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SOME FIXED POINT THEOREMS IN METRIC SPACES FOR ALTERING DISTANCE

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1. ABSTRACT

In the present paper we established some fixed point theorems in complete metric spaces. Our results generalized the previous results on complete metric spaces given by Banach, and other mathematicians.

2. INTRODUCTION AND PRELIMINARIES

In 1984, M.S. Khan, M. Swalech and S.Sessa [10] expanded the research of the metric fixed point theory to a new category by introducing a control function which they called an altering distance function.

Definition 2A ([19]): A function $\psi : \Re_+ \to \Re_+$ is called an altering distance function if the following properties are satisfied:

(ψ_1) $\psi(t) = 0 \Leftrightarrow t = 0$

(ψ_2) ψ is monotonically non-decreasing.

(ψ_3) ψ is continuous.

By ψ we denote the set of the all altering distance functions.

Theorem 2B ([19]): Let (M, d) be a complete metric space, let $\psi \in \Psi$ and let $S : M \to M$ be a mapping which satisfies the following inequality

$$\psi[d(Sx, Sy)] \le a\psi[d(x, y)]$$

For all $x, y \in M$ and for some $0 < a < 1$. Then S has a unique fixed point $z_0 \in M$ and moreover for each $x \in M$ $\lim_{n\to\infty} S^n x = z_0$

Lemma 2C: Let (M, d) be a metric space. Let $\{x_n\}$ be a sequence in M such that $\lim_{n\to\infty} \psi[d(x_n, x_{n+1})] = 0$

If $\{x_n\}$ is not a Cauchy sequence in M, then there exist an $\varepsilon_0 > 0$ and sequences of integers positive $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$

Such that

$$d(x_{m(k)}, x_{n(k)}) \ge \varepsilon_0 \text{ , } d(x_{m(k)-1} x_{n(k)}) < \varepsilon_0$$

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And

(i) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \varepsilon_0$

(ii) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon_0$

(iii) $\lim_{n\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon_0$

Remark 2D: Form Lemma 3Cis easy to get

$$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon_0$$

In this paper we prove some fixed point and common fixed point theorems for rational expression. Our result is generalization of various known results.

3. SOME BASIC RESULTS

Since Banach's fixed point theorem in 1922, because of its simplicity and useful ness, it has become a very popular tool in solving the existence problems in many branches of nonlinear analysis. For some more results of the generalization of this principle.

Theorem 3.1: Banach [1] The well known Banach contraction principle states that "If X is complete metric space and T is a contraction mapping on X into itself, then T has unique fixed point in X".

Theorem 3.2: Kanan [16] proved that "If T is self mapping of a complete metric space X into itself satisfying:
$d(Tx,Ty) \le \eta\,[d(x,Tx) + d(y,Ty)]\ for\ all\ \ x,y \in X,\ and\ \ \eta \in \left[0,\frac{1}{2}\right]$. Then T has unique fixed point in X.

Theorem 3.3: Fisher [9] proved the result with
$d(Tx,Ty) \le\ \mu\,[d(Tx,x) +\ d(Ty,y)] + \delta\, d(x,y)\ for\ all\ \ x,y \in X,\ and\ \ \mu,\delta \in \left[0,\frac{1}{2}\right]$. Then T has unique fixed point in X.

Theorem 3.4: A similar conclusion was also obtained by Chaterjee [3].
$d(Tx,Ty) \le \mu[\ d(Ty,x) +\ d(Tx,y)]\ for\ all\ \ x,y \in X,\ and\ \ \eta \in \left[0,\frac{1}{2}\right]$. Then T has unique fixed point in X.

Theorem 3.5: Ciric [5] proved the result
$d(Tx,Ty) \le \eta[\ d(x,Tx) + d(y,Ty)] + \mu[d(x,Ty) + d(y,Tx)] + \delta\, d(x,y)\ for\ all\ \ x,y \in X,\ and\ \ \eta,\mu,\delta \in [\ 0,1)$. Then T has unique fixed point in X.

Theorem 3.6: Reich [22] proved the result
$d(Tx,Ty) \le\ \mu[d(x,Ty) +\ d(y,Tx)]\ + \delta\, d(x,y)\ for\ all\ \ x,y \in X,\ and\ \ \mu,\delta \in [\ 0,1)$. Then T has unique fixed point in X.

