International Journal of Mathematical Archive-8(7), 2017, 179-183
Available online through www.ijma.info ISSN 2229 – 5046

THE EXISTENCE OF FIXED POINT THEOREMS IN COMPLEX VALUED b-METRIC SPACES

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(Received On: 22-06-17; Revised & Accepted On: 29-07-17)

ABSTRACT

In this paper, we consider complex valued b-metric spaces which was generalized form of complex valued metric spaces. We propose to derive the existence of fixed point theorems in complex valued b-metric spaces.

AMS Subject Classification: *47H10, 54H25.*

Key Words: *common fixed point, complex valued b-metric spaces.*

1. INTRODUCTION

One of the most influential spaces is complex valued b-metric spaces, introduced by Rao *et.al* [10] in 2013, which was more general than the complex valued metric spaces [1]. They proved some fixed point results for rational type mappings in complex valued b-metric spaces. Since then, this notion has been used by many authors to obtain various fixed point theorems (see [2], [3], [4], [5], [6], [7], [8], [9], [11]).

The purpose of this paper is to prove common fixed point theorem for two self-mappings in a complete complex valued b-metric spaces.

2. PRELIMINARIES

Let us start by defining some important notations and definitions.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows: $z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$. Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(1) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$;
(2) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$;
(3) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$;
(4) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$.

In particular, we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied, also we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that

(a) if $0 \precsim z_1 \precnsim z_2$ then $|z_1| < |z_2|$;
(b) if $z_1 \precsim z_2$ and $z_2 \prec z_3$ then $z_1 \prec z_3$;
(c) if $a, b \in \mathbb{R}$ and $a \leq b$ then $az \precsim bz$ for all $z \in \mathbb{C}_+$.

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The following definition is recently introduced by Rao *et.al* [10].

Definition 2.1[10]: Let Y be a nonempty set and let $p \geq 1$ be a given real number. A function $d: Y \times Y \to \mathbb{C}$ is called a complex valued b-metric on Y if for all $x, y, z \in Y$ the following conditions are satisfied:

(i) $0 \precsim d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \precsim p[d(x, z) + d(z, y)]$.

The pair (Y, d) is called a complex valued b-metric space.

Example 2.2[10]: If $Y = [0,1]$, define the mapping $d: Y \times Y \to \mathbb{C}$ by $d(x, y) = |x - y|^2 + i|x - y|^2$ for all $x, y \in Y$. Then (Y, d) is a complex valued b-metric space with $p = 2$.

Definition 2.3[10]: Let (Y, d) be a complex valued b-metric space.

(i) A point $x \in Y$ is called interior point of a set $A \subseteq Y$ whenever there exists $0 \prec r \in \mathbb{C}$ such that $B(x, r) = \{y \in Y: d(x, y) \prec r\} \subseteq A$.
(ii) A point $x \in Y$ is called limit point of a set A whenever for every $0 \prec r \in \mathbb{C}$, $B(x, r) \cap (A - \{x\}) \neq \emptyset$.
(iii) A subset $A \subseteq Y$ is called open set whenever each element of A is an interior point of A.
(iv) A subset $A \subseteq Y$ is called closed set whenever each element of A belongs to A.
(v) The family $F = \{B(x, r): x \in Y \; and \; 0 \prec r\}$ is a sub-basis for a Hausdorff topology τ on Y.

Definition 2.4[10]: Let (Y, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in Y and $x \in Y$.

(i) If for every $c \in \mathbb{C}$, with $0 \prec c$, there is $N \in \mathbb{N}$ such that for all $n > N, d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent and converges to x. We denote this by $\lim_{n\to\infty} x_n = x$ or $\{x_n\} \to x$ as $n \to \infty$.
(ii) If for every $c \in \mathbb{C}$, with $0 \prec c$, there is $N \in \mathbb{N}$ such that for all $n > N, d(x_n, x_{n+m}) \prec c$, where $m \in \mathbb{N}$, then $\{x_n\}$ is said to be Cauchy sequence.
(iii) If every Cauchy sequence in Y is convergent in Y, then (Y, d) is said to be a complete complex valued b-metric space.

