International Journal of Mathematical Archive-8(7), 2017, 184-187
Available online through www.ijma.info ISSN 2229 – 5046

A COMMON FIXED POINT THEOREMS FOR CONTRACTIVE MAPPINGS IN DISLOCATED QUASI METRIC SPACE

VISHNU BAIRAGI*[1], V. H. BADSHAH[2] AND AKLESH PARIYA[3]

[1]Department of Mathematics, Govt. M. L. B. Girls, P. G. College, Indore - (M.P), India.

[2]School of Studies in Mathematics, Vikram University, Ujjain - (M.P.), India.

[3]Department of Mathematics, Medi - Caps University, Indore - (M.P), India.

(Received On: 24-06-17; Revised & Accepted On: 17-07-17)

ABSTRACT

Aage and Salunke [1], proved the result on fixed point theorem in dislocated and dislocated quasi metric space. Dass and Gupta [2], given an extentionsion of Banach contraction principle through rational expression. In this paper we establish a common fixed point theorem for continuous contractive mapping in dislocated quasi metric space which is the generalized result of Isufati [4], Mujeeb Ur Rahman and Muhammad Sarwar [11], and Badshah, et al. [12].

Keywords: *Dislocated quasi metric space, Common fixed point, Continuous contractive mapping.*

AMS Subject Classification: *47H10, 54H25.*

1.1. INTRODUCTION AND PRELIMINARIES

In 1922, Banach proved fixed point theorem for contraction mapping in complete metric space. It is well known as a Banach fixed point theorem. In 1975 Dass and Gupta [2], generalized Banach contraction principle in metric space. In 1977 Rohades [7], introduced a comparison of various definitions of contractive mappings. In 2005 Zeyada *et al.* [10], given a generalization of fixed point theorem due to Hiltzler and Seda [3], in dislocated quasi metric space. In 2008 Aage and Saluke [1] proved result on fixed point theorem in dislocated & dislocated quasi metric space. After this in 2010 Isufati [4], established a fixed point theorem in qislocated quasi metric space, also in 2010 Kohli *et al.* [5], in 2011 Shrivastava and Gupta [8], Pagey and Nighojkar [6] and in 2014 Shrivastava *et al.* [9], Mujeeb Ur Rahman and Muhammad sarwar [11], worked on a common fixed point theorem in dislocated quasi metric space. In this paper we establish a common fixed point theorem for continuous contractive mapping in dislocated quasi metric space which is the generalized result of Isufati [4], Mujeeb Ur Rahman and Muhammad sarwar [11] and Badshah, *et al.* [12].

Definition 1.1 [3&10]: Let X be a non-empty set and let $d: X \times X \to [0, \infty)$ be a function satisfying the following conditions :
(d_1) $d(x\,,x) = 0$
(d_2) $d(x\,,y) = d(y\,,x) = 0$ implies $x = y$.
(d_3) $d(x\,,y) = d(y\,,x)$ for all $x, y \in X$
(d_4) $d(x\,,y) \leq d(x\,,z) + d(z\,,y)$ for all $x\,,y, z \in X$
If d satisfies conditions only (d_2) and (d_4), then d is called a dislocated quasi metric on X.

If d satisfies conditions (d_1), (d_2) and (d_4), then d is called a quasi metric on X. If d satisfies conditions (d_2), (d_3) and (d_4), then d is called a dislocated metric on X. If d satisfies all the conditions (d_1), (d_2) (d_3) and (d_4), then d is called a metric on X.

Definition 1.2 [10]: A sequence $\{x_n\}$ in a dq metric space (dislocated quasi metric space) (X, d) is called a Cauchy sequence if for given $\epsilon > 0$, there corresponds $n_0 \in N$ such that for all $m, n \geq n_0$, implies $d(x_n , x_m) < \epsilon$.

Corresponding Author: Vishnu Bairagi*[1],
[1]Department of Mathematics, Govt. M. L. B. Girls, P. G. College, Indore - (M.P), India.

Definition 1.3 [10]: A sequence in dq metric space converges to a point x if there exists $x \in X$ such that $d(x_{n,}x) \to 0$ as $n \to \infty$ or $lim_{n \to\infty} d(x_{n,}x) = 0$.

Definition 1.4 [3]: A dislocated quasi metric space *(X, d)* is a complete metric space if every Cauchy sequence in *(X, d)* is convergent sequence with respect to d.

Definition 1.5 [10]: Let *(X, d)* and *(Y, ρ)* be any two dislocated quasi metric spaces and Let $T : X \to Y$ be a function then T is a continuous function at $x_0 \in X$, if for each sequence $\{x_n\}$ which is convergent to x_0 in X, the sequence $\{T(x_n)\}$ is convergent to $\{T(x_0)\}$ in Y.

