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CONNECTED REGULAR DOMINATION NUMBER OF A GRAPH G

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ABSTRACT

The connected dominating set was first suggested by S. T. Hedetniemi and elabourate treatment of this parameter appears in E. Sampathkumar and H. B. Walikar [5]. The concept of regular domination was first introduced by prof. E. Sampathkumar. On combining these two parameters we obtain a new parameter namely connected regular domination as, let G be a graph, V is a vertex set of G and D ⊆ V, then D is said to be connected regular domination, if it satisfies the following conditions, (i) D is Dominating set (ii) D is regular (iii) D is connected. In this paper, we find the connected regular domination number for complete graph and for some standard graphs.

Subject Classification: *05C25, 05C69.*

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1. INTRODUCTION

Domination in graphs has been an extensively researched branch of graph theory. There are many origins to the theory of domination. Historically, the first domination type problems came from chess. Apart from chess, domination in graphs has applications to several other fields. There are more than a hundred models of dominating and related types of sets in graphs. The topic of domination was given a formal mathematical definition, first by C. Berge in [3] and later by Ore [6] in 1962. Berge called the domination as external stability and domination number as coefficient of external stability. Ore introduced the word domination in his famous book Theory of Graphs published in 1962. A recent book on the topic of domination [5] lists over 1,200 papers related to domination in graphs [1, 2, 4, 7] and several hundred papers [9, 10] on the topic have been written since the publication of the book few years ago.

The connected dominating set was first suggested by S. T. Hedetniemi and elabourate treatment of this parameter appears in E. Sampathkumar and H. B. Walikar [8]. The concept of regular domination was first introduced by prof. E. Sampathkumar.

In this paper we introduce the new parameter connected regular domination and evaluate the connected regular domination for complete graph, complete bipartite graph and for some standard graphs.

Definition 1.1 (A dominating set): is a set of vertices such that each vertex of V is either in D or has atleast one neighbour in D. The minimum cardinality of such a set is called the domination number of *G* denoted by $\gamma(G)$.

Definition 1.2 (A **connected dominating):** set D is a set of vertices of a graph G such that every vertex in V - D is adjacent to atleast one vertex in D and the subgraph ⟨D⟩ induced by the set D is connected. The connected domination number $\gamma_c(G)$ is the minimum cardinality of the connected dominating sets of *G*.

Definition 1.3 (A **regular dominating):** set is a dominating set D of V (G) if ⟨D⟩ is regular. The minimum cardinality of a regular dominating set is called regular domination number of G and is denoted by $\gamma_r(G)$.

Corollary 1.4: A graph *G* has a connected dominating set if and only if *G* is connected. [8]

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Theorem 1.5: A dominating set D of a connected graph *G* is a minimal dominating set if and only if for each vertex in u ∈ D, one of two conditions holds: i) u is not adjacent to any vertex in D. ii) there exists a vertex v ∈ V - D for which N (v) ∩ D = u.[8]

Corollary 1.6: Since any maximal independent set is a regular dominating set, the existence of a regular dominating set is always guaranteed in any graph. Also for any graph G, $\gamma_r(G)$. = 1 if and only if *G* has a full degree vertex.

On combining the two parameters we obtain a new parameter namely connected regular domination.

Definition 1.7 Let *G* be a graph, V is a vertex set of *G* and D ⊆ V, then D is said to be **Connected Regular dominating** set, if it satisfies the following conditions, (i) D is dominating set (ii) D is regular (iii) D is connected. The minimum cardinality of connected regular domination set is denoted by $\gamma_{cr}(G)$.

Theorem 1.8: Let *T* be a tree, then anyone of the following conditions hold
(i) $\gamma_{cr}(T)$ = 1, if ($T = K_{1,n}$) *T* is a star graph.
(ii) $\gamma_{cr}(T)$ = 1, if $T = P_2$; P_3
(iii) $\gamma_{cr}(T)$ = 2, if $T = P_4$
(iv) If *T* is any tree ($T \notin K_{1,n}, P_2, P_3, P_4$), then the connected regular domination does not exists.

Proof: Let *T* be a tree, by the definition of tree *T* is connected without cycle.

