International Journal of Mathematical Archive-8(6), 2017, 227-231
Available online through www.ijma.info ISSN 2229 – 5046

ON EDGE-EDGE DOMINATION IN GRAPHS

D. K. THAKKAR[1], N. P. JAMVECHA*[2]

[1]Department of Mathematics,
Saurashtra University, Rajkot-360005, Gujarat, India.

[2]Department of Mathematics,
Shree M. & N. Virani Science College, Rajkot-360005, Gujarat, India.

(Received On: 16-06-17; Revised & Accepted On: 30-06-17)

ABSTRACT

In this paper we further study the concept of edge-edge domination in graphs. We observe that the edge-edge domination number of a graph may increase or decrease or remains same when a edge is removed from a graph. We proved a necessary and sufficient condition under which the edge-edge domination number of a graph increases and also we proved a necessary and sufficient condition under which the edge-edge domination number of a graph decreases. For this purpose we introduce two new concepts namely e-dominating neighbourhood of an edge and private edge-edge neighbourhood of an edge with respect to a set containing the edge. Some examples also have been given.

Key words: *edge-edge dominating set, minimal edge-edge dominating set, minimum edge-edge dominating set, edge-edge domination number, e-dominating neighbourhood, Private edge-edge neighbourhood.*

AMS subject Classification (2010): *05C69.*

INTRODUCTION

The concept of edge-edge domination was introduced by R. S. Bhat, S. S. Kamath and S. R. Bhat in [4]. An edge $g = uv$ be e-dominates the edge $h = xy$ if $x, y \in N[u] \cup N[v]$. A set F of edges is said to be an edge-edge dominating(EED) set of G if for every edge h not in F, there is an edge g in F such that h is e-dominated by g. An edge-edge dominating set with minimum cardinality is called minimum edge-edge dominating set. The cardinality of minimum edge-edge dominating set is called edge-edge domination number and it is denoted by $\gamma_{ee}(G)$ [4].

It may happens that g e-dominates h but h does not e-dominate g. We may note that if F is an edge dominating set then every edge g which is not in F is adjacent to some member of F and thus F is an EED set of G. However, an EED set need not be edge dominating set.

The concept of vertices dominates edges and edges dominate vertices was introduced in 1985 by R. Laskar and Ken Peters [3] and then in 1992 by, Sampathkumar and S. S. Kamath [2]. A vertex v of a graph G m-dominates an edge xy if xy is an edge of the subgraph induced by the vertices of the $N[v]$. An edge x m-dominates a vertex v if $v \in N[x]$. Suppose xy & uv are two edges and suppose x m-dominates the edge uv then obviously xy e-dominates uv. Suppose xy and uv are two edges and suppose edge xy m-dominates u and xy m-dominates v then obviously xy e-dominates uv (edge). Converse is also true. Also note that $\gamma_{ee}(G) \le \gamma'(G)$.

Corresponding Author: N. P. Jamvecha[2]*
[2]Department of Mathematics, Shree M. & N. Virani Science College, Rajkot-360005, Gujarat, India.

PRELIMINARIES AND NOTATIONS

If G is a graph then $E(G)$ denotes the edge set and $V(G)$ denotes the vertex set of the graph. S is any set then $|S|$ denotes the cardinality of S and $E(G)\setminus S$ is a subgraph of G obtained by removing the edges of S. If f is an edge of G then $G\setminus f$ denotes the subgraph of G obtained by removing the edge f. $N[x]$ denotes the set of adjacent vertices of v including v and $N(V)$ denotes the set of vertices which are adjacent to v.

In this paper we consider only simple graphs with finite vertex set.

Proposition 1: Let G be a graph and $g = uv$ be an isolated edge of G. Let F be any EED set of G then $g \in F$.

Proof: Suppose $g \notin F$. Now there is an edge $f = xy \in F$ such that g is e-dominated by f and therefore $u = x$ or u is adjacent to x and $v = y$ or v is adjacent to y. But this implies that g is not an isolated edge of G and therefore $g \in F$.

Proposition 2: Let G be a graph. g be a pendant edge of G and F be a minimum EED set of G then $g \in F$ or for some edge f adjacent to g, $f \in F$.

