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SECURE RESTRAINED CONVEX DOMINATION IN GRAPHS

ENRICO L. ENRIQUEZ*

Department of Mathematics,
School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines.

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ABSTRACT

Let G be a connected simple graph. A restrained convex dominating set S in a connected graph G is a secure restrained convex dominating set, if for each element u in $V(G)\setminus S$ there exists an element v in S such that $uv \in E(G)$ and $(S \setminus \{ v \}) \cup \{ u \}$ is a restrained convex dominating set. The secure restrained convex domination number of G, denoted by $\gamma_{src}(G)$, is the minimum cardinality of a secure restrained convex dominating set in G. A secure restrained convex dominating set of cardinality $\gamma_{src}(G)$ will be called a γ_{src}-set. In this paper, we give some realization problems will be given. In particular, we show that given positive integers k and n such that $n \geq 5$ and $k \in \{ 1,2,\ldots, n-2, n \}$, there exists a connected graph G with $|V(G)| = n$ and $\gamma_{src}(G) = k$. Further, we characterize the secure restrained convex dominating sets in the join of two graphs and give some important results.

Mathematics Subject Classification: *05C69.*

Keywords: *convex domination, restrained convex dominating set, secure restrained convex dominating set.*

1. INTRODUCTION

Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [11]. However, it was not until 1977, following an article [4] by Ernie Cockayne and Stephen Hedetniemi, that domination in graphs became an area of study by many researchers. One type of domination parameter is the secure domination number in a graph. This was studied and introduced by E.J. Cockayne [5]. Secure dominating sets can be applied as protection strategies by minimizing the number of guards to secure a system so as to be cost effective as possible. Other type of domination parameter is the restrained domination number in a graph. This was introduced by Telle and Proskurowski [9] indirectly as a vertex partitioning problem. Moreover, restrained dominating set can found in [8]. One practical application of restrained domination is that of prisoners and guards. Here, each vertex not in the restrained dominating set corresponds to a position of a prisoner, and every vertex in the restrained dominating set corresponds to a position of a guard. To effect security, each prisoner's position is observed by a guard's position. To protect the rights of prisoners, each prisoner's position is seen by at least one other prisoner's position. To be cost effective, it is desirable to place a few guards as possible. In [12], Pushpam and Suseendran paper's "Secure Restrained Domination in Graphs" studied few properties of secure restrained domination number of certain classes of graphs and evaluate $\gamma_{sr}(G)$ values. Convexity in graphs has been discussed and studied in [7]. On the other hand, convex domination in graphs has been defined and studied in [10]. In [2], Enriquez and Canoy, introduced the concepts of secure convex and restrained convex domination in graphs. In this paper, we give some realization problems and characterize the secure restrained convex dominating sets in the join of two graphs. For general concepts we refer the reader to [6].

2. RESULTS

From the definitions above, the following result is immediate.

Remark 2.1: Let G be a nontrivial connected graph of order $n \geq 2$. Then
(i) $\gamma(G) \leq \gamma_{rcon}(G) \leq \gamma_{src}(G)$; and
(ii) $\gamma_{src} \in \{1,2,..,n-3,n-2,n\}$.

It is worth mentioning that the upper bound in Remark 2.1(ii) is sharp. For example, $\gamma_{src}(P_n) = n$ for all $n \geq 2$. The lower bound is also attainable as the following result shows.

Corresponding Author: Enrico L. Enriquez, Department of Mathematics, School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines.*

Theorem 2.2 (Realization Problem 1): Given positive integers k and n such that $n \geq 5$ and $k \in \{1,2,\ldots,n-2,n\}$, there exists a connected graph G with $|V(G)| = n$ and $\gamma_{src}(G) = k$.

Proof: Consider the following cases:

Case-1: Suppose that $k = 1$. Let $G = K_n$. Then $|V(G)| = n$ and $\gamma_{src}(G) = 1$.

Case-2: Suppose that $k = 2$. Let $G = K_n - e$ for some $e \in E(K_n)$ (see Figure 1).