Lemma 2.5 [10]: Let (Y, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in Y. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 2.6 [10]: Let (Y, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in Y. Then $\{x_n\}$ is Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$, where $m \in \mathbb{N}$.

3. MAIN RESULT

Theorem 3.1: Let (Y, d) be a complete complex valued b-metric space with the coefficient $p \geq 1$ and let $P, Q: Y \to Y$ be a mapping satisfying:

$$d(Px, Qy) \precsim \alpha d(x, y) + \beta[d(x, Px) + d(y, Qy)] + \gamma[d(x, Qy) + d(y, Px)], \quad (1)$$

for all $x, y \in Y$, where $\propto, \beta, \gamma$ are nonnegative reals with $\alpha + 2\beta + 2p\gamma < 1$.

Then $P \; and \; Q$ have a unique common fixed point in Y.

Proof: For any arbitrary point $x_o \in Y$, define sequence $\{x_n\}$ in Y such that

$$x_{2n+1} = Px_{2n},$$
$$x_{2n+2} = Qx_{2n+1}, \text{ for } n = 0,1,2,3 \ldots \ldots \quad (2)$$

Now, we show that the sequence $\{x_n\}$ is Cauchy.

Let $x = x_{2n}$ and $y = x_{2n+1}$ in (1), we have

$$\begin{aligned} d(Px_{2n}, Qx_{2n+1}) &= d(x_{2n+1}, x_{2n+2}) \\ &\precsim \alpha d(x_{2n}, x_{2n+1}) + \beta[d(x_{2n}, Px_{2n}) + d(x_{2n+1}, Qx_{2n+1})] \\ &+\gamma[d(x_{2n}, Qx_{2n+1}) + d(x_{2n+1}, Px_{2n})] \\ &= \alpha d(x_{2n}, x_{2n+1}) + \beta[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \\ &+\gamma[d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})] \\ &\precsim \alpha d(x_{2n}, x_{2n+1}) + \beta[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \\ &+p\gamma[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})], \end{aligned}$$

which implies that $|d(x_{2n+1}, x_{2n+2})| \leq \delta |d(x_{2n}, x_{2n+1})|$,

where $\delta = \frac{\alpha+\beta+p\gamma}{1-\beta-p\gamma} < 1$.

Similarly, we have $|d(x_{2n+2},x_{2n+3})| \leq \delta|d(x_{2n+1},x_{2n+2})|$,
where $\delta = \frac{\alpha+\beta+p\gamma}{1-\beta-p\gamma} < 1$.

Thus for all n,
$$\begin{aligned}|d(x_n,x_{n+1})| &\leq \delta|d(x_{n-1},x_n)| \\ &\leq \delta^2|d(x_{n-2},x_{n-1})| \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\leq \delta^n|d(x_0,x_1)|. \end{aligned} \tag{3}$$

Now for any $m > n, m, n \in \mathbb{N}$, we have
$$\begin{aligned}|d(x_n,x_m)| &\leq p|d(x_n,x_{n+1})| + p|d(x_{n+1},x_m)| \\ &\leq p|d(x_n,x_{n+1})| + p^2|d(x_{n+1},x_{n+2})| + p^2|d(x_{n+2},x_m)| \\ &\leq p|d(x_n,x_{n+1})| + p^2|d(x_{n+1},x_{n+2})| + p^3|d(x_{n+2},x_{n+3})| + p^3|d(x_{n+3},x_m)| \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\leq p|d(x_n,x_{n+1})| + p^2|d(x_{n+1},x_{n+2})| + p^3|d(x_{n+2},x_{n+3})| + \\ &\ldots\ldots\ldots + p^{m-n-2}|d(x_{m-3},x_{m-2})| + p^{m-n-1}|d(x_{m-2},x_{m-1})| \\ &+p^{m-n}|d(x_{m-1},x_m)|.\end{aligned}$$

