

**CONVOLUTION PROPERTY OF MULTIVALENT FUNCTIONS
 WITH COEFFICIENT OF ALTERNATING TYPE USING q -DERIVATIVE**

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ABSTRACT

By applying the concept of fractional q -calculus, we investigate coefficient bounds and convolution results of multivalent functions with coefficients of alternating type

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1. INTRODUCTION

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$.

For $-1 \leq A < B \leq 1$, let $P(A, B)$ [3] denote the class of functions which are of the form

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)},$$

where ω is a bounded analytic function satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$.

We consider another subclass M_p which consists of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} z^{k+1}, \quad a_{k+1} \geq 0.$$

The q -shifted factorial is defined for $\alpha, q \in \mathbb{C}$ as a product of n factors by

$$(\alpha, q)_n = \begin{cases} 1, & n = 0; \\ (1 - \alpha)(1 - \alpha q) \cdots (1 - \alpha q^{n-1}), & n \in \mathbb{N} \end{cases} \tag{1.2}$$

and in terms of the basic analogue of the gamma function

$$(q^\alpha; q)_n = \frac{\Gamma_q(\alpha + n)(1 - q)^n}{\Gamma_q(\alpha)}, \quad (n > 0), \tag{1.3}$$

where the q -gamma functions [1, 2] is defined by

$$\Gamma_q(x) = \frac{(q; q)_\infty (1 - q)^{1-x}}{(q^x; q)_\infty} \quad (0 < q < 1). \tag{1.4}$$

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Note that, if $|q| < 1$, the q -shifted factorial (1.2) remains meaningful for $n = \infty$ as a convergent infinite product

$$(\alpha; q)_\infty = \prod_{m=0}^{\infty} (1 - \alpha q^m).$$

Now recall the following q -analogue definitions given by Gasper and Rahman [1]. The recurrence relation for q -gamma function is given by

$$\Gamma_p(x+1) = [x]_q \Gamma_p(x), \text{ where, } [x]_q = \frac{(1-q^x)}{(1-q)}, \tag{1.5}$$

and called q -analogue of x .

Jackson's q -derivative and q -integral of a function f defined on a subset of \mathbb{C} are, respectively, given by (see Gasper and Rahman [1])

$$D_q f(z) = \frac{f(z) - f(zq)}{z(1-q)}, (z \neq 0, q \neq 0). \tag{1.6}$$

$$\int_0^z f(t) d_q(t) = z(1-q) \sum_{m=0}^{\infty} q^m f(zq^m). \tag{1.7}$$

In view of the relation

$$\lim_{q \rightarrow 1^-} \frac{(q^\alpha; q)_n}{(1-q)^n} = (\alpha)_n, \tag{1.8}$$

we observe that the q -shifted factorial (1.1) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n+1)$.

Consider the following definitions.

$$S_q^*(A, B) = \left\{ f \mid f \in A_p \text{ and } \frac{zD_q(f(z))}{f(z)} \in P(A, B) \right\}$$

$$H_q(A, B) = \left\{ f \mid f \in A_p \text{ and } \frac{D_q(zD_q(f(z)))}{D_q(f(z))} \in P(A, B) \right\}$$

$$M_q^*(A, B) = \left\{ f \mid f \in M_p \text{ and } \frac{zD_q(f(z))}{f(z)} \in P(A, B) \right\}$$

$$C_q(A, B) = \left\{ f \mid f \in M_p \text{ and } \frac{D_q(zD_q(f(z)))}{D_q(f(z))} \in P(A, B) \right\}.$$

Note that these classes generalize the classes of Padamanabhan and Ganeshan [5], Silverman [6], Khairnar and Meena More [4].

If $f(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} z^{k+1}$ and $g(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} b_{k+1} z^{k+1}$, then their modified hadamard product is defined by

$$h(z) = f(z) * g(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} b_{k+1} z^{k+1}.$$

In this paper we discuss some properties of convolution for the class $M_q^*(A, B)$ and $C_q(A, B)$.

