COMPARATIVE STUDY OF HIGHER DEGREE SPLINE SOLUTIONS OF LINEAR SEVENTH ORDER BOUNDARY VALUE PROBLEMS

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ABSTRACT

In this paper tenth degree spline approximations are developed following cubic spline Bickley's procedure for the numerical solutions of linear boundary value problems of order seven. Two numerical examples have been considered to illustrate the efficiency and implementation of the method. Numerical solutions are computed at different step lengths and compared with exact solutions. Absolute errors are also calculated. The results are tabulated and pictorially illustrated. Further the results are compared with the methods developed in [10 and 11].

Key words: Spines; Boundary value problems; Numerical results.

1. INTRODUCTION

Boundary value problems (BVPs) for ordinary differential equations (ODEs) are used as mathematical models in a wide variety of disciplines including biology, physics, and engineering [3]. Over the years, there are several authors who study on these types of boundary value problems. There are several methods, such as Finite difference method, Finite element method, Orthogonal splines, Collocation, and etc, to discrete the boundary value problems. Boundary value problems (BVPs) can be used to model several physical phenomena. For example, when an infinite horizontal layer of fluid is heated from below and is subject to the action of rotation, instability sets in. When this instability sets in as ordinary convection, the ordinary differential equation is sixth order and when this instability sets in as over stability, it is modeled by an eighth-order ordinary differential equation [15].

The theory of seventh order boundary value problems is not much available in the numerical analysis literature. The seventh order boundary value problems generally arise in modeling induction motors with two rotor circuits [12]. Siddiqi and Akram [13] presented the solutions of fifth and sixth order boundary value problems using non-polynomial spline. Mohyud-Din *et al.* [9] used variation of parameters method for solving fifth order boundary value problems. In [1] Akram and Rehman solved the fifth order boundary value problems using Reproducing Kernel space method. Akram and Siddiqi [2] presented the solution of seventh order boundary value problem using octic spline. Rashidina *et al.* [8] used spline collocation method for the solution of higher order linear boundary value problems. Sastry [14], Schoenberg [5, 6] made great contributions in the development of splines. Convergence properties of the cubic spline method have been discussed by Ahlberg and Nilson [7]. Numerical solution of boundary-value and initial boundary-value problems using spline functions has been given in [4].

In this work we construct tenth degree splines by using Bickley's method [16] and apply it to the linear seventh order Boundary value problem. The work has been illustrating through examples with different step lengths.

2. CONSTRUCTION OF TENTH DEGREE SPLINE FUNCTION

Assume the interval $[x_0, x_n]$ is divided in to n sub intervals with grid points $x_0, x_1, x_2, x_3, \dots x_n$

The function $u(x)$ in the interval $[x_0, x_1]$, is represent by tenth degree spline in the form
$$s(x) = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3 + g(x - x_0)^4 + j(x - x_0)^5 + k(x - x_0)^6$$
$$+ l(x - x_0)^7 + t(x - x_0)^8 + v(x - x_0)^9 + w_0(x - x_0)^{10}. \tag{1}$$

For the next interval $[x_1, x_2]$ we add a term $w_1(x - x_1)^{10}$,

Going on in to the next interval $[x_2, x_3]$ add another term $w_2(x - x_2)^{10}$ and so on until we reach x_n hence the function $s(x)$ represents in the form

$$s(x) = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3 + g(x - x_0)^4 + j(x - x_0)^5 + k(x - x_0)^6$$
$$+ l(x - x_0)^7 + t(x - x_0)^8 + v(x - x_0)^9 + \sum_{i=0}^{n-1} w_i \ (x - x_i)^{10}. \tag{2}$$

$$s^{(7)}(x) = 5040l + 40320t(x - x_0) + 181440v(x - x_0)^2 + 604800 \sum_{i=0}^{n-1} w_i \ (x - x_i)^3 \tag{3}$$

