^{**} \wedge x^{***})^* = (x^* \vee x^{**})^{**} \in < x^* \vee x^{**}]$ and since $0^*$ is maximal, we get that $< x^* \vee x^{**}] = A = < 0^*]$. This implies $x^* \vee x^{**} = 0^*$ since $x^*$ and $x^{**} \leq 0^*$. Thus $A$ is a Stone ADL.

**Theorem 3.11:** Let $A$ be a Stone ADL. Then $\beta(I)$ is a filter of $A$, for all ideals $I$ of $A$.

## 4. PRIME IDEALS AND FILTERS

Recall from [2], a proper ideal (filter) $P$ of an ADL $A$ is said to be prime, if for any $a, b \in A, a \wedge b(a \vee b) \in P \Rightarrow$ either $a \in P$ or $b \in P$. A prime ideal $P$ of an ADL $A$ is called minimal if there is no prime ideal $Q$ of $A$ such that $Q \subset P$.

**Remark 4.1:** In general, a prime ideal may not be an $\alpha-$ideal and $\alpha-$ideal need not be prime. For example, in 2.9(1) the prime ideals $\{0, c, a\}$ and $\{0, c, b\}$ are not $\alpha-$ideals, since $c^{**} = 0^* = 1$ and in 2.5 $\{0\}$ is an $\alpha-$ideal but not prime since $a \wedge b = 0$. Also, a prime filter may not be a $\beta-$filter. For, in 2.9(1), $\{b, 1\}$ is a prime filter but not a $\beta-$filter since $a^{**} = 0^* = 1$.

**Theorem 4.2 ([2]):** Let $P$ be a prime ideal of an ADL $A$. Then $P$ is minimal prime ideal if and only if $\{a\}^* \nsubseteq P$ for all $a \in P$.

In the following, minimal prime ideals of a weakly pseudo-complemented ADL are characterized in terms of their $\alpha-$ideals.

**Theorem 4.3:** Let $A$ be an ADL and $*$ a weak pseudo-complementation on $A$ and $P$ a prime ideal of $A$. Then the following conditions are equivalent.

(1) $P$ is minimal
(2) $x \in P$ implies that $x^* \notin P$
(3) $x \in P$ implies that $x^{**} \in P$
(4) $P \cap D(A) = \phi$.

**Proof:**
$(\mathbf{1}) \Rightarrow (\mathbf{2})$: Let $P$ be minimal and $x \in P$. Then by theorems 4.2 and 1.5(2), $(x^*] = \{x\}^* \nsubseteq P$. This implies $x^* \notin P$.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$: Assume (2). Let $x \in P$. Then $x^* \notin P$. Since $x^* \wedge x^{**} = 0 \in P$ and $P$ is prime, we get $x^{**} \in P$.

$(\mathbf{3}) \Rightarrow (\mathbf{4})$: Assume (3). If $x \in P \cap D(A)$ for some $x \in A$, then $x \in P$ and $x^* = 0$ and hence $x^{**} = 0^* \notin P$, since $0^*$ is maximal, a contradiction to (3).

$(\mathbf{4}) \Rightarrow (\mathbf{1})$: If $P$ is not minimal, then $Q \subset P$ for some prime ideal $Q$ of $A$. Let $x \in P - Q$. Then $x \wedge x^* = 0 \in Q$ and $x \notin Q$; therefore $x^* \in Q \subset P$, which implies that $x \vee x^* \in P$. Also, $x \vee x^*$ is dense, thus we obtain $x \vee x^* \in P \cap D(A)$, a contradiction to (4). Hence $P$ is minimal.

**Theorem 4.4:** Let $A$ be an ADL and $*$ a weak pseudo-complementation on $A$ and let $P$ be a prime ideal of $A$. Then the following conditions are equivalent.

(1) $A - P$ is maximal filter
*(2)* $A - P$ is prime filter and $a \vee a^* \in A - P$ for each $a \in A$
(3) $P$ is a minimal prime ideal
(4) $P$ is an $\alpha-$ideal
(5) $P \cap D(A) = \phi$.