Theorem 3.7: In 1977, the mathematician Jaggi [14] introduced the rational expression first
$d(Tx,Ty) \le \beta\ \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \delta d(x,y)$ for all $x,y \in X, x \ne y\ ,\ \ \beta,\delta \in [\ 0,1)$ and $0 \le\ \delta + \beta < 1$. Then T has unique fixed point in X.

Theorem 3.8: In 1980 the mathematicians Jaggi and Das [15] obtained some fixed point theorems with the mapping satisfying:
$d(Tx,Ty) \le \alpha\ d(x,y) + \beta\ \frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(y,Tx)+d(x,Ty)}$ for all $x,y \in X, x \ne y\ ,\ \ \beta,\delta \in [\ 0,1)$ and $0 \le\ \delta + \beta < 1$. Then T has unique fixed point in X.
These are extensions of Banach contraction principle [1] in terms of a new symmetric rational expression. Recently many other mathematicians viz. Dubey and Pathak [8], Murthy and Sharma [19], Rani and Chug [23] Nair and Shrivastwa [20], Kundu and Tiwari [17], Imdad and Khan [12], Yadava, Rajput and Bhardwaj [27], Bhardwaj, Rajput and Yadava [2], Choudhary, Wadhwa and Bhardwaj [4] Singh, Kumar and Hashim [27] gave very valuable results in complete metric spaces.

In the present paper we shall establish some unique fixed point and common fixed point theorems, through new symmetric rational expressions in complete metric spaces for altering distance function. Our theorems include the fundamental result of Banach [1], Kanan [16], Fisher [9], Reich [22], Chatterjee [3] and Ciric [5]. In second part we prove some fixed point and common fixed point theorems in 2- metric spaces which is motivated by Sharma and Iskey [26]

MAIN RESULTS

Theorem 3.10: Let T be a continuous self mapping defined on a complete metric space. A function $\psi : \Re_+ \to \Re_+$ is an altering distance function. Further T satisfies the following conditions:

$$\psi d(Tx,Ty) \le \alpha\,\psi\left[\frac{d(x,Tx)d(y,Ty)d(x,Ty)+d(x,y)}{1+d(x,Ty)d(y,Ty)d(x,y)}\right]+\beta\psi[\,d(x,Tx)+d(y,Ty)]$$
$$+\gamma\psi[d(x,Ty)+\ d(y,Tx)]+\delta\ d(x,y) \tag{3.10.1}$$

For all $x,y \in\ X, x\ \neq\ y$, and for some $\alpha,\beta,\gamma,\delta \in [0,1)$ with $\alpha+2\beta+2\gamma+\delta<1$, then T has a unique fixed point in X.

Proof: Let x_0 be an arbitrary point in X, and we define a sequence $\{x_n\}$ such that $T^n x_0 = x_n$, where n is positive integer.

If $x_n = x_{n+1}$ for some n then x_n is a fixed point of T. Taking $x_n \neq x_{n+1}$, for all n

Now

$$d(x_{n+1},x_n)=d(Tx_n,Tx_{n-1})$$

$$\psi d(Tx_n,Tx_{n-1})\ \le \alpha\psi\left[\frac{\begin{array}{c}d(x_n,Tx_n)d(x_{n-1},Tx_{n-1})d(x_n,Tx_{n-1})\\ +d(x_n,x_{n-1})\end{array}}{1+d(x_n,Tx_{n-1})d(x_{n-1},Tx_{n-1})d(x_n,x_{n-1})}\right]$$
$$+\ \beta\psi[\,d(x_n,Tx_n)+d(x_{n-1},Tx_{n-1})]+\ \gamma\psi[d(x_n,Tx_{n-1})+\ d(x_{n-1},Tx_n)]$$
$$+\delta\ \psi d(x_n,x_{n-1})$$

$$\psi d(x_{n+1},x_n)\le \alpha\psi\left[\frac{\begin{array}{c}d(x_n,x_{n+1})d(x_{n-1},x_n)d(x_n,x_n)\\ +d(x_n,x_{n-1})\end{array}}{1+d(x_n,x_n)d(x_{n-1},x_n)d(x_n,x_{n-1})}\right]$$
$$+\ \beta\psi[\,d(x_n,x_{n+1})+d(x_{n-1},x_n)]+\gamma\psi[d(x_n,x_n)+d(x_{n-1},x_{n+1})]+\delta\ \psi d(x_n,x_{n-1})$$