2.1 Method of obtaining the solution of seventh order boundary value problem using tenth degree spline function

Consider the linear seventh order differential equation
$$y^{(7)}(x) + f(x)y(x) = r(x), \tag{4}$$

with boundary conditions
$$y(x_0) = \beta, \ y(x_n) = \gamma, \ y'(x_0) = \beta', \ y'(x_n) = \gamma', \ y''(x_0) = \beta'', \ y''(x_n) = \gamma'', \ y'''(x_0) = \beta''', \tag{5}$$

From (5), and taking spline approximation in (4) at $x = x_i$ for $i = 0, 1, 2, 3, 4, 5, 6, \ldots, n$ we get $(n + 8)$ equations in $(n + 10)$ unknowns $a, b, c, d, g, j, k, l, t, v, w_0, w_1, w_2, w_3, w_4 \ldots, w_{n-1}$.

After determining these unknowns we substitute them in (2) and thus get the tenth degree spline approximation of $y(x)$. Substituting (2) and (3) in the differential equation (4) at $x = x_m$ where $m = 0, 1, 2, 3, \ldots n$ then we get the following

$$af(x_m) + bf(x_m)(x_m - x_0) + cf(x_m)(x_m - x_0)^2 + df(x_m)(x_m - x_0)^3 + gf(x_m)(x_m - x_0)^4$$
$$+ jf(x_m)(x_m - x_0)^5 + kf(x_m)(x_m - x_0)^6 + l[f(x_m)(x_m - x_0)^7 + 5040]$$
$$+ t[f(x_m)(x_m - x_0)^8 + 40320(x_m - x_0)] + v[f(x_m)(x_m - x_0)^9 + 181440(x_m - x_0)^2]$$
$$+ \sum_{i=0}^{n-1} w_i \ [f(x_m)(x_m - x_i)^{10} + 604800(x_m - x_i)^3] = s(x_m). \tag{6}$$

From the boundary conditions (5) we obtain
$$y(x_0) = \beta,$$

$$\beta = a + b(x - x_0) + c(x - x_0)^2 + d(x - x_0)^3 + g(x - x_0)^4 + j(x - x_0)^5 + k(x - x_0)^6$$
$$+ l(x - x_0)^7 + t(x - x_0)^8 + v(x - x_0)^9 + \sum_{i=0}^{n-1} w_i \ (x - x_i)^{10},$$

$$a = \beta. \tag{7}$$

$$y(x_n) = \gamma,$$

$a + b(x_n - x_0) + c(x_n - x_0)^2 + d(x_n - x_0)^3 + g(x_n - x_0)^4 + j(x_n - x_0)^5 + k(x_n - x_0)^6 + l(x_n - x_0)^7 + t x_n - x_0^8 + v x_n - x_0^9 + \sum_{i=0}^{n-1} w_i \ x_n - x_i^{10} = \gamma.$ (8)

$$y'(x_0) = \beta',$$

$$\beta' = b + 2c(x - x_0) + 3d(x - x_0)^2 + 4g(x - x_0)^3 + 5j(x - x_0)^4 + 6k(x - x_0)^5$$
$$+ 7l(x - x_0)^6 + 8t(x - x_0)^7 + 9v(x - x_0)^8 + 10 \sum_{i=0}^{n-1} w_i \ (x - x_i)^9,$$

$$b = \beta'. \tag{9}$$

$$y'(x_n) = \gamma',$$

$$b + 2c(x_n - x_0) + 3d(x_n - x_0)^2 + 4g(x_n - x_0)^3 + 5j(x_n - x_0)^4 + 6k(x_n - x_0)^5$$
$$+ 7l(x_n - x_0)^6 + 8t(x_n - x_0)^7 + 9v(x_n - x_0)^8 + 10 \sum_{i=0}^{n-1} w_i \ (x_n - x_i)^9 = \gamma'. \tag{10}$$

$$y''(x_0) = \beta'',$$

$$\beta'' = 2c + 6d(x - x_0) + 12g(x - x_0)^2 + 20j(x - x_0)^3 + 30k(x - x_0)^4 + 42l(x - x_0)^5$$
$$+ 56t(x - x_0)^6 + 72v(x - x_0)^7 + 90 \sum_{i=0}^{n-1} w_i \ (x - x_i)^8,$$