Definition 1.6 [10]: Let *(X, d)* be a d-metric space. A map $T: X \to X$ is called contraction mapping if there exists a number λ with $0 \le \lambda < 1$ such that $d(Tx, Ty) \le \lambda\, d(x, y)$ for all $x, y \in X$.

Lemma 1.1 [10]: Limits in a dq metric space are unique.

Theorem 1.1 [1]: Let *(X, d)* be a complete dq metric space and suppose there exist non negative constants $\alpha, \beta, \gamma > 0$ with $\alpha + \beta + \gamma < 1$. Let $T: X \to X$ be a continuous mapping satisfying condition,

$$d(Tx, Ty) \le \alpha\, d(x,y) + \beta\, d(x,Tx) + \gamma\, d(y,Ty) \text{ for all } x, y \in X.$$

Then T has a unique fixed point**.**

Theorem 1.2 [4]: Let *(X, d)* be a dq metric space and let $T: X\to X$ be a continuous mapping satisfying the following condition,

$$d(Tx, Ty) = \alpha \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \beta\, d(x,y) \quad \forall x, y \in X,$$

and $\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$. Then T has a unique fixed point.

Theorem 1.3 [9]: Let *(X, d)* be a dq metric space and $T: X\to X$ be a continuous mapping
Satisfying the following condition,

$$d(Tx, Ty) = \alpha \frac{d(y,Ty)[1+d(x,Tx)]}{(d(x,Ty))[1+d(x,Ty)]} + \beta\, d(x,y) + \gamma\, d(x,Ty) \quad \forall x, y \in X,$$

and $\alpha > 0, \beta > 0, \gamma > 0, \alpha + \beta + \gamma < 1$; Then T has a unique fixed point.

Theorem 1.5[11]: Let *(X, d)* be a complete dq metric space and and let $T : X \to X$ be a continuous self-mapping satisfying the condition,

$$d(Tx, Ty) \le \alpha\, d(x,y) + \beta \frac{d(x,Ty)d(y,Ty)}{d(x,y)+d(y,Ty)} + \gamma \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)} + \mu \frac{d(x,Tx)d(x,Ty)}{1+d(x,y)} \text{ for all } x, y \in X,$$

and $\alpha, \beta, \gamma, \mu \ge 0$ *with* $\alpha + \beta + \gamma + 2\mu < 1$.
Then T has a unique fixed point.

Theorem 1.6 [12]: Let *(X, d)* be a complete dq metric space and $T: X\to X$ be a continuous mapping satisfying the following condition,

$$d(Tx, Ty) \le \alpha \frac{d(y,Ty)d(x,Tx)}{[1+d(x,Tx)][1+d(y,Ty)]} + \beta \frac{d(x,y)d(x,Tx)}{1+d(x,Tx)} + \gamma \frac{d(x,y)d(y,Ty)}{1+d(x,y)} \quad \forall x, y \in X$$

and $\alpha, \beta, \gamma > 0$, $\alpha + \beta + \gamma < 1$; Then T has a unique fixed point.

2. MAIN RESULT

Theorem 2.1: Let *(X, d)* be a complete dq metric space and $S, T: X\to X$ be two continuous mapping satisfying the following condition,

$$d(Sx, Ty) \le \alpha \frac{d(y,Ty)d(x,Sx)}{[1+d(x,Sx)][1+d(y,Ty)]} + \beta \frac{d(x,y)d(x,Sx)}{1+d(x,Sx)} + \gamma \frac{d(x,y)d(y,Ty)}{1+d(x,y)} \quad \forall x, y \in X \tag{1}$$

and $\alpha, \beta, \gamma > 0$, $\alpha + \beta + \gamma < 1$. Then S and T have a unique common fixed point in X.

Proof: Let $\{x_n\}$ be a sequence in dq metric space *(X, d)* and let x_0 be arbitrary in X. We define a sequence $\{x_n\}$ by the rule x_0,

$$x_1 = Sx_0,\ x_3 = Sx_2 \ldots x_{2n+1} = Sx_{2n} \text{ and } x_2 = Tx_1,\ x_4 = Tx_3 \ldots x_{2n+2} = Tx_{2n+1} \ \forall n \in N \tag{2}$$