2. MAIN RESULTS

Lemma 2.1 A function $f(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} z^{k+1}$, $a_{k+1} \geq 0$ is in $M_q^*(A, B)$ if and only if

$$\sum_{k=p+1}^{\infty} \left\{ \frac{[k+1]_q(B+1) - (A+1)}{(A+1) - (B+1)[p]_q} \right\} a_{k+1} \leq 1. \tag{2.1}$$

Proof: Consider

$$f(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} z^{k+1}$$

$$\frac{zD_q f(z)}{f(z)} = \frac{[p]_q z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} [k+1]_q z^{k+1}}{z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} z^{k+1}}.$$

Now $\frac{zD_q f(z)}{f(z)} \in P(A, B)$ if and only if

$$\frac{[p]_q z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} [k+1]_q z^{k+1}}{z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} z^{k+1}} = \frac{1 + A\omega(z)}{1 + B\omega(z)}$$

$$\omega(z) \left[(B[p]_q - A)z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} (B[k+1]_q - A)a_{k+1} z^{k+1} \right] = (1 - [p]_q)z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} (1 - [k+1]_q)a_{k+1} z^{k+1}$$

by using the condition $|\omega(z)| \leq 1$, we get

$$\left| \frac{(1 - [p]_q)z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} (1 - [k+1]_q)a_{k+1} z^{k+1}}{(B[p]_q - A)z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} (B[k+1]_q - A)a_{k+1} z^{k+1}} \right| \leq 1.$$

Allowing $|z| = r \rightarrow 1$

$$\left\{ \frac{(1 - [p]_q) + \sum_{k=p+1}^{\infty} (1 - [k+1]_q)a_{k+1}}{(B[p]_q - A) + \sum_{k=p+1}^{\infty} (B[k+1]_q - A)a_k} \right\} \leq 1$$

$$\sum_{k=p+1}^{\infty} [1 - (B+1)[k+1]_q + A]a_{k+1} \leq (B+1)[p]_q - (A+1)$$

$$\sum_{k=p+1}^{\infty} \left\{ \frac{[[k+1]_q(B+1) - (A+1)]}{(A+1) - (B+1)[p]_q} \right\} a_{k+1} \leq 1.$$

and the result follows.

As a consequence we have the following result.

Lemma 2.2: A function $f(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} z^{k+1}$, $a_{k+1} \geq 0$ is in $C_q(A, B)$ if and only if

$$\sum_{k=p+1}^{\infty} \left\{ \frac{[k+1]_q \{ [k+1]_q (B+1) - (A+1) \}}{[p]_q \{ (A+1) - (B+1)[p]_q \}} \right\} a_{k+1} \leq 1. \tag{2.2}$$

Theorem 2.3: If $f(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} z^{k+1}$ and $g(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} b_{k+1} z^{k+1}$, $a_{k+1}, b_{k+1} \geq 0$ are elements of classes $M_q^*(A, B)$ then $h(z) = f(z) * g(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} b_{k+1} z^k$ is an element of $M_q^*(A_1, B_1)$ with $-1 \leq A_1 < B_1 \leq 1$ where $A_1 \geq -1$, $B_1 \leq \frac{A_1 + 1 - s}{s}$.

Proof: Since $f, g \in M_q^*(A, B)$, by Lemma 2.1 we have,

$$\sum_{k=p+1}^{\infty} \left\{ \frac{[[k+1]_q(B+1) - (A+1)]}{(A+1) - (B+1)[p]_q} \right\} a_{k+1} \leq 1$$

$$\sum_{k=p+1}^{\infty} \left\{ \frac{[[k+1]_q(B+1) - (A+1)]}{(A+1) - (B+1)[p]_q} \right\} b_{k+1} \leq 1.$$

We need A_1, B_1 such that $-1 \leq A_1 < B_1 \leq 1$ and

$$h(z) = f(z) * g(z) \in M_q^*(A_1, B_1).$$

Now $h(z) \in M_q^*(A_1, B_1)$ if

$$\sum_{k=p+1}^{\infty} \left\{ \frac{[[k+1]_q(B_1+1) - (A_1+1)]}{(A_1+1) - (B_1+1)[p]_q} \right\} a_{k+1} b_{k+1} \leq 1. \tag{2.3}$$

i.e., $\sum_{k=p+1}^{\infty} u_1 a_{k+1} b_{k+1} \leq 1$, where $u_1 = \frac{[[k+1]_q(B_1+1) - (A_1+1)]}{(A_1+1) - (B_1+1)[p]_q}$.