(i) Suppose *T* is a star graph namely $K_{1,n}$. Clearly, there exists a centre vertex in *T* and is adjacent to all other vertices. Then the centre vertex will act as a dominating vertex. This dominating centre vertex is both connected and regular. Thus $\gamma_{cr}(T)$ = 1.
(ii) Suppose *T* is a path. Let $T = P_2$, contains two vertices and one edge. Then clearly any one edge will act as a dominating vertex. Also this dominating vertex is both connected and regular. Thus$\gamma_{cr}(T)$ = 1. Let $T = P_3$, contains 3 vertices and two edges. In P_3 the centre vertex dominates the other two vertexes. Then clearly the centre vertex is the dominating vertex. Also it act as both connected and regular. Thus $\gamma_{cr}(T)$ = 1.
(iii) Suppose $T = P_4$, clearly two end points of P_4 is adjacent to the internal graph K_2. That is V_1, V_2, V_3, V_4 is a path P_4. Then $V_1$and V_4 is the end points which are adjacent to V_2 and V_3 respectively. Thus $\gamma(P_4) =|V_2, V_3|$ = 2. Also these two dominating vertices are both connected and regular. Thus$\gamma_{cr}(T)$ = 2.
(iv) Suppose *T* is a tree with $T \neq$ star, P_2, P_3, P_4. Let $T = K_{2,n}$. Then clearly *T* contains a cycle which is a contradiction that *T* does not contain a cycle. Thus $T \neq K_{2,n}$. Suppose let $T= K_n$, where n ≥ 3, then once again *T* contains a cycle which is again a contradiction that *T* does not contains a cycle. Thus $T \neq K_n$, n ≥ 3. Suppose let $T = P_n$ where n ≥ 5. That is *T* is a path P_5containing $V_1V_2V_3V_4V_5$ with $V_1$and V_5 as the end vertices. In P_5 the dominating set contains $V_2V_3V_4$ that is γ (P_5) = 3. Clearly the path $V_2V_3V_4$ are connected and the dominating set D is connected dominating set of T. But 1 = d(V_2) ≠ d(V3) = 2 ≠ d(V4) = 1. Clearly D is not regular dominating set and hence P and P_n where n ≥ 5 does not have connected regular dominating set.

2. CONNECTED REGULAR DOMINATION FOR A COMPLETE GRAPH G

In this section we calculate the connected regular domination for a complete multipartite graph

Corollary 2.1:

1. For any K_n, $\gamma_{cr}(K_n)$ = 1.
2. For any $K_{n,m}$, $\gamma_{cr}(K_{n,m})$ = 2.
3. For any $K_{n1,n2,\ldots,nn}$, $\gamma_{cr}(K_{n1;n2,\ldots,nn})$ = 2.

3. CONNECTED REGULAR DOMINATION FOR SOME STANDARD GRAPHS

In this section we calculate the connected regular domination for some standard graphs.

Definition 3.1: A **Wheel graph,** W_n of order n is a graph that contains an outer cycle of order n -1 and for which every vertex in the cycle is connected to one other vertex(which is known as the hub). The number of vertices in W_n is n and the number of edges is 2(n-1).

Theorem 3.2: For any wheel graph W_n, $\gamma_{cr}(W_n)$ = 1.

Proof: Let *G* = (V, E) be the wheel graph W_n with the vertex set V ={v_1, v_2, v_3,…, v_{n-1},v_n } and the edge set as E = {v_1v_i : 2 ≤ i ≤ n} ∪ {v_iv_{i+1}: 2 ≤ i ≤ (n-1)} ∪ {v_nv_2}. From the definition a wheel graph W_n is obtained when a additional vertex v_1 not on C_{n-1} is joined to each of the n-1 vertices namely v_2, v_3, …, v_n in c_{n-1} by new edges. Then clearly that v_1 dominates all other vertices in c_{n-1}. Thus the domination number for the wheel graph W_n is one. Also that particular vertex v_n is both connected and regular domination. Therefore we have the connected regular domination number for any wheel graph with n vertices is one. Thus $\gamma_{cr}(W_n)$ = 1.

Definition 3.3 (A **Helm graph):** H_n of order n is a graph that obtained from a wheel graph W_n by attaching a pendent edge at each vertex of the n cycle of the wheel. The number of vertices in H_n is 2n -1 and the number of edges is 3(n- 1).