$$\psi d(x_{n+1},x_n)\le \alpha\psi[d(x_n,x_{n-1})\,]+\beta\psi[\,d(x_n,x_{n+1})+d(x_{n-1},x_n)]+\gamma\psi d(x_{n-1},x_{n+1})+\psi\delta\ d(x_n,x_{n-1})$$

$$\psi d(x_{n+1},x_n)\le(\beta+\gamma+\delta+\alpha)\psi d(x_{n-1},x_n)+(\beta+\gamma)\psi d(x_{n+1},x_n)$$

$$\psi d(x_{n+1},x_n)\le\frac{(\alpha+\beta+\gamma+\delta)}{1-(\beta+\gamma)}\psi d(x_{n-1},x_n)$$

$$\psi d(x_{n+1},x_n)\le k\psi\ d(x_{n-1},x_n)$$

$$where\ k=\frac{(\alpha+\beta+\gamma+\delta)}{1-(\beta+\gamma)}<1$$

On the same way we can write,

$$\psi d(x_{n+1},x_n)\le k^n\ \psi d(x_0,x_1)$$

By triangular inequality, we have, for $m\ >\ n$

$$\psi d(x_n,x_m)\le\psi\ d(x_n,x_{n+1})+\ \psi\ d(x_{n+1},x_{n+2})+\ \dots\dots..+\psi\ \ d(x_{m-1},x_m)$$

$$\psi d(x_n,x_m)\le[k^n+k^{n+1}+k^{n+2}+\cdots\dots..+k^{m-1}]\psi d(x_0,x_1)$$

$$\psi d(x_n,x_m)\le\ \frac{k^n}{1-k}\psi d(x_0,x_1)$$

Therefore

$$\psi d(x_n,x_m)\le\ \frac{k^n}{1-k}\psi d(x_0,x_1)\ \ \to 0\ \text{ as } m,n\to\infty$$

So $\{x_n\}$ is Cauchy sequence in X. So by completeness of X there is a point $u \in X$ such that $x_n \to u$, as $n \to \infty$. Further the continuity of T in X implies

$$\psi T(u)=T\lim_{n\to\infty}x_n=\ \lim_{n\to\infty}\psi T(x_n)=\lim_{n\to\infty}x_{n+1}=u$$

Therefore u is fixed point of T in X. Now if there is any other $v\ (\neq u)$ in X such that $T(v) = v$, then

$$d(u,v) = d(Tu,Tv)$$

$$\psi d(Tu,Tv) \leq \alpha\psi\left[\frac{d(u,Tu)d(v,Tv)d(u,Tv)+d(u,v)}{1+d(u,Tv)d(v,Tv)d(u,v)}\right] + \beta\psi[\,d(u,Tu) + d(v,Tv)]$$
$$+ \gamma\psi[d(u,Tv) + d(v,Tu)] + \delta\psi\, d(u,v)$$

$$\psi d(u,v) \leq (2\,\gamma + \delta)\psi\, d(u,v)$$

Which is contradiction, because $(2\,\gamma + \delta) < 1$ .Hence u is the unique fixed point of T.

We now prove another theorem in which T is not necessarily continuous on X. but T^{p} is continuous for some positive integer p , in fact we prove

Theorem 3.11: Let T be self map defined on a complete metric space (X,d) such that (3.10.1) holds. A function $\psi : \Re_+ \to \Re_+$ is an altering distance function. If for some positive integer p , T^p is continuous, then T has a unique fixed point.

Proof: We define sequence $\{x_n\}$ as in theorem 3.10. Clearly it converges to some fixed point u in X. Therefore its subsequence $\{x_{n_k}\}$, $(n_k = k_p)$also converges to u.