Now we claim that $\{x_n\}$ is a Cauchy sequence. For this consider,

$d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})$

$$\le \alpha \frac{d(x_{2n+1}, Tx_{2n+1})d(x_{2n}, Sx_{2n})}{[1+d(x_{2n}, Sx_{2n})][1+d(x_{2n+1}, Tx_{2n+1})]} + \beta \frac{d(x_{2n}, x_{2n+1})d(x_{2n}, Sx_{2n})}{1+d(x_{2n}, Sx_{2n})} + \gamma \frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, Tx_{2n+1})}{1+d(x_{2n}, x_{2n+1})}$$

$$\le \alpha \frac{d(x_{2n+1}, x_{2n})d(x_{2n}, x_{2n+1})}{[1+d(x_{2n}, x_{2n+1})][1+d(x_{2n+1}, x_{2n+2})]} + \beta \frac{d(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+1})}{1+d(x_{2n}, x_{2n+1})} + \gamma \frac{d(x_{2n}, x_{2n+1})d(x_{2n+1}, x_{2n+2})}{1+d(x_{2n}, x_{2n+1})}$$

$$< \alpha \frac{d(x_{2n+1}, x_{2n})}{[1+d(x_{2n+1}, x_{2n+2})]} + \beta\, d(x_{2n}, x_{2n+1}) + \gamma\, d(x_{2n+1}, x_{2n+2})$$

Since $d(x_{2n}, x_{2n+1}) < 1 + d(x_{2n}, x_{2n+1}) \Rightarrow \frac{d(x_{2n}, x_{2n+1})}{1+d(x_{2n}, x_{2n+1})} < 1$

$$< \alpha\, d(x_{2n+1}, x_{2n}) + \beta\, d(x_{2n}, x_{2n+1}) + \gamma\, d(x_{2n+1}, x_{2n+2})$$

This gives,

$$d(x_{2n+1}, x_{2n+2}) < (\alpha + \beta) d(x_{2n}, x_{2n+1}) + \gamma\, d(x_{2n+1}, x_{2n+2})$$
$$\Rightarrow d(x_{2n+1}, x_{2n+2}) < \frac{(\alpha+\beta)}{1-\gamma} d(x_{2n}, x_{2n+1})$$

Therefore we have,

$$d(x_{2n+1}, x_{2n+2}) < \delta(x_{2n}, x_{2n+1}), \text{ where } \delta = \frac{(\alpha+\beta)}{1-\gamma} \in (0, 1)$$

Similarly we have,

$$(x_{2n}, x_{2n+1}) < \delta\, d(x_{2n-1}, x_{2n}),$$
$$d(x_{2n-1}, x_{2n}) < \delta\, d(x_{2n-2}, x_{2n-1}),$$
$$\Rightarrow d(x_2, x_1) < \delta\ d(x_1, x_0).$$

Therefore we have,

$$d(x_n, x_{n+1}) < \delta\, d(x_{n-1}, x_n)\ ,$$

Similarly we have,

$$d(x_{n-1}, x_n) < \delta\, d(x_{n-2}, x_{n-1}),$$
$$d(x_{n-2}, x_{n-1}) < \delta\, d(x_{n-3}, x_{n-2}),$$
$$\Rightarrow d(x_2, x_1) < \delta\, d(x_1, x_0).$$

Finally, we have,

$$d(x_n, x_{n+1}) < \delta^n\, d(x_1, x_0).$$
$$\Rightarrow \ |\, d(x_n, x_{n+1}) \,| < \delta^n \,|\, d(x_1, x_0) \,|$$

Since $0 < \delta < 1$ and letting $n \to \infty \Rightarrow \delta^n \to 0$, implies that $|\, d(x_n, x_{n+1}) \,| \to 0$ as $n \to \infty$

Hence the sequence $\{x_n\}$ is Cauchy sequence in the complete dislocated quasi metric space (X, d).

Thus the sequence $\{x_n\}$ is a convergent sequence in dislocated quasi metric space (X, d) to the point $z \in X$. i.e. $x_n \to z$ as $n \to \infty$. Also sub sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converges to z. Since T is continuous mapping therefore,

$$\lim_{n\to\infty} x_{2n+1} = z \Rightarrow \lim_{n\to\infty} Tx_{2n+1} = Tz \Rightarrow \lim_{n\to\infty} x_{2n+2} = Tz$$

Hence, $Tz = z$ *i.e.* z is the fixed point of T.

Similarly, using the continuity of S we can show that $Sz = z$.

Finally we have $Tz = z = Sz$. *i.e.* z is the common fixed point of S and T.

This completes the proof of theorem 2.1

For uniqueness:

To prove S and T have unique fixed point we suppose z and w are any two common fixed point of S and T with $z \neq w$
i.e. $Tz = z$ and $Tw = w$ and $Sz = z$ and $Sw = w$

Consider

$$d(z, w) = d(Sz, Tw)$$
$$\leq \alpha \frac{d(w,Tw)d(z,Sz)}{[1+d(z,Sz)][1+d(w,Tw)]} + \beta \frac{d(z,w)d(z,Sz)}{1+d(z,Sz)} + \gamma \frac{d(z,w)d(w,Tw)}{1+d(z,w)}$$

$d(z, w) \leq 0$ [$\because$ z and w are any two common fixed point of T, i.e. $Tz = z$ and $Tw = w$ *also* $Sz = z$ and $Sw = w$ and $d(z, z) = 0$ & $d(w, w) = 0$] but $d(z, w) \geq 0$