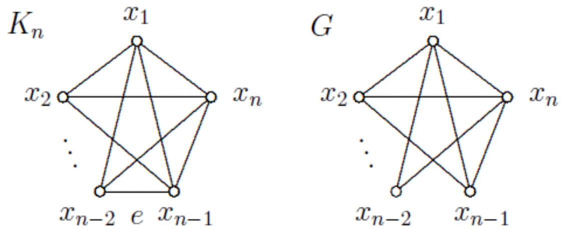

Figure-1: A graph G with $|V(G)| = n$ and $\gamma_{src}(G) = 2$

The set $S = \{\, x_1, x_n \,\}$ is a γ_{src}-*set* of G. Hence, $|V(G)| = n$ and $\gamma_{src}(G) = 2$.

Case-3: Suppose $3 \leq k \leq n-2$. Let $r = n-k+1$ $(r \geq 3)$. Consider the graph $G = \langle \{\, v \,\} \rangle + (K_r \cup \overline{K}_{k-2})$ (see Figure 2).

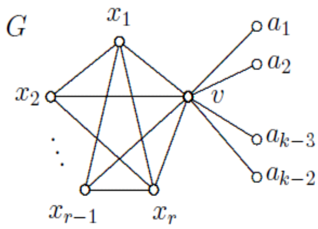

Figure-2: A graph G with $|V(G)| = n$ and $\gamma_{src}(G) = k$

The set $S = \{\, x_1, v, a_1, a_2, \ldots, a_{k-2} \,\}$ is a γ_{src}-*set* of G. Hence, $|V(G)| = r + (k-2) + 1 = n$ and $\gamma_{src}(G) = (k-2) + 2 = k$.

Case-4: Suppose $k = n$. Let $G = K_{1,n-1}$. Then $|V(G)| = n$ and $\gamma_{src}(G) = k$. This proves the assertion.

Theorem 2.3 (Realization Problem 2): Given positive integers $k, m \geq 2$ and $n \geq 6$ such that $1 \leq k \leq m-1$ and $m \in \{\, 2,3,\ldots,n-2,n \,\}$, there exists a connected graph G with $|V(G)| = n$, $\gamma_{src}(G) = m$, and $\gamma_{rcon}(G) = k$.

Proof: Consider the following cases:

Case-1: Suppose that $m = n$. Let the path $P_k = [v_1, v_2, \ldots, v_k]$ and the path $P_p = [x_1, x_2, \ldots, x_p]$ with $k \geq 2$, $p \geq 2$. Consider the graph G obtained from P_k by adding the edges $x_i v_1$, where $i = 1,2,\ldots,p$ (see Figure 3).

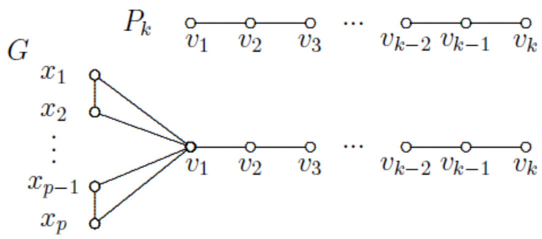

Figure-3: A graph G with $\gamma_{src}(G) = n = m$ and $\gamma_{rcon}(G) = k$

The sets $A_1 = \{\, v_1, v_2, \ldots, v_k \,\}$ and $B_1 = V(G)$ are, respectively, a γ_{rcon}-*set* and a γ_{src}-*set* of G. It follows that $|V(G)| = n = m = \gamma_{src}(G)$ and $\gamma_{rcon}(G) = k$.

Case-2: Suppose that $m < n$. Let $r = n - k$ $(r \geq 3)$ and suppose that $k = m-1$. Let H_2 be the complete graph K_{r+1} and let G be the graph obtained from H_2 by adding the edges $v_1v_2, v_2v_3, \ldots, v_{k-1}v_k$ (see Figure 4).

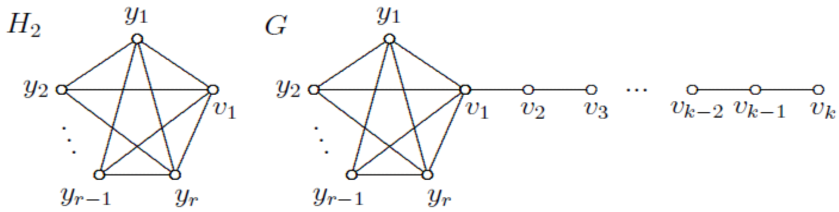

Figure-4: A graph G with $\gamma_{src}(G) = k + 1 = m$ and $\gamma_{rcon}(G) = k$

The set $A_2 = \{ v_1, v_2, \ldots, v_k \}$ is a γ_{rcon}-set of G and the set $B_2 = \{ y_1, v_1, v_2, \ldots, v_k\}$ is a γ_{src}-set of G. Hence, $\gamma_{rcon}(G) = k$, $\gamma_{src}(G) = k + 1 = m$, and $|V(G)| = r + k = n$.