Also $T_u{}^p = T^p(\lim_{k\to\infty} x_{n_k}) = \lim_{k\to\infty} T^p(x_{n_k}) = \lim\limits_{k\to\infty}(x_{n_{k+1}}) = u$

Therefore u is fixed point of T^p .Now we show that $Tu = u$. Let m be the smallest positive integer such that $T^m u = u$ but $T^q u \neq u$, for $q = 1,2,3,4 \dots, m-1$. If $m > 1$ then by (3.10.1)

$$\psi d(Tu,u) = \psi d(Tu\,,T^m u)$$

$$\psi d(Tu,u) = \psi d\big(Tu,T(T^{m-1}u)\big)$$

$$\psi d\big(Tu,T(T^{m-1}u)\big) \leq \alpha\psi\left[\frac{d(u,Tu)d\big((T^{m-1}u),T(T^{m-1}u)\big)d\big(u,T(T^{m-1}u)\big) + d\big(u,(T^{m-1}u)\big)}{1+d\big(u,T(T^{m-1}u)\big)d\big((T^{m-1}u),T(T^{m-1}u)\big)}\right]$$
$$+ \beta\psi\big[\,d(u,Tu) + d\big((T^{m-1}u),T(T^{m-1}u)\big)\big]$$
$$+ \gamma\psi\big[d\big(u,T(T^{m-1}u)\big) + d\big((T^{m-1}u),Tu\big)\big] + \delta\,\psi d\big(u,(T^{m-1}u)\big)$$

$$\psi d\big(Tu,T(T^{m-1}u)\big) \leq \alpha\psi\left[\frac{d(u,Tu)d\big((T^{m-1}u),u\big)d(u,u) + d\big(u,(T^{m-1}u)\big)}{1+d(u,u)\,d\big((T^{m-1}u),u\big)d\big(u,(T^{m-1}u)\big)}\right]$$
$$+ \beta\psi\big[\,d(u,Tu) + d\big((T^{m-1}u),u\big)\big] + \gamma\psi\big[d(u,u) + d\big((T^{m-1}u),Tu\big)\big]$$
$$+\delta\,\psi d\big(u,(T^{m-1}u)\big)$$

$$\psi d\big(Tu,T(T^{m-1}u)\big) \leq \alpha\psi\big[d\big(u,(T^{m-1}u)\big)\big]$$
$$+ \beta\psi\big[\,d(u,Tu) + d\big((T^{m-1}u),u\big)\big]$$
$$+ \psi\,\gamma\big[d(u,u) + d\big((T^{m-1}u),Tu\big)\big] + \delta\,\psi d\big(u,(T^{m-1}u)\big)$$

$$\psi d\big(Tu,T(T^{m-1}u)\big) \leq \alpha\psi\, d(u,T^{m-1}u) + \beta\psi\big[\,d(u,Tu) + d\big((T^{m-1}u),u\big)\big]$$
$$+ \gamma\psi\big[d(u,Tu) + d\big((T^{m-1}u),Tu\big)\big] + \delta\psi\, d\big(u,(T^{m-1}u)\big)$$

$$\psi d(Tu,u) \leq (\beta+\gamma)\psi d(u,Tu) + (\alpha+\beta+\gamma+\delta)\psi d(T^{m-1}u,u)$$

This implies

$$\psi d(Tu,u) \leq \frac{(\alpha+\beta+\gamma+\delta)}{1-(\beta+\gamma)}\psi d(T^{m-1}u,u)$$

$$\psi d(Tu,u) \leq q.\psi d(T^{m-1}u,u)$$

Where $q = \frac{(\alpha+\beta+\gamma+\delta)}{1-(\beta+\gamma)} < 1$

Since $\alpha + 2\beta + 2\gamma + \delta < 1$, thus we write

$$\psi d(u, Tu) \leq q^m \psi d(u, Tu)$$

Since $k^m < 1$, therefore $d(u, Tu) \leq \ d(u, Tu)$, this is contradiction. Hence $Tu = u$, i.e. u is fixed point of T. The uniqueness of u follows as in theorem 3.10.

We further generalize the result of theorem-3.10 in which T is neither continuous nor satisfies (3.10.1). In what follows T^m , for some positive integer m, satisfying the same rational expression and continuous, still T has unique fixed point .In fact we prove,

Theorem 3.12: A function $\psi : \Re_+ \to \Re_+$ is an altering distance function. Let T be self map defined on a complete metric space (X, d) such that for some positive integer m, satisfy the condition:

$$\psi d(T^m x, T^m y) \leq \alpha\psi\left[\frac{d(x,T^m x)d(y,T^m y)d(x,T^m y)+d(x,y)}{1+d(y,T^m y)d(y,T^m y)d(x,y)}\right] + \beta\psi[\, d(x, T^m x) + d(y, T^m y)] + \gamma\psi[d(x, T^m y) + d(y, T^m x)] + \delta\, \psi d(x, y)$$

For all $x, y \in X,\ x \neq y$, and for some $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + 2\beta + 2\gamma + \delta < 1$. If T^m is continuous, then T has a unique fixed point.