**Proof:**
$(\mathbf{1}) \Rightarrow (\mathbf{2})$: Clearly $A - P$ is a prime filter since every maximal filter is prime filter. Let $a \in A$ such that $a \notin A - P$ then $A - P \neq [(A - P) \cup \{a\}) = (A - P) \vee [a\rangle$, so by the maximality, we get $(A - P) \vee [a\rangle = A$. In particular, $0 = x \wedge a$ for some $x \in A - P$ and hence $a^* \wedge x = x$. But $x \in A - P$ implies $a^* \in A - P$ and hence $a \vee a^* \in A - P$.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$: Since $A - P$ is a prime filter, $P$ is a prime ideal. Let $Q$ be a prime ideal and $Q \subset P$ with $a \in P - Q$. Then $a \wedge a^* = 0 \in Q$ and hence $a^* \in Q \subset P$, this implies $a \vee a^* \in P$, which is a contradiction to hypothesis. Thus $P$ is minimal prime ideal.

$(\mathbf{3}) \Rightarrow (\mathbf{4})$ and $(4) \Rightarrow (5)$ follow from the above theorem.

$(\mathbf{5}) \Rightarrow (\mathbf{1})$: Since $P$ is a prime ideal, $A - P$ is a prime filter. Let $F$ be a filter of $A$ and $A - P \subset F$ with $a \in F - (A - P)$. Since $a \vee a^*$ is dense, $a \vee a^* \in D(A)$ and hence $a \vee a^* \notin P$. But $a \in P$ and therefore $a^* \in A - P \subset F$. Also $a \in F$ and thus $a \wedge a^* = 0 \in F$ which implies that $F = A$.

For any $a$ and $b \in A$ with $a \leq b$, the interval $[\mathrm{a}, \mathrm{b}] = \{\mathrm{x} \in \mathrm{A} : \mathrm{a} \leq \mathrm{x} \leq \mathrm{b}\}$ is bounded distributive lattice with respect to the operations induced by those in the ADL $A$.

Recall from [3, 6] that, an ADL with a maximal element is said to be an Almost Boolean algebra if for any $a, b \in A$ with $a < b$, the interval $[a, b]$ is a complemented lattice.

**Theorem 4.5 ([6]):** Let $A$ be an ADL with a maximal element. Then, the following are equivalent.

(1) $A$ is an Almost Boolean algebra
(2) For any $a \in A$, there exist $b \in A$ such that $a \wedge b = 0$ and $a \vee b$ is maximal
*(3)* $[0, m]$ is a Boolean algebra for all maximal elements $m$
(4) There exists a maximal element $m$ such that $[0, m]$ is a Boolean algebra.

In general, an Almost Boolean algebra is pseudo-complemented but converse is not true. However, in the following we characterize an Almost Boolean algebra in terms of $\alpha-$ideals and $\beta-$filters.

**Theorem 4.6:** Let $A$ be a pseudo-complemented ADL. Then, $A$ is an Almost Boolean algebra if and only if every ideal of $A$ is an $\alpha-$ideal.

**Proof:** Let $*$ be a pseudo-complementation on $A$. We assume that $A$ is an Almost Boolean algebra and let $I$ be an ideal of $A$. Let $x \in I$. Then $x \vee 0^*$ is maximal since $0^*$ is maximal. By assumption, the interval $[0, x \vee 0^*]$ is a Boolean algebra and $x \leq x \vee 0^*$. Therefore, there exists $y \in [0, x \vee 0^*]$ such that $x \wedge y = 0$ and $x \vee y = x \vee 0^*$, hence $x \vee y$ is maximal and $y \wedge x^{**} = 0$.
Now, $x^{**} = (x \vee y) \wedge x^{**} = (x \wedge x^{**}) \vee (y \wedge x^{**}) = x \wedge x^{**} \in I$ (since $x \in I$ and $I$ is an ideal). Thus, $I$ is an $\alpha-$ideal.

Conversely suppose that every ideal of $A$ is an $\alpha-$ideal. Then, for any $x \in A$, $\langle x]$ is an $\alpha-$ideal and hence $x^{**} \in \langle x]$. This implies $x^{**} = x \wedge x^{**}$.

Now, $x \wedge 0^* = x^{**} \wedge x \wedge 0^* = x \wedge x^{**} \wedge 0^* = x^{**} \wedge 0^* = x^{**}$. (since $x^* \wedge x = 0$ and hence $x^{**} \wedge x = x$ and $x^{**} \leq 0^*$). Let $x \in [0, 0^*]$. Then $x = x \wedge 0^* = x^{**}$. This shows that $[0, 0^*] = A^*$, which is a Boolean algebra (refer [4]). By theorem 4.5, $A$ is an Almost Boolean algebra.

The following is similar to above theorem.

**Theorem 4.7:** $A$ is an Almost Boolean algebra if and only if every filter of $A$ is a $\beta-$filter.

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