By using (3), we get
$$\begin{aligned}|d(x_n,x_m)| &\leq p\delta^n|d(x_0,x_1)| + p^2\delta^{n+1}|d(x_0,x_1)| + p^3\delta^{n+2}|d(x_0,x_1)| \\ &+ \ldots\ldots\ldots + p^{m-n-2}\delta^{m-3}|d(x_0,x_1)| + p^{m-n-1}\delta^{m-2}|d(x_0,x_1)| \\ &+p^{m-n}\delta^{m-1}|d(x_0,x_1)| \\ &= \textstyle\sum_{i=1}^{m-n} p^i\delta^{i+n-1}|d\,(x_0,x_1)|.\end{aligned}$$

Therefore,
$$\begin{aligned}|d(x_n,x_m)| &\leq \textstyle\sum_{i=1}^{m-n} p^{i+n-1}\delta^{i+n-1}|d\,(x_0,x_1)| \\ &= \textstyle\sum_{t=n}^{m-1} p^t\delta^t|d\,(x_0,x_1)| \\ &\leq \textstyle\sum_{t=n}^{\infty}(p\delta)^t|d\,(x_0,x_1)| \\ &= \frac{(p\delta)^n}{1-p\delta}|d(x_0,x_1)|\end{aligned}$$
and hence
$$|d(x_n,x_m)| \leq \frac{(p\delta)^n}{1-p\delta}|d(x_0,x_1)| \to 0 \; as \; m,n \to \infty. \tag{4}$$

Thus, $\{x_n\}$ is a Cauchy sequence in Y. Since Y is complete, there exists some $w \in Y$ such that $x_n \to w$ as $n \to \infty$. Assume not, then there exists $z \in Y$ such that
$$|d(w,Pw)| = |z| > 0. \tag{5}$$

So by using the triangular inequality and (1), we get
$$\begin{aligned}z = d(w,Pw) &\precsim pd(w,x_{2n+2}) + pd(x_{2n+2},Pw) = pd(w,x_{2n+2}) + pd(Qx_{2n+1},Pw) \\ &\precsim pd(w,x_{2n+2}) + p\alpha d(w,x_{2n+1}) + p\beta[d(w,Pw) + d(x_{2n+1},Qx_{2n+1})] \\ &+p\gamma[d(w,Qx_{2n+1}) + d(x_{2n+1},Pw)] \\ &= pd(w,x_{2n+2}) + p\alpha d(w,x_{2n+1}) + p\beta[d(w,Pw) + d(x_{2n+1},x_{2n+2})] + p\gamma[d(w,x_{2n+2}) + d(x_{2n+1},Pw)]\end{aligned}$$

which implies that
$$\begin{aligned}|z| &= |d(w,Pw)| \\ &\leq p|d(w,x_{2n+2})| + p\alpha|d(w,x_{2n+1})| + p\beta|d(w,Pw) + d(x_{2n+1},x_{2n+2})| \\ &+p\gamma|d(w,x_{2n+2}) + d(x_{2n+1},Pw)|.\end{aligned} \tag{6}$$

Taking the limit of (6) as $n \to \infty$, we obtain that $|z| = |d(w,Pw)| \leq 0$, a contradiction with (5). So $|z| = 0$. Hence $Pw = w$. Similarly, we obtain $Qw = w$.

Now, we show that $P \; and \; Q$ have unique common fixed point of $P \; and \; Q$. To prove this, assume that w^* is another common fixed point of $P \; and \; Q$. Then,
$$\begin{aligned}d(w,w^*) &= d(Pw,Qw^*) \\ &\precsim \alpha d(w,w^*) + \beta[d(w,Pw) + d(w^*,Qw^*)] + \gamma[d(w,Qw^*) + d(w^*,Pw)]\end{aligned}$$
So that
$$\begin{aligned}|d(w,w^*)| &\leq \alpha|d(w,w^*)| + \beta|d(w,Pw) + d(w^*,Qw^*)| + \gamma|d(w,Qw^*) + d(w^*,Pw)| \\ &\leq \alpha|d(w,w^*)|\end{aligned}$$

So that $w = w^*$, which proves the uniqueness of common fixed point.

Corollary 3.2: Let (Y, d) be a complete complex valued b-metric space with the coefficient $p \geq 1$ and let $Q: Y \to Y$ be a mapping satisfying:

$$d(Qx, Qy) \precsim \alpha d(x, y) + \beta[d(x, Qx) + d(y, Qy)] + \gamma[d(x, Qy) + d(y, Qx)], \qquad (7)$$

for all $x, y \in Y$, where α, β, γ are nonnegative reals with $\alpha + 2\beta + 2p\gamma < 1$. Then Q has a unique fixed point in Y.