Theorem 3.4: For any Helm graph H_n , $\gamma_{cr}(H_n)$ = n - 1.

Proof: Let *G* = (V, E) be the helm graph with the vertex set V ={v_1, v_2, v_3, …, v_{n-1}, v_n, u_1, u_2, u_3, …, u_{n-1}} and the edge set as E = {v_1v_i: $2 \leq i \leq n$} ∪ {v_iv_{i+1},v_iu_{i-1}: $2 \leq i \leq (n-1)$} ∪ {v_nv_2}. From the definition, we know that a helm graph H_n is obtained by attaching one vertex by an edge to each n-1 vertices in the outer circuit of the wheel graph. The vertex v_1 is adjacent to all the n-1 vertices in the wheel graph. Thus the connected regular domination for a wheel graph is 1. But that vertex v_1 dominates only wheel graph but not for the helm graph. Since the helm graph contains n-1pendent vertices, the dominating set must contain at least n-1 vertices. Thus the n-1 vertices in the outer circuit of the helm graph H_n dominates all the 2n-1 vertices of the helm graph. Thus the domination number of the helm graph is n-1. Also these n-1 vertices are both connected domination and regular domination for H_n. Therefore the connected regular domination for H_n is n-1.Thus $\gamma_{cr}(H_n)$ = n - 1.

Definition 3.5 (A Flower graph): Fn is a graph that obtained from a helm graph H_n by joining each pendent vertex to the central vertex of the helm. The number of vertices in F_n is 2n-1 and the number of edges is 4(n-1).

Theorem 3.6: For any flower graph F_n, γ_{cr}(Fn) = 1 for all n ≥5.

Proof: Let G = (V, E) be the Flower graph F_n with the vertex set V = { v_1, v_2, v_3,...,v_{n-1}, v_n, u_1, u_2, u_3,...,u_{n-1}} and the edge set as E = {v_1v_i: $2 \leq i \leq n$} ∪ {v_iv_{i+1},v_iu_{i-1}: $2 \leq i \leq (n-1)$} ∪ {v_nv_2} ∪ {v_1u_i: $1 \leq i \leq (n-1)$}. From the definition, we know that a flower graph F_n is obtained by attaching all the pendent vertices of the helm graph to the centre vertex v_1 each by an edge. Thus v_1 dominates all the other vertices in the flower graph. Thus the domination number for the flower graph is 1. Also this center vertex v_1 is connected and regular. Therefore the connected regular domination number for flower graph is one . Thus γ_{cr}(Fn) = 1 for all n ≥5.

Definition 3.7 (A **Friendship graph):** is the planar undirected graph with (2n + 1) vertices and 3n edges. It can be constructed by joining n copies of the cycle graph C_3 with a common vertex. Let us denote it by Fr_n. The number of vertices in Fr_n is 2n + 1 and the number of edges is 3n.

Theorem 3.8: For any friendship graph Fr_n, $\gamma_{cr}(Fr_n)$ = 1 for all n ≥ 3.

Proof: Let G=(V,E) be the Friendship graph Fr_n with the vertex set V ={ v_0,v_1,v_2,...,v_{2n-1},v_{2n}}and the edge set as E = {v_0v_i: $1 \leq i \leq 2n$} ∪ {$v_{2i-1}v_{2i}$: $1 \leq i \leq n$}. From the definition, we know that a friendship graph Fr_n contains n copies of cycle graph c_3 with a common vertex. Let v_{2n+1} be the common vertex for Fr_n. Clearly this center vertex is adjacent to all other remaining vertices in Fr_n. Thus the domination number for the friendship graph is 1. Also this center vertex is connected and regular. Therefore the connected regular domination number for flower graph is one . Thus $\gamma_{cr}(Fr_n)$ = 1 for all n ≥ 3.

Definition 3.9: The **Lollipop graph** $L_{m,n}$, is the graph obtained by joining a complete graph K_m to a path P_n with a bridge.

Theorem 3.10: For any lollipop graph $L_{m,n}$, we have the following,

(i) $\gamma_{cr}\left(L_{m,1}\right) = 1$

(ii) $\gamma_{cr}\left(L_{m,2}\right) = 2$

(iii) $\gamma_{cr}\left(L_{m,n}\right)$ does not exist for n>2.