$$2c = \beta''. \tag{11}$$

$$y''(x_n) = \gamma'',$$

$$\gamma'' = 2c + 6d(x - x_0) + 12g(x - x_0)^2 + 20j(x - x_0)^3 + 30k(x - x_0)^4 + 42l(x - x_0)^5$$
$$+56t(x - x_0)^6 + 72v(x - x_0)^7 + 90\sum_{i=0}^{n-1} w_i (x - x_i)^8,$$
$$2c + 6d(x_n - x_0) + 12g(x_n - x_0)^2 + 20j(x_n - x_0)^3 + 30k(x_n - x_0)^4 + 42l(x_n - x_0)^5$$
$$+56t(x_n - x_0)^6 + 72v(x_n - x_0)^7 + 90\sum_{i=0}^{n-1} w_i (x - x_i)^8 = \gamma''. \tag{12}$$

$$y'''(x_0) = \beta''',$$

$$\beta''' = 6d + 24g(x - x_0) + 60j(x - x_0)^2 + 120k(x - x_0)^3 + 210l(x - x_0)^4$$
$$+336t(x - x_0)^5 + 504v(x - x_0)^6 + 720\sum_{i=0}^{n-1} w_i (x - x_i)^7,$$

$$6d = \beta'''. \tag{13}$$

From the boundary conditions and taking spline approximation in the linear seventh order boundary value problem at $x = x_i$ for $i = 0,1,2,3,4,5,...,n$ in $(n + 8)$ equations and assuming $w_{n-1} = w_{n-2} = w_{n-3}$ (the last three unknowns are equal), we determine the unknowns.

3. NUMERICAL RESULTS

In this part we consider two linear boundary value problems. The numerical solution and absolute errors are given at different step lengths. The approximate solution, exact solution and absolute errors at the grid points are summarized in the tabular form and shown graphically.

Example 1: Consider the linear seventh order boundary value problem
$$u^{(7)}(x) = -u(x) - e^x(35 + 12x + 2x^2), \quad 0 \le x \le 1, \tag{14}$$
with boundary conditions
$$u(0) = 0, \; u'(0) = 1, \; u^{(2)}(0) = 0, u^{(3)}(0) = -3$$
$$u(1) = 0, \; u'(1) = -e, \; u^{(2)}(1) = -4e.$$

The exact solution of the example 1 is $u(x) = x(1 - x)e^x$

Solution with $h = 0.2$
The values of $s(x)$, $u(x)$ and the corresponding absolute errors at x_1, x_2, x_3, x_4 have been given in the Table 1 and the comparison has been shown in Figure 1.

Table-1: Numerical solution $s(x)$, exact solution $u(x)$ and absolute error of the example 1 with $h = 0.2$

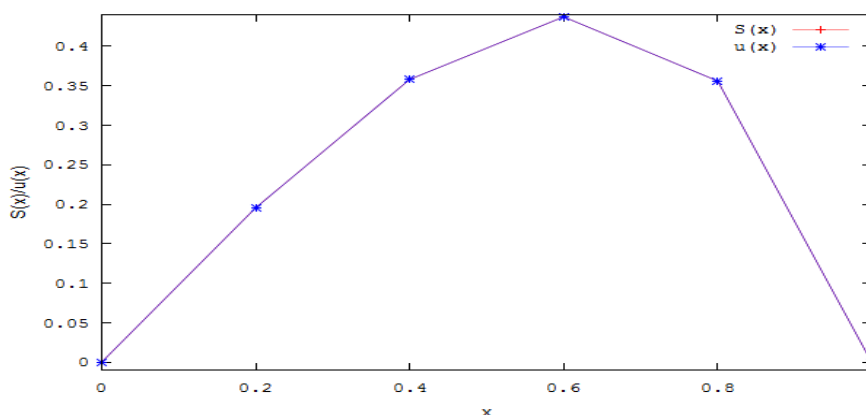| x	$s(x)$	$u(x)$	Absolute error
0.2	0.195424441238452	0.19542444130562	$6.717462697203E - 011$
0.4	0.358037926880896	0.3580379274339	$5.530090274596E - 010$
0.6	0.437308511249002	0.43730851209372	$8.447200161576E - 010$
0.8	0.356086548221534	0.356086548558795	$3.3726099690767E - 010$