Proof: By theorem 3.11, we assume that T^m has unique fixed point.

Also $\quad \psi Tu = \psi T(T^m u) = \psi T^m(Tu)$

Which implies $Tu = u$, further since a fixed point of T is also a fixed point of T^m and T^m has a unique fixed point u, it follows that u is a unique fixed point of T.

SOME FIXED POINT THEOREMS IN 2-METRIC SPACES

Theorem 3.13: Let T be a continuous self map defined on a complete 2-metric space. A function $\psi : \Re_+ \to \Re_+$ is an altering distance function. Further T satisfies the following conditions:

$$\psi d(Tx, Ty, a) \leq \alpha\psi\left[\frac{d(x,Tx,a)d(y,Ty,a)d(x,Ty,a)+d(x,y,a)}{1+d(x,Ty,a)d(y,Ty,a)d(x,y,a)}\right] + \beta\psi[\, d(x, Tx, a) + d(y, Ty, a)] + \gamma\psi[d(x, Ty, a) + d(y, Tx, a)] + \delta\, \psi d(x, y, a) \qquad (3.13.1)$$

For all $x, y \in X, x \neq y$, and for some $\alpha, \beta, \gamma, \delta \in [0, 1)$, $\alpha + 2\beta + 2\gamma + \delta < 1$, then T has a unique fixed point in X.

Proof: Let x_0 be an arbitrary point in X, and we define a sequence $\{x_n\}$ such that, $T^n x_0 = x_n$, where n is positive integer.

If $x_n = x_{n+1}$ for some n then x_n is a fixed point of T. Taking $x_n \neq x_{n+1}$, for all n

Now $\ d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1})$

$$\psi d(Tx_n, Tx_{n-1}, a) \leq \alpha\psi\left[\frac{\begin{array}{c}d(x_n,Tx_n,a)d(x_{n-1},Tx_{n-1},a)d(x_n,Tx_{n-1},a)\\ +d(x_n,x_{n-1},a)\end{array}}{1+d(x_n,Tx_{n-1},a)d(x_{n-1},Tx_{n-1},a)d(x_n,x_{n-1},a)}\right] + \beta\psi[\, d(x_n, Tx_n, a) + d(x_{n-1}, Tx_{n-1}, a)] + \gamma\psi[d(x_n, Tx_{n-1}, a) + d(x_{n-1}, Tx_n, a)] + \delta\, \psi d(x_n, x_{n-1}, a)$$

$$\psi\ d(x_{n+1}, x_n, a) \leq \alpha\psi\left[\frac{\begin{array}{c}d(x_n, x_{n+1},a)d(x_{n-1},x_n,a)d(x_n,x_n,a)\\ +d(x_n,x_{n-1},a)\end{array}}{1+d(x_n,x_n,a)d(x_{n-1},x_n,a)d(x_n,x_{n-1},a)}\right] + \beta\psi[\, d(x_n, x_{n+1}, a) + d(x_{n-1}, x_n, a)] + \gamma\psi[d(x_n, x_n, a) + d(x_{n-1}, x_{n+1}, a)] + \delta\, \psi d(x_n, x_{n-1}, a)$$

$$\psi d(x_{n+1}, x_n, a) \leq \alpha\psi d(x_n, x_{n-1}, a) + \beta\psi[\, d(x_n, x_{n+1}, a) + d(x_{n-1}, x_n, a)] + \gamma\psi d(x_{n-1}, x_{n+1}, a) + \delta\psi\ d(x_n, x_{n-1}, a)$$