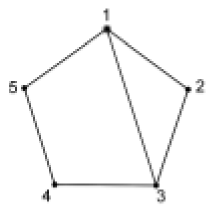
Proof: Suppose $g \notin F$. Now e is e-dominated by some edge f in F. Let $g = uv$ and $f = xy$. Suppose, v is the pendant vertex of g.

Case-I : $u = x$ or $u = y$
Then obviously $f = xy$ is adjacent to g.

Case II: $u \neq x$, $u \neq y$
Then $v \notin N[x] \cup N[y]$ and this contradicts the fact that g is e-dominated by $f = xy$. Thus g is adjacent to $f = xy$.

Definition 3: (e-dominating neighbourhood) Let G be a graph and f be an edge of G. Then e-dominating neighbourhood of f in G is the set $N_{ee}[f] = \{g \in E(G) / g \text{ e-dominates } f\}$.

Example 1: Consider the graph G_1. Let $f = \{15\}$ then the e-dominating neighbourhood of f will be the set $\{12, 13, 45\}$.



G_1

Theorem 4: Let G be a graph and f be any edge of G. Then $\gamma_{ee}(G\setminus f) > \gamma_{ee}(G)$ iff

1. f is not an isolated edge of G.
2. For every minimum EED set F of G, $f \in F$.
3. There is no subset F of $E(G)\setminus N_{ee}[f]$ such that $|F| \leq \gamma_{ee}(G)$ and F is an EED set of $G\setminus f$.

Proof: First suppose that $\gamma_{ee}(G\setminus f) > \gamma_{ee}(G)$.

1. Suppose f is an isolated edge of G. Let F be any minimum EED set of G. Then $f \in F$. Let $F_1 = F\setminus\{f\}$. Now consider the subgraph $G\setminus f$. Let h be any edge of $G\setminus f$ such that h does not belongs to F_1. Then $h \notin F$. Now h is e-dominated by some member g of F. Obviously, $g \neq f$ because f is isolate edge of G. Therefore h is e-dominated by some member of F_1. Thus F_1 is EED set of $G\setminus f$. Hence, $\gamma_{ee}(G\setminus f) \leq |F_1| < |F| = \gamma_{ee}(G)$. Which is a contradiction. Therefore f is not isolated edge of G.

2. Suppose there is a minimum EED set F of G such that $f \notin F$. Now, consider the subgraph $G \setminus f$. It can be easily proved that F is an EED set of $G \setminus f$. Therefore, $\gamma_{ee}(G \setminus f) < |F| = \gamma_{ee}(G)$. Which is a contradiction. Therefore, condition (2) holds.
3. Suppose there is a subset F of $E(G) \setminus N_{ee}[f]$ and F is an EED set of $G \setminus f$. Then again $\gamma_{ee}(G \setminus f) \leq |F| \leq \gamma_{ee}(G)$. Which is again a contradiction. Therefore, condition (3) also holds.

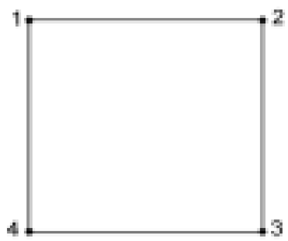
Conversely, suppose (1), (2) & (3) hold.

First suppose that $\gamma_{ee}(G \setminus f) = \gamma_{ee}(G)$. Let F be any minimum EED set of $G \setminus f$. Suppose F is also an EED set of G. Then F is a minimum EED set of G not containing f. Which contradicts (2).

Suppose F is not an EED set of G. Therefore, f is not e-dominated by any member of F and therefore, $F \cap N_{ee}[f] = \phi$. Thus F is an EED set of $G \setminus f$ such that $|F| \leq \gamma_{ee}(G)$ and F is subset of $E(G) \setminus N_{ee}[f]$. This contradicts condition (3). Thus, in both the cases we have contradiction. Therefore, $\gamma_{ee}(G \setminus f) = \gamma_{ee}(G)$ is not possible.
Suppose, $\gamma_{ee}(G \setminus f) < \gamma_{ee}(G)$. Let F be any minimum EED set of $G \setminus f$ then F cannot be an EED set of G. Therefore, f is not e-dominated by any member of F. Which means that F is a subset of $E(G) \setminus N_{ee}[f]$. Also, $|F| \leq \gamma_{ee}(G)$ and F is an EED set of $G \setminus f$. This again contradicts (3). Thus $\gamma_{ee}(G \setminus f) < \gamma_{ee}(G)$ is also not possible. Hence, it must be true that $\gamma_{ee}(G \setminus f) > \gamma_{ee}(G)$.