Figure-1: Comparison of approximate solution and exact solution for example 1 with $h = 0.2$

Solution with $h = 0.1$

The values of numerical solution *s(x)*, exact solution *u(x)* and absolute errors at the points x_0 , x_1 , x_2, x_3, x_4 , x_5 , x_6 , x_7 , x_8, x_9, x_{11} of example1 with $h = 0.1$ have been given in the table 2 and the comparison has been shown in Figure 2.

Table-2: Numerical solution $s(x)$, exact solution $u(x)$ and absolute error of the example 1 with $h = 0.1$

| x	$s(x)$	$u(x)$	*Absolute error*
0.1	0.0994653826190540	0.0994653826268083	9.02899977006655E-012
0.2	0.195424441218261	0.195424441305627	8.73660033207102E-011
0.3	0.283470349338641	0.283470349590961	2.52319998228501E-010
0.4	0.358037926984394	0.358037927433905	4.49510983990820E-010
0.5	0.412180317056123	0.412180317675032	6.18908979621580E-010
0.6	0.437308511367867	0.437308512093722	7.25854987226882E-010
0.7	0.422888067802474	0.422888068568800	7.66326002654694E-010
0.8	0.356086547786610	0.356086548558795	7.72185038133699E-010
0.9	0.221364279221554	0.221364280004125	7.82571007995614E-010
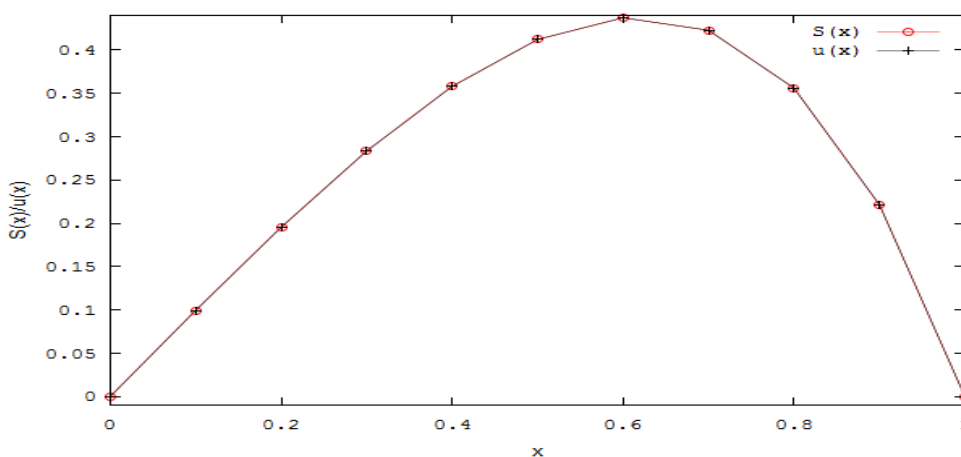
Figure-2: Comparison of approximate solution and exact solution for example 1 with $h = 0.1$

Example 2: The following seventh order linear boundary value problem is considered.

$$u^7(x) = xu(x) + e^x(x^2 - 2x - 6), \quad 0 \le x \le 1 \tag{15}$$

with the boundary conditions

$$u(0) = 1, \quad u'(0) = 0, \quad u''(0) = -1, \quad u^3(0) = -2,$$
$$u(1) = 0, \quad u'(1) = -e, \quad u''(1) = -2e,$$

The analytical solution of the above-mentioned problem is

$$u(x) = (1 - x)e^x$$

Solution with $h = 0.2$

The values of numerical solution *s(x)*, exact solution *u(x)* and absolute errors at the points x_0, x_1, x_2, x_3, x_4 , x_5 of example 2 with $h = 0.2$ have been given in the table 3 and the comparison has been shown in Figure 3.