Using Cauchy-Schwarz inequality we have,

$$\sum_{k=p+1}^{\infty} \sqrt{u a_{k+1} b_{k+1}} \leq \left\{ \sum_{k=p+1}^{\infty} u a_{k+1} \right\}^{\frac{1}{2}} \left\{ \sum_{k=p+1}^{\infty} u b_{k+1} \right\}^{\frac{1}{2}} \leq 1,$$

where $u = \frac{[[k+1]_q(B+1) - (A+1)]}{(A+1) - (B+1)[p]_q}$.

(2.3) is true if

$$u_1 a_{k+1} b_{k+1} \leq u \sqrt{a_{k+1} b_{k+1}}$$

using (2.3) we have,

$$u_1 \sqrt{a_{k+1} b_{k+1}} \leq 1 \text{ for } k = 2, 3, \dots$$

Therefore it is enough to find u_1 such that

$$\frac{1}{u} \leq \frac{u}{u_1}.$$

i.e., $\frac{[[k+1]_q(B_1+1) - (A_1+1)]}{(A_1+1) - (B_1+1)[p]_q} \leq u^2$

$$A_1 \geq -1 + \frac{(B_1+1)([k+1]_q + [p]_q u^2)}{u^2 + 1}.$$

Consider $B_1 = 1$ and $k = 2$ to obtain,

$$A_1 \geq -1 + \frac{2([3]_q + [p]_q u^2)}{u^2 + 1}$$

$$= -1 + 2 \frac{[3]_q ((A+1) - (B+1)[p]_q)^2 + [p]_q ([3]_q (B+1) - (A+1))^2}{((A+1) - (B+1)[p]_q)^2 + ([3]_q (B+1) - (A+1))^2}$$

where $s = \frac{-1 + 2s}{[3]_q ((A+1) - (B+1)[p]_q)^2 + [p]_q ([3]_q (B+1) - (A+1))^2}$.

Theorem 2.4: If $f(z) \in M_q^*(A, B)$ and $g(z) \in M_q^*(A', B')$ then $f(z) * g(z) \in M_q^*(A_1, B_1)$ where

$$A_1 \geq -1, B_1 \leq \frac{A_1 + 1 - x}{x}, \text{ with}$$

$$x = \frac{[3]_q ((A+1) - (B+1)[p]_q)((A'+1) - (B'+1)[p]_q) + [p]_q ([3]_q (B+1) - (A+1))([3]_q (B'+1) - (A'+1))}{((A+1) - (B+1)[p]_q)((A'+1) - (B'+1)[p]_q) + ([3]_q (B+1) - (A+1))([3]_q (B'+1) - (A'+1))}$$

Proof: Analogously proceeding as developed in Theorem 2.3, we require

$$\left\{ \frac{[[k+1]_q (B_1 + 1) - (A_1 + 1)]}{(A_1 + 1) - (B_1 + 1)[p]_q} \right\} \leq \left\{ \frac{[[k+1]_q (B + 1) - (A + 1)]}{(A + 1) - (B + 1)[p]_q} \right\} \left\{ \frac{[[k+1]_q (B' + 1) - (A' + 1)]}{(A' + 1) - (B' + 1)[p]_q} \right\} = \delta.$$

i.e., $\frac{B_1 + 1}{A_1 + 1} \leq \frac{\delta + 1}{[k+1]_q + \delta[p]_q}$

$$\frac{A_1 + 1}{B_1 + 1} \geq \frac{[k+1]_q + \delta[p]_q}{\delta + 1}.$$

$$\frac{A_1 + 1}{B_1 + 1} \geq$$

$$\frac{[k+1]_q ((A+1) - (B+1)[p]_q)((A'+1) - (B'+1)[p]_q) + [p]_q ([k+1]_q (B+1) - (A+1))([k+1]_q (B'+1) - (A'+1))}{((A+1) - (B+1)[p]_q)((A'+1) - (B'+1)[p]_q) + ([k+1]_q (B+1) - (A+1))([k+1]_q (B'+1) - (A'+1))}$$