Proof: We can prove this result by applying Theorem 3.1 with $P = Q$.

Corollary 3.3: Let (Y, d) be a complete complex valued b-metric space with the coefficient $p \geq 1$ and let $Q: Y \to Y$ be a mapping satisfying (for some fixed n):

$$d(Q^n x, Q^n y) \precsim \alpha d(x, y) + \beta[d(x, Q^n x) + d(y, Q^n y)] + \gamma[d(x, Q^n y) + d(y, Q^n x)], \qquad (8)$$

for all $x, y \in Y$, where α, β, γ are nonnegative reals with $\alpha + 2\beta + 2p\gamma < 1$. Then Q has a unique fixed point in Y.

Proof: Set $P = Q^n$ and $Q = Q^n$ in inequality (1) and use the Theorem 3.1 and Corollary 3.2.

Following results is obtained from Corollary 3.2.

Corollary 3.4: Let (Y, d) be a complete complex valued b-metric space with the coefficient $p \geq 1$ and let $Q: Y \to Y$ be a mapping satisfying:

$$d(Qx, Qy) \precsim \alpha d(x, y), \qquad (9)$$

for all $x, y \in Y$, where $p\alpha \in [0,1)$. Then Q has a unique fixed point in Y.

Proof: We can prove this result applying Corollary 3.2 with $\beta = \gamma = 0$. Corollary 3.4 is the Banach type version of a fixed point results for contractive mappings in a complex valued b-metric space.

Corollary 3.5: Let (Y, d) be a complete complex valued b-metric space with the coefficient $p \geq 1$ and let $Q: Y \to Y$ be a mapping satisfying:

$$d(Qx, Qy) \precsim \alpha d(x, y) + \beta[d(x, Qx) + d(y, Qy)], \qquad (10)$$

for all $x, y \in Y$, where α, β are nonnegative reals with $p(\alpha + 2\beta) < 1$. Then Q has a unique fixed point in Y.

Proof: We can prove this result by applying Corollary 3.2 with $\gamma = 0$.

Corollary 3.6: Let (Y, d) be a complete complex valued b-metric space with the coefficient $p \geq 1$ and let $Q: Y \to Y$ be a mapping satisfying:

$$d(Qx, Qy) \precsim \alpha d(x, y) + \gamma[d(x, Qy) + d(y, Qx)], \qquad (11)$$

for all $x, y \in Y$, where α, γ are nonnegative reals with $\alpha + 2p\gamma < 1$. Then Q has a unique fixed point in Y.

Proof: We can prove this result by applying Corollary 3.2 with $\beta = 0$.

Corollary 3.7: Let (Y, d) be a complete complex valued b-metric space with the coefficient $p \geq 1$ and let $Q: Y \to Y$ be a mapping satisfying:

$$d(Qx, Qy) \precsim \alpha_1 d(x, y) + \alpha_2 d(x, Qx) + \alpha_3 d(y, Qy) + \alpha_4 d(x, Qy) + \alpha_5 d(y, Qx), \qquad (12)$$

for all $x, y \in Y$, where $\alpha_i \geq 0$ for every $i \in \{1,2, \dots \dots \dots 5\}$ and $\alpha_1 + \alpha_2 + \alpha_3 + 2p\alpha_4 + \alpha_5 < 1$. Then Q has a unique fixed point in Y.

Proof: In (12) interchanging the roles of x and y, and adding the new inequality to (12), gives (7) with

$$\alpha = \alpha_1, \beta = \frac{\alpha_2 + \alpha_3}{2} \text{ and } \gamma = \frac{\alpha_4 + \alpha_5}{2}.$$

4. CONCLUSION

In this attempt, we prove some fixed point theorems in complex valued b-metric spaces. These results generalize and improve the recent results of [8], [9], [10], [11], which extend the further scope of our results.

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Source of support: Nil, Conflict of interest: None Declared.