Table- 3: Numerical solution using tenth degree spline approximation *s(x)*, exact solution *u(x)* and absolute errors of example 2 with $h = 0.2$

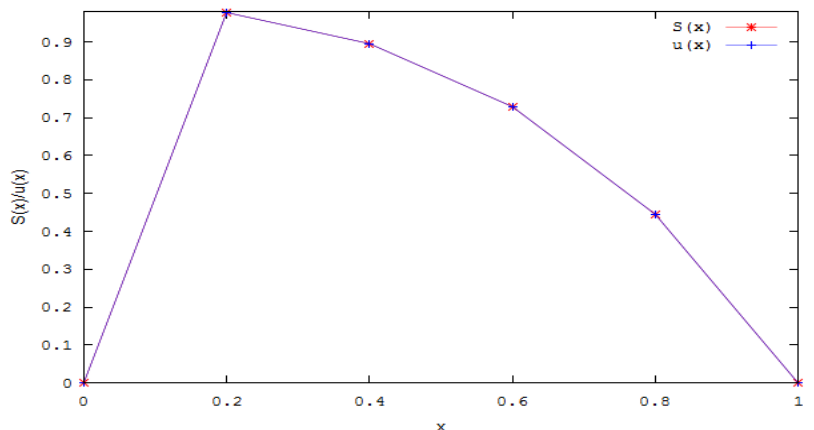| x	$S(x)$	$u(x)$	*Absolute error*
0. 2	0.977122206521071	0.977122206528136	$7.06534830641203E - 012$
0.4	0.895094818530000	0.895094818584762	$5.47620837565432E - 011$
0.6	0.728847520074112	0.728847520156204	$8.20919998645309E - 011$
0.8	0.445108185666043	0.445108185698494	$3.2450986342524E - 011$

Figure 3: Comparison of approximate solution and exact solution for example 2 with $h = 0.2$

Solution with $h = 0.1$

The values of numerical solution $s(x)$, exact solution $u(x)$ and absolute errors at the points x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{11} of example 2 with $h = 0.1$ have been given in the table 4 and the comparison has been shown in Figure 4.

Table- 4: Numerical solution using tenth degree spline approximation $s(x)$, exact solution $u(x)$
and absolute errors of example 2 with $h = 0.1$

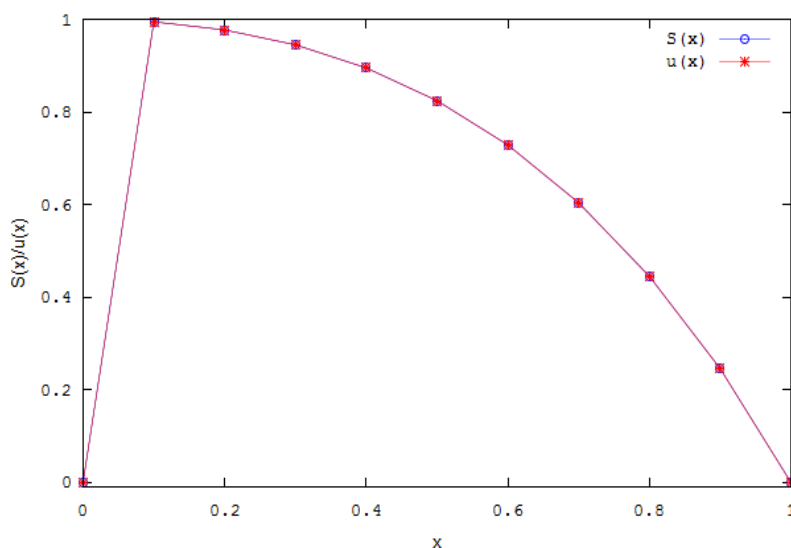| x	$S(x)$	$u(x)$	Absolute error
0.1	0.994653826269063	0.994653826268083	$9.7999386383662E - 013$
0.2	0.977122206538923	0.977122206528136	$1.0787037929560E - 011$
0.3	0.944901165332005	0.944901165303202	$2.8802960017060E - 011$
0.4	0.895094818630416	0.895094818584762	$4.5654258151728E - 011$
0.5	0.824360635403098	0.824360635350064	$5.3034354685621E - 011$
0.6	0.728847520203677	0.728847520156204	$4.7473025510669E - 011$
0.7	0.604125812272944	0.604125812241143	$3.1800895250455E - 011$
0.8	0.445108185712051	0.445108185698494	$1.3557044376000E - 011$
0.9	0.245960311116585	0.245960311115695	$8.8973273193460E - 013$