Taking $k = 2$, we get $\frac{A_1 + 1}{B_1 + 1} \geq$

$$\frac{[3]_q ((A+1) - (B+1)[p]_q)((A'+1) - (B'+1)[p]_q) + [p]_q ([3]_q (B+1) - (A+1))([3]_q (B'+1) - (A'+1))}{((A+1) - (B+1)[p]_q)((A'+1) - (B'+1)[p]_q) + ([3]_q (B+1) - (A+1))([3]_q (B'+1) - (A'+1))} = x.$$

i.e., $\frac{A_1 + 1}{B_1 + 1} \geq x,$

$$B_1 \leq \frac{A_1 + 1 - x}{x}. \text{ But } B_1 \geq -1, \text{ we get } A_1 > -1.$$

Theorem 2.5: If $f(z) \in C_q(A, B)$ and $g(z) \in C_q(A', B')$ then $f(z) * g(z) \in C_q(A_1, B_1)$ where $A_1 \geq -1,$

$$B_1 \leq \frac{A_1 + 1 - y}{y} \text{ with}$$

$$y = \frac{[3]_q [p]_q \left[\begin{aligned} & [3]_q ((A+1) - (B+1)[p]_q)((A'+1) - (B'+1)[p]_q) \\ & + [p]_q ([3]_q (B+1) - (A+1))([3]_q (B'+1) - (A'+1)) \end{aligned} \right]}{[p]_q ((A+1) - (B+1)[p]_q)((A'+1) - (B'+1)[p]_q) + [k+1]_q ([3]_q (B+1) - (A+1))([3]_q (B'+1) - (A'+1))}$$

Proof: The proof of the theorem follows the pattern of that in Theorem 2.4

$$\frac{A_1 + 1}{B_1 + 1} \geq \frac{[3]_q [p]_q \left\{ [3]_q \left((A+1) - (B+1)[p]_q \right) \left((A'+1) - (B'+1)[p]_q \right) + [p]_q \left([3]_q (B+1) - (A+1) \right) \left([3]_q (B'+1) - (A'+1) \right) \right\}}{[p]_q \left((A+1) - (B+1)[p]_q \right) \left((A'+1) - (B'+1)[p]_q \right) + [k+1]_q \left([3]_q (B+1) - (A+1) \right) \left([3]_q (B'+1) - (A'+1) \right)} = y.$$

$$\frac{\gamma_1 - B_1 p}{B_1 + 1} \geq y$$

$$B_1 \leq \frac{A_1 + 1 - y}{y} \tag{2.4}$$

$B_1 \geq -1$, using (2.4) we get $A_1 \geq -1$.

Theorem 2.6: If $f(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} z^{k+1}$, $a_{k+1} \geq 0 \in M_q^*(A, B)$ and $g(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} b_{k+1} z^{k+1}$ with $|b_{k+1}| \leq 1$ for $k \geq 2$ then $f(z) * g(z) \in S_q^*(A, B)$.

Proof: Since $f \in M_q^*(A, B)$ we have,

$$\sum_{k=p+1}^{\infty} \left\{ \frac{[[k+1]_q (B+1) - (A+1)]}{(A+1) - (B+1)[p]_q} \right\} a_{k+1} b_{k+1} \leq \sum_{k=p+1}^{\infty} \left\{ \frac{[[k+1]_q (B+1) - (A+1)]}{(A+1) - (B+1)[p]_q} \right\} a_{k+1} |b_{k+1}| \leq 1.$$

This shows that

$$f(z) * g(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} b_{k+1} z^{k+1} \in S_q^*(A, B).$$

The proof of Theorem 2.7 below follows the pattern of that in Theorem 2.6.

Theorem 2.7: If $f(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} a_{k+1} z^{k+1}$, $a_{k+1} \geq 0 \in C_q(A, B)$ and $g(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} b_{k+1} z^{k+1}$ with $|b_{k+1}| \leq 1$ for $k \geq 2$ then $f(z) * g(z) \in H_q(A, B)$.

Theorem 2.8: If $f, g \in M_q^*(A, B)$ then $h(z) = z^p + \sum_{k=p+1}^{\infty} (-1)^{k+1} (a_{k+1}^2 + b_{k+1}^2) z^{k+1} \in M_q^*(A_1, B_1, p, \alpha)$

where $A_1 \geq -1, B_1 \leq \frac{A_1 + 1 - s}{s}$, with $s = \frac{2[3]_q \left((A+1) - (B+1)[p]_q \right)^2 + [p]_q \left([3]_q (B+1) - (A+1) \right)^2}{2 \left((A+1) - (B+1)[p]_q \right)^2 + \left([3]_q (B+1) - (A+1) \right)^2}$.