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$$\psi d(x_{n+1}, x_n, a) \leq (\alpha + \ \beta + \gamma + \delta)\psi d(x_{n-1}, x_n, a) + (\beta + \gamma)\psi d(x_{n+1}, x_n, a)$$

$$\psi d(x_{n+1}, x_n, a) \le \frac{(\alpha+\beta+\gamma+\delta)}{1-(\beta+\gamma)} \psi d(x_{n-1}, x_n, a)$$

$$\psi d(x_{n+1}, x_n, a) \le k\psi\, d(x_{n-1}, x_n, a)$$

$$where\ k = \frac{(\alpha+\beta+\gamma+\delta)}{1-(\beta+\gamma)} < 1$$

On the same way we can write,

$$\psi d(x_{n+1}, x_n, a) \le k^n \psi\, d(x_0, x_1, a)$$

By triangular inequality, we have, for $m > n$

$$\psi d(x_n, x_m, a) \le \psi\, d(x_n, x_{n+1}, a) + \psi\, d(x_{n+1}, x_{n+2}, a) + \dots\dots.. + \psi\, d(x_{m-1}, x_m, a)$$

$$\psi d(x_n, x_m, a) \le [k^n + k^{n+1} + k^{n+2} + \cdots \dots.. + k^{m-1}]\psi d(x_0, x_1, a)$$

$$\psi d(x_n, x_m, a) \le \frac{k^n}{1-k} \psi d(x_0, x_1, a)$$

Therefore

$$\psi d(x_n, x_m, a) \le \tfrac{k^n}{1-k} \psi d(x_0, x_1, a) \ \to 0 \text{ as } \mathrm{m}, \mathrm{n} \to \infty$$

So $\{x_n\}$ is Cauchy sequence in X. So by completeness of X there is a point $u \in X$ such that $x_n \longrightarrow u$, as $n \to \infty$. Further the continuity of T in X implies

$$\psi T(u) = \psi T \lim_{n\to\infty} x_n = \lim_{n\to\infty} \psi T(x_n) = \lim_{n\to\infty} \psi x_{n+1} = u$$

Therefore u is fixed point of T in X. Now if there is any other $v\ (\neq u)$ in X such that $T(v) = v$, then

$$\psi d(u, v, a) = d(Tu, Tv, a)$$

$$d(Tu, Tv, a) \le \alpha\psi \left[\frac{\begin{matrix} d(u, Tu, a) d(v, Tv, a) d(u, Tv, a) \\ + d(u, v, a)\end{matrix}}{1 + d(u, Tv, a) d(v, Tv, a) d(u, v, a)}\right]$$
$$+ \beta\psi[\, d(u, Tu, a) + d(v, Tv, a)] + \gamma\psi[d(u, Tv, a) + d(v, Tu, a)] + \delta\, \psi d(u, v, a)$$

$$\psi d(u, v, a) \le (\alpha + 2\,\gamma + \delta)\, \psi d(u, v, a)$$

Which is contradiction, because $(\alpha + 2\,\gamma + \delta) < 1$. Hence u is the unique fixed point of T.

We now prove another theorem in which T is not necessarily continuous on X. But $T^{\boldsymbol{p}}$ is continuous for some positive integer p , in fact we prove

Theorem 3.14: Let T be a self mapping defined on a complete $2-$metric space (X, d) such that (3.13.1) holds. A function $\psi : \Re_+ \to \Re_+$ is an altering distance function. If for some positive integer p , T^p is continuous, then T has a unique fixed point.

Proof: We define sequence $\{x_n\}$ as in theorem 3.13 Clearly it converges to some fixed point u in X. therefore its subsequence $\{x_{n_k}\}$, $(\, n_k = k_p)$ also converges to u.

Also $\quad \psi T_u{}^p = \psi T^p(\lim_{k\to\infty} x_{n_k}) = \lim_{k\to\infty} \psi T^p(x_{n_k}) = \lim_{k\to\infty} \psi(x_{n_{k+1}}) = u$

Therefore u is fixed point of T^p .Now we show that Tu =u. Let m be the smallest positive integer such that $T^m u = u$ but $T^q u \neq u$, for q = 1, 2, 3, 4…, m-1. If $m > 1$ then by (3.13.1)