Suppose that $k < m - 1$. Let $r = n - m + 1$ and $p = m - k - 1$. Consider the graph G in Figure 4 and let G_1 be the graph obtained from G and the path $P_p = [x_1, x_2, \ldots, x_p]$ with $p \geq 2$ by adding the $x_i v_1$, where $i = 1,2,\ldots,p$. The set $A_3 = \{ v_1, v_2, \ldots, v_k \}$ is a γ_{rcon}-set of G_1 and the set $B_3 = \{y_1, x_1, x_2, \ldots, x_p, v_1, v_2, \ldots, v_k\}$ is a γ_{src}-set of G_1 (see Figure 5).

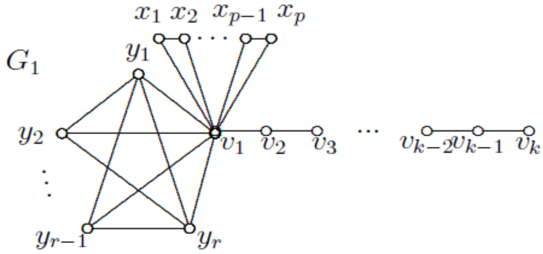

Figure-5: A graph G_1 with $\gamma_{src}(G_1) = p + k + 1 = m$ and $\gamma_{rcon}(G_1) = k$

Hence, $\gamma_{rcon}(G_1) = k$, $\gamma_{src}(G_1) = p + k + 1 = m$, and $|V(G_1)| = r + p + k = n$. This proves the assertion.

Corollary 2.4: The difference $\gamma_{src} - \gamma_{rcon}$ can be made arbitrarily large.

Proof: Let n be a positive integer. By Theorem 2.3, there exists a connected graph G such that $\gamma_{src}(G) = n + 1$ and $\gamma_{rcon}(G) = 1$. Thus, $\gamma_{src}(G) - \gamma_{rcon}(G) = n$. Therefore, $\gamma_{src} - \gamma_{rcon}$ can be made arbitrarily large.

Lemma 2.5: If S is a secure restrained convex dominating set in a graph G, then S is a secure convex dominating set in G.

Proof: Suppose that S is a secure restrained convex dominating set in G. Then S is a restrained convex dominating set in G, that is, S is a convex dominating set in G. Let $u \in V(G) \setminus S$. Then there exists $v \in S$ such that $uv \in E(G)$ and $S_u = (S \setminus \{v\}) \cup \{u\}$ is a restrained convex dominating set in G, that is, S_u is a convex dominating set in G. Hence, S is a secure convex dominating set in G.

The converse of the above Lemma is not necessarily true. For example, consider the graph G below (see Figure 6). The set $S = \{x, y, z, w\}$ is a secure convex dominating set in G but not a secure restrained convex dominating set in G.

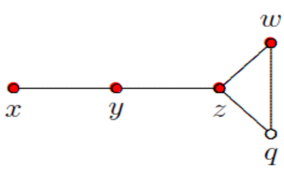

Figure-6: A graph G with $\gamma_{scon}(G) = 4$

We need the following Theorems and Corollary for our next result.

Theorem 2.6 [2]: Let G be any connected graph. Then $\gamma_{scon}(G) = 1$ if and only if G is complete.

Theorem 2.7: Let G be a connected graph of order $n \geq 3$. Then $\gamma_{src}(G) = 1$ if and only if G is complete.

Proof: Suppose that $\gamma_{src}(G) = 1$, say $S = \{ v \}$ is a γ_{src}-set of G. Then S is a secure convex dominating set of G by Lemma 2.5. Thus, G is complete by Theorem 2.6.