Proof:

(i) From the definition of the lollipop graph, $L_{m,1}$ contains m complete graphs connecting one pendent vertex by an edge. Clearly the vertex connecting the pendent vertex is a full degree vertex. Thus by using the corollary (1.7) we have $\gamma_{cr}\left(L_{m,1}\right) = 1$.

(ii) From the definition of the lollipop graph, $L_{m,2}$ contains m complete graphs connected with a path of length 2. Thus the longest path between the end vertex from the path to any vertex in the complete graph is 4. Thus $L_{m,2}$ contains a path P_4 in it. By using the theorem(1.9) the connected regular domination of P_4 is 2. Thus we have $\gamma_{cr}\left(L_{m,2}\right) = 2$.

(iii) Let us consider the lollipop graph $L_{m,3}$, which contains m complete graph connected with a path of length 3. Thus the longest path between the end vertex of the path to any vertex in the complete graph is 5. Thus $L_{m,3}$, contains the path P_5. By using the theorem (1.9) the connected regular domination of P_5 does not exist. Thus $\gamma_{cr}\left(L_{m,3}\right)$ does not exist. In the same way, $L_{m,4}$ contains a path P_6. By using the theorem (1.9) $\gamma_{cr}\left(L_{m,4}\right)$ does not exist.

In general, for any lollipop graph of $L_{m,n}$ n > 2, the connected regular domination does not exist. Thus $\gamma_{cr}(L_{m,n})$ does not exist for n > 2.

Definition 3.11: In graph theory, the **Petersen graph**, is an undirected graph with 10 vertices and 15 edges.

It is a small graph that serves as a useful example and counter example for many problems in graph theory. It is the smallest bridge less cubic graph.

Theorem 3.12: For a Petersen graph $\gamma_{cr}(G) = 5$

Definition 3.13: In the mathematical field of graph theory a **Prism graph**, is a graph that has one of the prisms as its skeleton.

Prism graphs are examples of generalized Petersen graphs with parameters GP(n,1).

Theorem 3.14: For any Prism graph GP (n, 1) we have the following,
(i) γ_{cr}(GP(3,1)) = 2
(ii) γ_{cr}(GP(n,1)) = n, for n > 3

Proof:

(i) Let us consider the triangular prism. From the definition we know that GP(3,1) contains 6 vertices and 9 edges. Also they are regular and cubic graphs. Since the prism has symmetries taking each vertex to each other vertex the prism graphs are vertex-transitive graphs. Choose any vertex v_1 from the outer circuit and choose other vertex v_2 from the inner circuit in such a way that v_2 is adjacent to v_1. This vertex set {v_1, v_2} dominates all other vertices in the triangular prism graph. Since v_1 is adjacent to v_2 we have v_1v_2 is connected and having same degree as one. Thus this vertex set {v_1, v_2} act as the connected regular domination set. Thus γ_{cr}(GP(3,1)) = 2.

(ii) Let us consider GP(4, 1) which is the cubical prism graph. From the definition it has 8 vertices and 12 edges in which 4 vertices are connected in the outer circuit attached with the remaining four vertices are connected in the inner circuit. Since the both outer and inner circuits are connected two vertices each one from outer and the other from inner circuits dominates all the remaining vertices in GP(4, 1). As this vertices having degree 0, it is non adjacent. Thus they are not connected. Therefore, in order to choose the connected domination set, it is possible to take all the four vertices either in the outer circuit or from the inner circuit. Fortunately this vertex set is regular with degree 2. Thus we have the connected domination number for prism graph is γ_{cr}(GP(4,1)) = 4.

Now let us consider GP(5,1), which is a pentagonal graph with 10 vertices and 15 edges. As the domination number for GP(5,1) is 3 and similarly proceeding as above we have γ_{cr}(GP(5,1)) = 5.

In the same way, we have the connected regular domination number for hexagonal prism graph as 6, whereas for heptagonal prism graph the connected regular domination number is 7 and for octagonal prism graph it is 8 and so on. Thus in general the connected regular domination number for GP(n, 1) is n for n > 3. Thus γ_{cr}(GP(n, 1)) = n, for n > 3.

4. CONCLUSION

In this chapter, we evaluated the connected regular domination for some standard graphs namely Wheel, Helm, Flower, Friendship, Lollipop, Petersen and Prism graph.

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