Figure-4: Comparison of approximate solution and exact solution for example 2 with $h = 0.1$

4. COMPARATIVE STUDY OF TENTH DEGREE WITH EIGHTH AND NINTH DEGREE SPLINE FUNCTIONS

The absolute error associated with yi for examples 1 and 2 of the present method is compared with the eighth and ninth degree spline approximation [10, 11] at different step lengths. Comparison is given in tabular form and shown graphically.

Table-5: Comparison of absolute errors for example 1 at h $= 0.2$

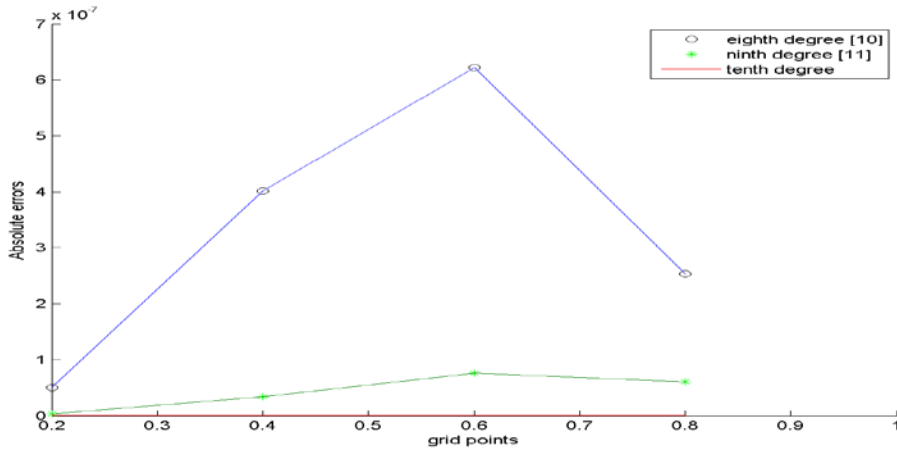| x	Exact solution	Absolute error for $8^{th}degree$[10]	Absolute error for $9^{th}degree$ [11]	Absolute error for $10^{th}degree$
0.2	0.195424441305627	$5.000000e-08$	$3.74400e-09$	$6.7174626972e-011$
0.4	0.358037927433905	$4.016999e-07$	$3.36110e-08$	$5.5300902746e-010$
0.6	0.437308512093722	$6.220000e-07$	$7.56990e-08$	$8.44720016158e-010$
0.8	0.356086548558795	$2.538000e-07$	$6.03450e-08$	$3.37260996908e-010$