For the converse, suppose that G is complete. Then for every $v \in V(G)$, the set $S = \{ v \}$ is a convex dominating set of G. Since $n \geq 3$, there exists $x, y \in V(G) \setminus S$ such that $vx, xy \in E(G)$. Hence, S is a restrained convex dominating set of G. Since $S \setminus \{ v \} \cup \{ z \}$ is also a restrained convex dominating set of G for all $z \in V(G) \setminus S$, it follows that S is a γ_{src}-set of G. Therefore, $\gamma_{src}(G) = 1$.

The following result is a quick consequence of Theorem 2.7.

Corollary 2.8: Let G and H be nontrivial connected graphs. Then $\gamma_{src}(G + H) = 1$ if and only if G and H are complete graphs.

Corollary 2.9: $\gamma_{src}(G \circ H) = 1$ if and only if G is a trivial graph and H is a nontrivial complete graph.

A nonempty subset S of $V(G)$, where G is any graph, is a *clique* in G if the graph $G[S] = \langle S \rangle$ induced by S is complete. A clique S in G is a \textit{clique dominating} set if it is a dominating set. It is a *secure clique dominating* set in G if for every $u \in V(G) \setminus S$, there exists $v \in S \cap N_G(u)$ such that $(S \setminus \{ v \}) \cup \{ u \}$ is a clique dominating set in G. The *secure clique domination number* of G, denoted by $\gamma_{scl}(G)$, is the minimum cardinality of a secure clique dominating set of G.

Theorem 2.10 [1]: Let G and H be non-complete graphs. Then $S \subseteq V(G + H)$ is a restrained convex dominating set in $G + H$ if and only if one of the following holds.

(i) S is a clique dominating set in G.
(ii) S is a clique dominating set in H.
(iii) $S = S_G \cup S_H$, where S_G and S_H are cliques in G and H, respectively.

Theorem 2.11 [2]: Let G and H be connected non-complete graphs. Then a proper subset S of $V(G + H)$ is a secure convex dominating set in $G + H$ if and only if one of the following statements holds:

(i) S is a secure clique dominating set in G and $|S| \geq 2$.
(ii) S is a secure clique dominating set in H and $|S| \geq 2$.
(iii) $S = \{ v \} \cup \{ w \}$ where $\{ v \}$ and $\{ w \}$ are dominating sets in G and H respectively.
(iv) $S = \{ v \} \cup S_H$ where $\{ v \}$ is a dominating set in G, and $|S_H| \geq 2$, and S_H is a secure clique dominating set in H.
(v) $S = S_G \cup \{ w \}$ where $\{ w \}$ is a dominating set in H, and $|S_G| \geq 2$, and S_G is a secure clique dominating set in G.
(vi)$S = S_G \cup S_H$ where $|S_G| \geq 2$, $|S_H| \geq 2$, and S_G and S_H are secure clique dominating sets in G and H respectively.

The following characterizes the secure restrained convex dominating sets in the join of two connected non-complete graphs.

Theorem 2.12: Let G and H be connected non-complete graphs. Then a proper subset S of $V(G + H)$ is a secure restrained convex dominating set in $G + H$ if and only if it is a secure convex dominating set in $G + H$.

Proof: Suppose that S is a secure restrained convex dominating set in $G + H$. Then S is a secure convex dominating set in $G + H$ by Lemma 2.5.

For the converse, suppose that S is a secure convex dominating set in $G + H$. Then statement (i) or (ii) or (iii) or (iv) or \$$(v)$\$ or (vi) of Theorem 2.11 holds. Suppose first that statement (i) holds. Then S is a restrained convex dominating set in $G + H$ by Theorem 2.10(i). Since $V(G + H) \setminus S \neq \emptyset$, let $z \in V(G + H) \setminus S$.