Definition 5: (Private edge-edge neighbourhood of g with respect to F) Let G be a graph. F be a set of edges and $g \in F$. Then the private edge-edge neighbourhood of g with respect to F is $prn_{ee}[g,F] = \{ h \in E(G) / h$ is e-dominated by only one member of F namely $g \} = \{h \in E(G) / N_{ee}[h] \cap F = \{g\}\}$

Example 2: Consider the graph G_2. If we take $F = \{12, 34\}$ and $g = \{12\}$ then the private edge-edge neighbourhood of g with respect to F is the set $\{12\}$.



G_2

Theorem 6: Let G be a graph and $g \in E(G)$. Then $\gamma_{ee}(G \setminus g) < \gamma_{ee}(G)$ if and only if there is a minimum EED set F of G containing g such that $prn_{ee}[g,F] = \{g\}$.

Proof: Suppose, $\gamma_{ee}(G \setminus g) < \gamma_{ee}(G)$. Let F_1 be a minimum EED set of $G \setminus g$. Then F_1 cannot be an EED set of G. This implies that there is no member of F_1 which e-dominates g.

Let $F = F_1 \cup \{g\}$. Then obviously F is an EED set of G. Since $\gamma_{ee}(G \setminus g) < \gamma_{ee}(G)$, F must be a minimum EED set of G. Also, $g \in F$. Since g is not e-dominated by any other member of F and g is e-dominated by g itself, $g \in prn_{ee}[g,F]$. Suppose h is an edge of G such that $h \neq g$ and $h \in prn_{ee}[g,F]$. Then, $h \notin F_1$ because if $h \in F_1$ then $h \in F$ and this implies that h is e-dominated by two distinct member of F namely g and h which is a contradiction and thus $h \notin F_1$.

Now, h is e-dominated by some member h' of F_1 because F_1 is a EED set of $G \setminus g$. Then $h' \in F$ and we have h is e-dominated by two distinct members of F namely h' and g. Which is a contradiction. Thus, we have proved that if $h \neq g$ then $h \notin prn_{ee}[g,F]$. Thus, $prn_{ee}[g,F] = \{g\}$.

Conversely, suppose there is a minimum EED set F of G such that $prn_{ee}[g,F]=\{g\}$. Let $F_1 = F \setminus \{g\}$. Let h be an edge of $G \setminus g$ such that $h \notin F_1$. Then $h \notin F$.

Suppose h is e-dominated by g. Since $h \notin prn_{ee}[g,F]$, h must be e-dominated by some $g' \in F \ni g' \neq g$. Then $g' \in F_1$ and h is e-dominated by g'. If h is not e-dominated by g then h must be e-dominated by some other member h'' of G. Obviously, $h'' \in F_1$. Thus F_1 is an EED set of $G \setminus g$. Therefore $\gamma_{ee}(G \setminus g) < \gamma_{ee}(G)$.

Corollary 7: Let G be a graph and $g \in E(G)$. If $\gamma_{ee}(G \setminus g) < \gamma_{ee}(G)$ then $\gamma_{ee}(G \setminus g) = \gamma_{ee}(G) - 1$.

Corollary 8: Let G be a graph, g and h be two edges of G such that $\gamma_{ee}(G \setminus g) < \gamma_{ee}(G)$ and $\gamma_{ee}(G \setminus h) > \gamma_{ee}(G)$. Then g is not e-dominated by h. In particular, g and h cannot be an adjacent edges.

Proof: There is a minimum EED set of G such that $g \in F$ and $prn_{ee}[g,F]=\{g\}$. This means that g is not e-dominated by any other member of F. In Particular, g is not adjacent to any other member of F. Now, $h \in F$ by theorem ($\gamma_{ee}(G \setminus h) > \gamma_{ee}(G)$). Therefore, g is not e-dominated by h. Also g and h are non-adjacent edges.