Proof: Since $f(z), g(z) \in M_q^*(A, B)$ we have,

$$\sum_{k=p+1}^{\infty} \left\{ \frac{[[k+1]_q (B+1) - (A+1)]}{(A+1) - (B+1)[p]_q} a_{k+1} \right\} \leq 1$$

$$\sum_{k=p+1}^{\infty} \left\{ \frac{[[k+1]_q (B+1) - (A+1)]}{(A+1) - (B+1)[p]_q} b_{k+1} \right\} \leq 1.$$

We see that

$$\sum_{k=p+1}^{\infty} \left\{ \frac{[[k+1]_q (B+1) - (A+1)]}{(A+1) - (B+1)[p]_q} \right\}^2 a_{k+1}^2 \leq \left\{ \sum_{k=p+1}^{\infty} \frac{[[k+1]_q (B+1) - (A+1)]}{(A+1) - (B+1)[p]_q} a_{k+1} \right\}^2 \leq 1. \tag{2.5}$$

$$\left\{ \sum_{k=p+1}^{\infty} \frac{[[k+1]_q (B+1) - (A+1)]}{(A+1) - (B+1)[p]_q} b_{k+1} \right\}^2 \leq 1 \tag{2.6}$$

Adding (2.5) and (2.6) we get

$$\sum_{k=p+1}^{\infty} \left\{ \frac{[[k+1]_q(B+1) - (A+1)]}{(A+1) - (B+1)[p]_q} \right\}^2 (a_{k+1}^2 + b_{k+1}^2) \leq 2. \tag{2.7}$$

Now $f(z), g(z) \in M_q^*(A_1, B_1, p, \alpha)$

$$\sum_{k=p+1}^{\infty} \left\{ \frac{[[k+1]_q(B_1+1) - (A_1+1)]}{(A_1+1) - (B_1+1)[p]_q} \right\} (a_{k+1}^2 + b_{k+1}^2) \leq 1$$

(2.7) implies that it is enough show that

$$\frac{[[k+1]_q(B_1+1) - (A_1+1)]}{(A_1+1) - (B_1+1)[p]_q} \leq \frac{1}{2} \left\{ \frac{[[k+1]_q(B+1) - (A+1)]}{(A+1) - (B+1)[p]_q} \right\}^2 = \frac{u^2}{2}.$$

i.e.,

$$\frac{k(B_1+1)}{A_1+1} \leq \frac{u^2 + 2}{2[k+1]_q + [p]_q u^2}$$

$$\frac{A_1+1}{k(B_1+1)} \geq \frac{2[k+1]_q + [p]_q u^2}{u^2 + 2} = \beta(k).$$

Notice that $\beta(k)$ decreases as k increases and replacing k by 2 and simplifying we obtain,

$$\gamma_1 \geq -p, B_1 \leq \frac{A_1+1-s}{s}, \text{ with } s = \frac{2[3]_q \left((A+1) - (B+1)[p]_q \right)^2 + [p]_q \left([3]_q (B+1) - (A+1) \right)^2}{2 \left((A+1) - (B+1)[p]_q \right)^2 + \left([3]_q (B+1) - (A+1) \right)^2}.$$

REFERENCES

1. G. Gasper and M. Rahman, *Basic Hypergeometric Series*, vol. 35 of Encyclopedia of Mathematics and Its Applications, Cambridge University Press, Cambridge, Mass, USA, 1990.
2. F. H. Jackson, On q-functions and a certain difference operator, *Transactions of the Royal Society of Edinburgh*, vol. 46, no. 2,(1909) pp.253-281.
3. Janowski W, Some extremal problems for certain families of analytic functions, *I. Ann. Polon. Math.*, 28, 298-326, 1973.
4. S.M. Khairnar and Meena More, Convolution properties of Univalent functions with missing second coefficient of alternating type, *Int.Journal of Math.Analysis*, Vol. 2, 2008, no. 12, 569-580.
5. K.S. Padmanban and M.S. Ganeshan, Convolution of certain class of univalent functions with negative coefficients *Indian J. Pure Appl. Math. Sept.* 19(9) (1988), 880-889.
6. H. Silverman, Univalent functions with varying raguments, *Houston Journal of Math.*, Vol.7, No. 2 (1981).

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