$$\psi d(Tu, u, a) = \psi d(Tu, T^m u, a)$$

$$\psi d(Tu, u, a) = \psi d(Tu, T(T^{m-1}u), a)$$

$$\psi d(Tu, T(T^{m-1}u), a) \le \alpha\psi\left[\frac{d(u,Tu,a)d(T^{m-1}u,T(T^{m-1}u),a)d(u,T(T^{m-1}u),a)+d(u,(T^{m-1}u),a)}{1+d(u,T(T^{m-1}u),a)d(T^{m-1}u,T(T^{m-1}u),a)d(u,(T^{m-1}u),a)}\right]$$
$$+\beta\psi\left[d(u,Tu,a)+d\left((T^{m-1}u),T(T^{m-1}u),a\right)\right]$$
$$+\gamma\psi\left[d(u,T(T^{m-1}u),a)+d\left((T^{m-1}u),Tu,a\right)\right]+\delta\,\psi d(u,(T^{m-1}u),a)$$

$$\psi d(Tu, T(T^{m-1}u), a) \le \alpha\psi d(u,(T^{m-1}u),a)$$
$$+\beta\psi\left[d(u,Tu,a)+d\left((T^{m-1}u),u,a\right)\right]$$
$$+\gamma\psi[d(u,u,a)+d(u,Tu,a)]+\delta\psi\, d(u,(T^{m-1}u),a)$$

$$\psi d(Tu, T(T^{m-1}u), a) \le \alpha\psi d(u,(T^{m-1}u),a)$$
$$+\beta\psi\left[d(u,Tu,a)+d\left((T^{m-1}u),u,a\right)\right]+\gamma\psi[d(u,Tu,a)]\;+\delta\psi\, d(u,(T^{m-1}u),a)$$

$$\psi d(Tu, T(T^{m-1}u), a) \le \alpha\psi d(u,T^{m-1}u,a)$$
$$+\beta\psi\left[d(u,Tu,a)+d\left((T^{m-1}u),u,a\right)\right]+\gamma\psi[d(u,Tu,a)]\;+\delta\, d(u,(T^{m-1}u),a)$$

$$\psi d(Tu,u,a) \le (\beta+\gamma)\psi d(u,Tu,a)+(\alpha+\beta+\gamma+\delta)\psi d(T^{m-1}u,u,a)$$

$$\psi d(Tu,u,a) \le \frac{(\alpha+\beta+\gamma+\delta)}{1-(\beta+\gamma)}\,\psi d(T^{m-1}u,u,a)$$

This implies
$$\psi d(Tu,u,a) \le k\psi\, d(T^{m-1}u,u,a)$$

Where $k = \frac{(\alpha+\beta+\gamma+\delta)}{1-(\beta+\gamma)}$

Since $\alpha+2\beta+2\gamma+\delta<1$, thus we write
$$\psi d(Tu,u,a) \le k^m\psi d(Tu,u,a)$$

Since $k^m<1$ $k^m<1$, therefore $\psi d(Tu,u,a) \le \psi d(Tu,u,a)$ this is contradiction. Hence $Tu=u$

$i.e.$ u is fixed point of T. The uniqueness of u follows as in theorem 3.1.

We further generalize the result of theorem-3.13 in which T is neither continuous nor satisfies (3.13.1). In what follows T^m, for some positive integer m, satisfying the same (3.13.1) rational expression and continuous, still T has unique fixed point .In fact we prove,

Theorem 3.15: A function $\psi:\Re_+\to\Re_+$ is an altering distance function. Let T be a self mapping defined on a complete 2-metric space (X, d) such that for some positive integer m, satisfy the condition:

$$\psi d(T^m x, T^m y, a) \le \alpha\psi\left[\frac{d(x,T^m x,a)d(y,T^m y,a)d(x,T^m y,a)+d(x,y,a)}{1+d(x,T^m y,a)d(y,T^m y,a)d(x,y,a)}\right]$$
$$+\beta\psi[d(x,T^m x,a)+d(y,T^m y,a)]$$
$$+\gamma\psi[d(x,T^m y,a)+d(y,T^m x,a)]+\delta\psi\, d(x,y,a)$$

For all $x,y\in X, x\neq y$, and for some α, β, γ, δ ≥ 0 with $\alpha+2\beta+2\gamma+\delta<1$. $a>0$, If T^m is continuous, then T has a unique fixed point.

Proof: By theorem 3.14 we assume that T^m has unique fixed point.

Also $\psi Tu=\psi T(T^m u)=\psi T^m(Tu)$

Which implies $Tu=u$, further since a fixed point of T is also a fixed point of T^m and T^m has a unique fixed point u, it follows that u is a unique fixed point of T.

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