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PRIME GAMMA RINGS WITH CENTRALIZING AND COMMUTING LEFT GENERALIZED DERIVATIONS

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ABSTRACT

Let M be a prime Γ-ring satisfying a certain assumption and D a nonzero derivation on M. Let $f: M \to M$ be a left generalized derivation such that f is centralizing and commuting on a left ideal J of M. Then we prove that M is commutative.

Key words: *Prime Γ-ring, Centralizing and Commuting, Derivation, Left derivation, Generalized derivations, Left generalized derivations.*

PRELIMINARIES

Let M and Γ be additive abelian groups. If there exists a mapping $(x, \alpha, y) \to x\alpha y$ of $M \times \Gamma \times M \to M$, which satisfies the conditions

(i) $x\alpha y \in M$
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z, x(\alpha + \beta)z = x\alpha z + x\beta z, x\alpha(y + z) = x\alpha y + x\alpha z$
(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, then M is called a Γ-ring.

Every ring M is a Γ-ring with $M = \Gamma$. However a Γ-ring need not be a ring. Let M be a Γ-ring. Then an additive subgroup U of M is called a left (right) ideal of M if $M\Gamma U \subset U (U\Gamma M \subset U)$. If U is both a left and a right ideal, then we say U is an ideal of M. Suppose again that M is a Γ-ring. Then M is said to be a 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in M$. An ideal P_1 of a Γ-ring M is said to be prime if for any ideals A and B of M, $A\Gamma B \subseteq P_1$ implies $A \subseteq P_1$ or $B \subseteq P_1$. An ideal P_2 of a Γ-ring M is said to be semiprime if for any ideal U of M, $U\Gamma U \subseteq P_2$ implies $U \subseteq P_2$. A Γ-ring M is said to be prime if $a\Gamma M\Gamma b = (0)$ with $a, b \in M$, implies $a = 0$ or $b = 0$ and semiprime if $a\Gamma M\Gamma a = (0)$ with $a \in M$ implies $a = 0$. Furthermore, M is said to be commutative Γ-ring if $x\alpha y = y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover, the set $Z(M) = \{x \in M: x\alpha y = y\alpha x$ for all $y \in M$ and $\alpha \in \Gamma\}$ is called the centre of the Γ-ring M. If M is a Γ-ring, then $[x, y]_\alpha = x\alpha y - y\alpha x$ is known as the commutator of x and y with respect to α, where $x, y \in M$ and $\alpha \in \Gamma$. We make the basic commutator identities:
$[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x\alpha[y, z]_\beta$ and $[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y\alpha[x, z]_\beta$, for all $x, y \in M$ and $\alpha \in \Gamma$. We consider the following assumption:
(A)..............$x\alpha y\beta z = x\beta y\alpha z$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. An additive mapping $D: M \to M$ is called a derivation if $D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$. A mapping f is said to be commuting on a left ideal J of M if $[f(x), x]_\alpha = 0$ for all $x \in J$ and $\alpha \in \Gamma$ and f is said to be centralizing if $[f(x), x]_\alpha \in Z(M)$for all $x \in J$ and $\alpha \in \Gamma$. An additive mapping $f: M \to M$ is said to be a generalized derivation on M, if $f(x\alpha y) = f(x)\alpha y + x\alpha D(y)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$, where D is a derivation on M. An additive mapping $f: M \to M$ is called a left generalized derivation on M, if $f(x\alpha y) = x\alpha f(y) + D(x)\alpha y$ holds for all $x, y \in M$ and $\alpha \in \Gamma$, where D is a derivation on M.

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INTRODUCTION

The concept of the Γ-ring was first introduced by Nobusawa[13] and also shown that Γ-rings, more general than rings. Bernes [1] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. Bresar[2] studied centralizing mappings and derivations in prime rings. Kyuno[9], Luh[10], Hoque and Paul[5], [6] and others were obtained a large numbers of important basic properties of Γ-rings in various ways and determined some more remarkable results of Γ-rings. Ceven[3] studied on Jordan left derivations on completely prime Γ-rings. Mayne[12] have developed some remarkable result on prime rings with commuting and centralizing. Jaya subba reddy.C *et.al* [8] studied centralizing and commutating left generalized derivation on prime ring is commutative. Hoque and paul [7] studied prime gamma rings with centralizing and commuting generalized derivations is a commutative. In this paper, we extended some results on prime gamma rings with centralizing and commuting left generalized derivations is a commutative.

Some preliminary results

We have to make some use of the following well-known results

Remark 1: Let M be a prime Γ-ring. If $a\alpha b \in Z(M)$ with $0 \neq a \in Z(M)$, then $b \in Z(M)$.

Remark 2: Let M be a prime Γ-ring and J a nonzero left ideal of M. If D is a nonzero derivation on M, then D is also a nonzero on J.