Suppose that $z \in V(G)$. Then by assumption, there exists $a \in S$ such $az \in E(G) \subset E(G + H)$ and $S_z = (S \setminus \{ a \}) \cup \{ z \}$ is a convex dominating set in G (and hence in $G + H$). To show that S_z is a restrained dominating set, let $u \in V(G + H) \setminus S_z$. If $u \in V(G)$, then $uw \in E(G + H)$ for all $w \in V(H) \setminus S_z \subset V(G + H) \setminus S_z$ and $ux \in E(G) \subset E(G + H)$ for some $x \in S_z$. If $u \in V(H)$, then $uy \in E(G + H)$ for all $y \in V(G) \setminus S_z \subset V(G + H) \setminus S_z$ and $ux \in E(G + H)$ for some $x \in S_z$. Thus, for each $u \in V(G + H) \setminus S_z$, there exists $x \in S_z$ such that $ux \in E(G + H)$ and there exists $t \in V(G + H) \setminus S_z$ such that $ut \in E(G + H)$. This implies that $\langle V(G + H) \setminus S_z \rangle$ has no isolated vertices. Hence S_z is a restrained dominating set of $G + H$, that is, S_z is a restrained convex dominating set of $G + H$. Similarly, if $z \in V(H)$, then S_z is a restrained convex dominating set in $G + H$. Accordingly, S is a secure restrained convex dominating set in $G + H$.

Next, if statement (ii) of Theorem 2.11 holds, then S is a restrained convex dominating set in $G + H$ by Theorem 2.10(ii). By using similar proofs when statement (i) of Theorem 2.11 holds, S is a secure restrained convex dominating set in $G + H$.

Now, suppose that statement (iii) of Theorem 2.11 holds. Let $S_G = \{ v \} \subset V(G)$ and $S_H = \{ w \} \subset V(H)$. Then $S = S_G \cup S_H$, where S_G and S_H are cliques in G and H, respectively. Thus, S is a restrained convex dominating set in $G + H$ by Theorem 2.10(iii). Since $V(G + H) \setminus S \neq \emptyset$, let $u \in V(G + H) \setminus S$.

Suppose that $u \in V(G)$. Since S is a secure convex dominating set in $G + H$, there exists $x \in S$, say $x = w$, such that $uw \in E(G + H)$ and $S_u = (S \setminus \{ w \}) \cup \{ u \}$ is a convex dominating set in $G + H$ and $S_u = \{ v, u \} \subset V(G)$. This implies that S_u is a clique dominating set in G. Thus, S_u is a restrained convex dominating set in G by Theorem 2.10(i). Similarly, if $u \in V(H)$, then S_u is a restrained convex dominating set of $G + H$. Accordingly S is a secure restrained convex dominating set in $G + H$.

Suppose that statement (iv) of Theorem 2.11 holds. Since $S = \{ v \} \cup S_H$, where $\{ v \}$ and S_H are cliques in G and H respectively, it follows that S is a restrained convex dominating set in $G + H$ by Theorem 2.10(iii). Since $V(G + H) \setminus S \neq \emptyset$, let $u \in V(G + H) \setminus S$.

If $u \in V(G)$, then $S_u = (S \setminus \{ v \}) \cup \{ u \} = S_H \cup \{ u \}$. Since $\{ u \}$ and S_H are cliques in G and H respectively, it follows that S_u is a restrained convex dominating set in $G + H$ by Theorem 2.10(iii). Now, if $u \in V(H)$, then there exists $w \in S_H$ such that $uw \in E(H)$ and $(S_H \setminus \{ w \}) \cup \{ u \}$ is a clique dominating set in H. This implies that $S_u = (S \setminus \{ w \}) \cup \{u\} = (\{ v \} \cup S_H \setminus \{ w \}) \cup \{ u \}$. Since $\{ v \}$ and $(S_H \setminus \{ w \}) \cup \{ u \}$ are cliques in G and H respectively, it follows that S_u is a restrained convex dominating set in $G + H$ by Theorem 2.10(iii). Accordingly, S is a secure restrained dominating set of $G + H$. Similarly, if statement (v) of Theorem 2.11 holds, then S is a secure restrained dominating set of $G + H$.

Finally, if statement (vi) of Theorem 2.11 holds, then S is a secure restrained convex dominating set in $G + H$ (using similar proofs).

The following result is a quick consequence of Theorem 2.12.

Corollary 2.13: Let G and H be connected non-complete graphs. Then

$$\gamma_{src}(G + H) = \begin{cases} 2, & if\ \gamma(G) = 1 = \gamma(H)\ or\ \gamma_{scl}(G) = 2\ or\ \gamma_{scl}(H) = 2 \\ V(G + H), & if\ otherwise \end{cases}$$

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