Figure-5: Comparison of absolute errors for example 1 at h = 0.2

Table-6: Comparison of absolute errors for example 1 at $h = 0.1$

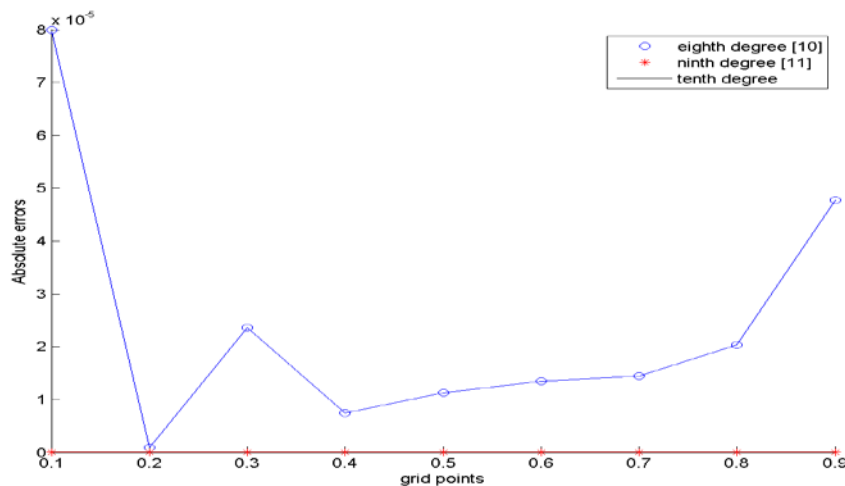| x	Exact solution	Absolute error for 8^{th}degree[10]	Absolute error for 9^{th}degree[11]	Absolute error for 10^{th}degree
0.1	0.09946538262	$7.9999E-08$	$3.3200E-10$	$9.02899977E-012$
0.2	0.19542444130	$9.8000E-07$	$3.5540E-09$	$8.73660033E-011$
0.3	0.283470349	$2.3657E-05$	$1.2366E-08$	$2.523199982E-010$
0.4	0.358037927	$7.4400E-06$	$2.7224E-08$	$4.495109839E-010$
0.5	0.412180317	$1.1289E-05$	$4.3750E-08$	$6.189089796E-010$
0.6	0.437308512	$1.3459E-05$	$5.4648E-08$	$7.258549872E-010$
0.7	0.422888068	$1.4430E-05$	$5.1837E-08$	$7.663260026E-010$
0.8	0.356086547	$2.0369E-05$	$3.3095E-08$	$7.721850381E-010$
0.9	0.221364280	$4.7770E-05$	$8.5130E-09$	$7.825710079E-010$

Figure-6: Comparison of absolute errors for example 1 at h = 0.1

Table-7: Comparison of absolute errors for example 2 at $h = 0.2$

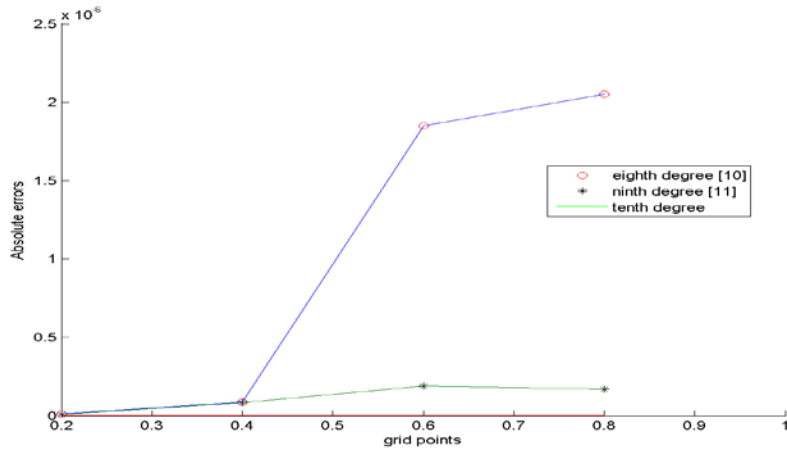| x	Exact solution	Absolute error for $8^{th}degree$[10]	Absolute error for $9^{th}degree$[12]	Absolute error for $10^{th}degree$
0.2	0.9771222	$1.2000E - 08$	$8.6590E - 09$	$7.0653483E - 012$
0.4	0.89509481	$8.6000E - 08$	$8.3135E - 08$	$5.47620837E - 011$
0.6	0.72884752	$1.8500E - 06$	$1.8903E - 07$	$8.20919998E - 011$
0.8	0.44510818	$2.0500E - 06$	$1.6752E - 07$	$3.24509863E - 011$