Remark 3: Let M be a prime Γ-ring and J a nonzero left ideal of M. If J is commutative, then M is also commutative.

Lemma 1: Suppose M is a prime Γ-ring satisfying the assumption (A) and $D: M \to M$ be a derivation. For an element $a \in M$, if $a\alpha D(x) = 0$, for all $x \in M$ and $\alpha \in \Gamma$, then either $a = 0$ or $D = 0$.

Proof: By our assumption, $a\alpha D(x) = 0$, for all $x \in M$, and $\alpha \in \Gamma$.

Replacing x by $x\beta y$ in above equation, we get

$a\alpha D(x\beta y) = 0$

$a\alpha(D(x)\beta y + x\beta D(y)) = 0$

$a\alpha D(x)\beta y + a\alpha x\beta D(y) = 0$

$a\alpha x\beta D(y) = 0$, for all $x, y \in M$, and $\alpha, \beta \in \Gamma$.

If D is not a zero, that is, if $D(y) \neq 0$, for some $y \in M$.

By definition of prime Γ-ring, then $a = 0$. Hence proved.

Lemma 2: Suppose M is a prime Γ-ring satisfying the assumption (A) and J a nonzero left ideal of M. If M has a derivation D which is zero on J, then D is zero on M.

Proof: By the hypothesis, $D(J) = 0$

Replacing J by $M\Gamma J$ in above equation then, we get

$D(M\Gamma J) = 0$

$D(M)\Gamma J + M\Gamma D(J) = 0$

$D(M)\Gamma J = 0.$

By Lemma 1, D must be zero, since J is nonzero.

Lemma 3[7]: Suppose M is a prime Γ-ring satisfying the assumption (A) and J a nonzero left ideal of M. If J is commutative on M, then M is commutative.

Lemma 4: Suppose M is a prime Γ-ring and $f: M \to M$ be a additive mapping. If f is centralizing on a left ideal J of M, then $f(a) \in Z(M)$, for all $a \in J \cup Z(M)$.

Proof: f is a centralizing a on left ideal J of M, we have $[f(a), a]_\alpha \in Z(M)$ for all $a \in J$ and $\alpha \in \Gamma$.

By linearization, we have

$a, b \in J \Rightarrow a + b \in J$, for all $\alpha \in \Gamma$.

$[f(a+b), a+b]_\alpha \in Z(M)$

f is a additive mapping then

$$[f(a)+f(b), a+b]_\alpha \in Z(M)$$
$$[f(a), a]_\alpha + [f(a), b]_\alpha + [f(b), a]_\alpha + [f(b), b]_\alpha \in Z(M)$$

f is a centralizing on left ideal J of M then, we get

$$[f(a), a]_\alpha = 0, [f(b), b]_\alpha = 0$$
$$[f(a), b]_\alpha + [f(b), a]_\alpha \in Z(M). \qquad (1)$$

If $a \in Z(M)$, then equation (1) becomes

$$[f(a), b]_\alpha \in Z(M).$$

Replacing b by $f(a)\beta b$ in above equation then, we get

$$[f(a), f(a)\beta b]_\alpha \in Z(M)$$
$$[f(a), f(a)]_\alpha \beta b + f(a)\beta[f(a), b]_\alpha \in Z(M)$$
$$f(a)\beta[f(a), b]_\alpha \in Z(M). \text{ If } [f(a), b]_\alpha = 0.$$

Then $f(a) \in C_{\Gamma M}(J)$.

The centralizer of J in M and hence $f(a) \in Z(M)$. Otherwise, if $[f(a), b]_\alpha \neq 0$, remark 1 follows that $f(a) \in Z(M)$. Hence the lemma.

Theorem 1: Let M be a prime Γ-ring satisfying the assumption (A) and D a nonzero derivation on M. If f is a left generalized derivation on a left ideal J of M such that f is commuting on J, then M is commutative.

Proof: Since f is commuting on J, we have

$$[f(a), a]_\alpha = 0, \text{ for all } a \in J \text{ and } \alpha \in \Gamma.$$

Replacing a by $a+b$ in above equation, we get

$$[f(a+b), a+b]_\alpha = 0$$
$$[f(a)+f(b), a+b]_\alpha = 0$$
$$[f(a), a]_\alpha + [f(a), b]_\alpha + [f(b), a]_\alpha + [f(b), b]_\alpha = 0$$
$$[f(a), b]_\alpha + [f(b), a]_\alpha = 0 \qquad (2)$$