This implies that

$$d(z, w) = 0$$

i.e. $z = w$, this proves the uniqueness of common fixed point of S and T in X

Corollary 2.2: Let *(X, d)* be a complete *dq* metric space and *S, T:* $X \to X$ be a continuous mapping. Satisfying the following condition,

$$d(Sx,Ty) \leq \beta \frac{d(x,y)d(x,Sx)}{1+d(x,Sx)} + \gamma \frac{d(x,y)d(y,Ty)}{1+d(x,y)} \qquad \forall\, x, y \in X$$

and $\beta > 0, \gamma > 0, \beta + \gamma < 1$; Then *S* and *T* have a unique common fixed point in *X*.

Proof: The proof of the corollary 2.2 follows immediately by putting $\alpha = 0$ in Theorem 2.1

Corollary 2.3: Let *(X, d)* be a complete *dq* metric space and *S, T* : $X \to X$ be a continuous mapping Satisfying the following condition,

$$d(Sx, Ty) \leq \alpha \frac{d(y,Ty)d(x,Sx)}{[1+d(x,Sx)][1+d(y,Ty)]} + \gamma \frac{d(x,y)d(y,Ty)}{1+d(x,y)} \qquad \forall\, x, y \in X$$

and $\alpha > 0, \gamma > 0, \alpha + \gamma < 1$; Then *S,T* have a unique common fixed point in *X*.

Proof: The proof of the corollary 2.3 follows immediately by putting $\beta = 0$ in Theorem 2.1

Corollary 2.4: Let *(X, d)* be a complete *dq* metric space and *S, T*: $X \to X$ be a continuous mapping Satisfying the following condition

$$d(Sx, Ty) \leq \alpha \frac{d(y,Ty)d(x,Sx)}{[1+d(x,Sx)][1+d(y,Ty)]} + \beta \frac{d(x,y)d(x,Sx)}{1+d(x,Sx)} \qquad \forall\, x, y \in X$$

and $\alpha > 0, \beta > 0, \alpha+\beta < 1$; Then *S, T* have a unique common fixed point in *X*.

Proof: The proof of the corollary 2.4 follows immediately by putting $\gamma = 0$ in Theorem 2.1

REFERENCES

1. Aage, C. T. and Salunke, J. N. "The results on fixed point theorem in dislocated and dislocated quasi-metric spaces", *Applied Mathematical Science* 2 (59); 2941-2948 (2008).
2. Dass, B. K. and Gupta, S., "An extension of Banach contraction principle through rational expression", *Indian J. Pure appl. Math.*, 6; 1455-1458 (1975).
3. Hitzler, P. and Seda, A. K. "*Dislocated Topologies, Journal of Electrical Engineering*" 51(12/s); 3-7(2000).
4. Isufati, A., "Fixed point theorem in dislocated quasi-metric space", *Applied Math. Sci.*, 4 (5); 217-223 (2010).
5. Kohli, M., Sriravastava, R. and Sharma, M., "Some results on fixed point theorem in dislocated quasi metric Space "*International Journal of Theoretical &Applied Sciences*, 2(1); 27-28 (2010).
6. Pagey, S. S. and Nighojkar, R. "A fixed point theorem for expansive type mapping in dislocated quasi metric space", *International Journal of Theoretical & Applied Sciences*, 3(1); 26-27 (2011).
7. Rhoades, B. E. "A comparison of various definition of contractive mapping", *Trans. Amer. Soc.*, 226; 257-290 (1977).
8. Shrivastava, R. and Gupta, R. "A fixed point theorem for random operator in dislocated Quasi Metric Spaces" *International Journal of theoretical & Applied Sciences*, 3(1); 47-48 (2011).
9. Shrivastava, R., Qureshi, K. and Rathore, K. "A common fixed point theorem for a pair of mapping in dislocated metric space". *Advances in Applied Science Research* 5(3); 417- 419 (2014).
10. Zeyada, F. M., Hassan, G. H. and Ahmed, M. A., "A generalization of a fixed point theorem due to Hitzler and Seda in dislocated quasi-metric spaces", *The Arabian Journal for Sci. Engg.*, 31 (1A); 111-114 (2006).
11. Mujeep, Ur Rahman and Muhammad Sarwar. "Fixed point results in dislocated quasi metric spaces" *International Mathematical Forum*, vol 9(14); 677-682 (2014).
12. Badshah, V. H., Bairagi, V. and Pariya, A. "A Fixed point theorem for a contractive mapping in dislocated quasi metric spaces" *International Journal of Scientific Research in Mathematical and Statistical Sciences*, Vol, 4(2); 14 -17(2017).

Source of support: Nil, Conflict of interest: None Declared.