Figure-7: Comparison of absolute errors for example 2 at $h = 0.2$

Table-8: Comparison of absolute errors for example 2 at $h = 0.2$

| x	Exact solution	Absolute error for 8^{th}degree [10]	Absolute error for 9^{th}degree[1	Absolute error for 10^{th}degre*e*
0.1	0.99465382	$7.99990E - 09$	$9.000E - 10$	$9.79993863E - 013$
0.2	0.97712220	$7.87990E - 08$	$1.410E - 10$	$1.07870379E - 011$
0.3	0.944901165	$2.32699E - 07$	$4.871E - 10$	$2.88029600E - 011$
0.4	0.89509481	$3.85500E - 07$	$3.937E - 09$	$4.56542581E - 011$
0.5	0.82436063	$4.10000E - 07$	$9.280E - 10$	$5.30343546E - 011$
0.6	0.72884752	$1.03900E - 07$	$5.709E - 09$	$4.74730255E - 011$
0.7	0.60412581	$3.87499E - 0.7$	$6.239E - 08$	$3.1800895E - 011$
0.8	0.44510818	$8.12699E - 07$	$1.298E - 08$	$1.35570443E - 011$
0.9	0.24596031	$7.56099E - 07$	$1.048E - 08$	$8.89732731E - 013$
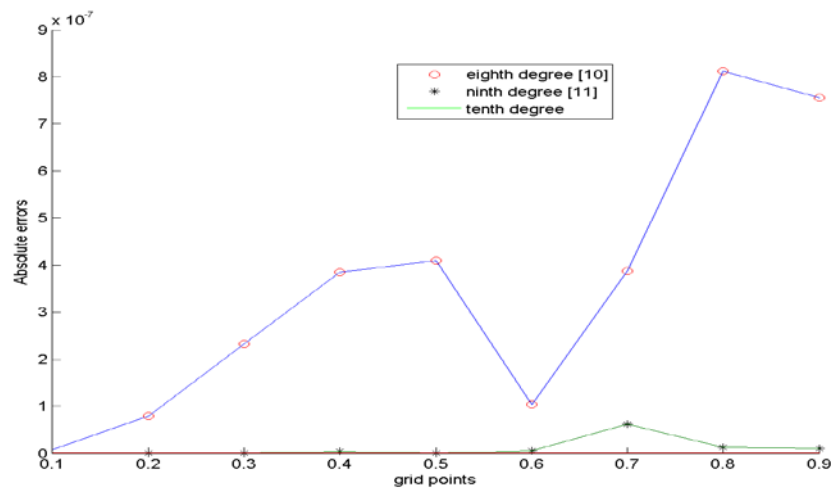
Figure-8: Comparison of absolute errors for example 2 at $h = 0.2$

5. CONCLUSION

In this paper we developed the numerical method to obtain the solution of seventh order linear boundary value problems using tenth degree spline. Tenth degree spline approximation has been employed on two problems at different step lengths. Numerical solution of the problems has been found with $h = 0.2$ and $h = 0.1$. Computational work has been carried out using MATLAB software. Approximate solution, exact solution and absolute errors with $h = 0.2$ and $h = 0.1$ of the example 1 and example 2 are summarized in Table 1, Table 2, Table 3 and Table 4 respectively. The comparison has been shown in Figures 1, 2, 3 and 4 respectively.

The maximum absolute errors at these step length are 6.7174×10^{-11} and 9.0289×10^{-12} for example 1 at $h = 0.2$ and $h = 0.1$ and 7.06534×10^{-12}, and 9.79993×10^{-13} for example 2 at $h = 0.2$ and $h = 0.1$ respectively. From this we understand that there is good agreement with the exact solution. It is also observed that the approximate solution is more close to the exact solution when h is small. Presented method is compared with the eighth and ninth degree spline approximation [10, 11] at different step lengths. It may be noted from Tables that the presented method is more efficient.

It is observed that the absolute errors are better than the methods in [10, 11]. It is also observed that our proposed method is well suited for the solution of higher order boundary value problems and reduces the computational work. Spline approximations converge to exact solutions more rapidly as the degree increases, if the step length decreases the numerical solution is more accurate.

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