Replacing b by $a\beta b$ in equation (2), we get

$$[f(a), a\beta b]_\alpha + [f(a\beta b), a]_\alpha = 0$$
$$[f(a), a]_\alpha \beta b + a\beta\, [f(a), b]_\alpha + [a\beta f(b) + D(a)\beta b, a]_\alpha = 0$$
$$[f(a), a]_\alpha \beta b + a\beta\, [f(a), b]_\alpha + [a\beta f(b), a]_\alpha + [D(a)\beta b, a]_\alpha = 0$$
$$[f(a), a]_\alpha \beta b + a\beta\, [f(a), b]_\alpha + [a, a]_\alpha \beta f(b) + a\beta[f(b), a]_\alpha + [D(a)\beta b, a]_\alpha = 0$$

f is centralizer then, $[f(a), a]_\alpha \beta b = 0,\ [a, a]_\alpha \beta f(b) = 0$.

$$a\beta\, [f(a), b]_\alpha + a\beta[f(b), a]_\alpha + [D(a)\beta b, a]_\alpha = 0$$
$$a\beta(\, [f(a), b]_\alpha + [f(b), a]_\alpha) + [D(a)\beta b, a]_\alpha = 0$$

Using equation (2) in above equation, we get

$$[D(a)\beta b, a]_\alpha = 0. \qquad (3)$$

Replacing b by $a\gamma r$ in above equation (3), we get

$$[D(a)\beta a\gamma r, a]_\alpha = 0$$
$$[D(a), a]_\alpha \beta a\gamma r + D(a)\beta[a\gamma r, a]_\alpha = 0$$
$$[D(a), a]_\alpha \beta a\gamma r + D(a)\beta\, [a, a]_\alpha \gamma r + D(a)\beta a\gamma\, [r, a]_\alpha = 0$$
$$D(a)\beta a\gamma\, [r, a]_\alpha = 0, \text{ for all } a \in J, r \in M \text{ and } \alpha, \beta, \gamma, \in \Gamma.$$

Since M is prime Γ-ring, thus $D(a) = 0$ or $[r, a]_\alpha = 0$.

Since D is nonzero derivation on M, then by lemma 2, D is nonzero on J.

Suppose $D(a) \neq 0$ for some $a \in J$, then $a \in Z(M)$.

Let $c \in J$ with $c \neq Z(M)$. Then $D(c) = 0$ and $a + c \notin Z(M)$, that is, $D(a+c) = 0$ and so $D(a) = 0$, which is a contradiction. Thus $c \in Z(M)$ for all $c \in J$. Hence J is commutative and hence by lemma 3, M is commutative. Hence the theorem.

Theorem 2: Let M be a prime Γ-ring satisfying the assumption (A) and J a left ideal of M with $J \cap Z(M) \neq 0$. If f is a left generalized derivation on M with associated nonzero derivation D such that f is commuting on J, then M is commutative.

Proof: we claim that, $Z(M) \neq 0$ because of f is commuting on J and the proof is complete.

Now from equation (1), we get

$$[f(a), b]_\alpha + [f(b), a]_\alpha \in Z(M)$$

We replace a by $b\beta c$ with $0 \neq c \in Z(M)$, then we get

$$[f(b\beta c), b]_\alpha + [f(b), b\beta c]_\alpha \in Z(M)$$
$$[b\beta f(c) + D(b)\beta c, b]_\alpha + [f(b), b]_\alpha \beta c + b\beta[f(b), c]_\alpha \in Z(M)$$
$$[b\beta f(c), b]_\alpha + [D(b)\beta c, b]_\alpha + b\beta[f(b), c]_\alpha \in Z(M)$$
$$[b, b]_\alpha \beta f(c) + b\beta\,[f(c), b]_\alpha + [D(b), b]_\alpha \beta c + D(b)\beta\,[c, b]_\alpha + [f(b), b]_\alpha \beta c + b\beta\,[f(b), c]_\alpha \in Z(M)$$
$$c \in Z(M) \Longrightarrow [c, b]_\alpha = 0 \text{ for all } b \in J,\ [b, b]_\alpha = 0$$

Since $c \in Z(M) \Rightarrow f$ is a centralizer on J.

$$f(b) \in Z(M) \Rightarrow [f(b), c]_\alpha = 0.$$
$$b\beta\,[f(c), b]_\alpha + [D(b), b]_\alpha \beta c + [f(b), b]_\alpha \beta c \in Z(M)$$

From lemma 1, $f(c) \in Z(M)$ and hence $[D(b), b]_\alpha \beta c + [f(b), b]_\alpha \beta c \in Z(M)$. Since f is a centralizing on J, we have $[f(b), b]_\alpha \beta c \in Z(M)$ and consequently $[D(b), b]_\alpha \beta c \in Z(M)$. As c is nonzero, remark 1 follows that $[D(b), b]_\alpha \in Z(M)$. This implies D is centralizing on J and hence we conclude that M is commutative.

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