Theorem 9: Let G be a graph. g be an edge of G such that $\gamma_{ee}(G \setminus g) > \gamma_{ee}(G)$. If F is a minimum EED set of G then $g \in F$ and $prn_{ee}[g,F]$ contains two non-adjacent edges.

Proof: Since, $\gamma_{ee}(G \setminus g) > \gamma_{ee}(G)$ and $g \in F$. Since F is a minimal EED of G, $prn_{ee}[g,F] \neq \phi$. If $prn_{ee}[g,F]=\{g\}$ then $\gamma_{ee}(G \setminus g) < \gamma_{ee}(G)$ (By theorem 6) and this is a contradiction. Therefore, there is an edge $h \neq g$ such that $h \in prn_{ee}[g,F]$. Obviously $h \notin F$. Suppose $prn_{ee}[g,F]=\{h\}$. Now consider the set $F_1 = (F \setminus \{g\}) \cup \{h\}$. Then $|F_1| = |F|$ and $g \notin F_1$.

Let h' be any edge of G such that $h' \in F_1$. Suppose $h' = g$. Now $g \notin prn_{ee}[g,F]$. Therefore, $h' = g$ is e-dominated by some member of F_1. Now suppose $h' \neq g$. Again $h' \notin prn_{ee}[g,F]$. Therefore h' is e-dominated by some member of F different from g. Therefore, h' is e-dominated by some member of F_1. Thus F_1 is a minimum EED set of G not containing g which is a contradiction. Thus, $prn_{ee}[g,F]$ must contain at least two edges.

Suppose, $prn_{ee}[g,F]=\{g,h\}$, where $h \neq g$. If g and h are non-adjacent edges then the statement of the theorem is proved.

Suppose g and h are adjacent edges. Let $F_1 = (F \setminus \{g\}) \cup \{h\}$. Note that $|F_1| = |F|$. Since g and h are adjacent edges, g is e-dominated by h which is in F_1.

Let f be any edge such that $f \notin F_1$ and $f \neq g$. If f is e-dominated by g then f is also e-dominated by some other member of F because $f \notin prn_{ee}[g,F]$. If f is not e-dominated by g then f is also e-dominated by some other member of F. Thus f is e-dominated by some member of F_1. Thus, F_1 is a minimum EED set of G such that $g \notin F_1$ which is again a contradiction. Thus it follows that $prn_{ee}[g,F]$ contains at least two edges h_1 and h_2 such that $h_1 \neq g$, $h_2 \neq g$.

Suppose any two edges in the $prn_{ee}[g,F]$ are adjacent. Let $h_1, h_2 \in prn_{ee}[g,F]$ such that $h_1 \neq g$ and $h_2 \neq g$. Then h_1 and h_2 are adjacent.

Suppose $g \in prn_{ee}[g,F]$. If h_1 or h_2 is not adjacent to g then we have a contradiction because g & h_1 or g & h_2 are non-adjacent edges in the $prn_{ee}[g,F]$. Therefore g & h_1 are adjacent and g & h_2 are adjacent. Let $F_1 = (F \setminus \{g\}) \cup \{h_1\}$. Obviously, g is e-dominated by h_1 because g & h_1 are adjacent. h_2 is e-dominated by h_1. Any other edge f which is not in F_1, if it is e-dominated by g or otherwise is e-dominated by some other member of F. This means that f is e-dominated by some member of F_1. Thus we have proved that F_1 is a minimum EED set of G not containing g which is a contradiction. Thus we have proved that $g \notin prn_{ee}[g,F]$.

Again let $F_1 = (F \setminus \{g\}) \cup \{h_1\}$. Then h_2 is e-dominated by h_1. Since $g \notin prn_{ee}[g,F]$, g is e-dominated by some member f of F which is also a member of F_1. Any other edge which is not in F_1 and if it is in the $prn_{ee}[g,F]$ then it is adjacent to h_1 and therefore it is e-dominated by some member of F_1. Any edge which is not in the $prn_{ee}[g,F]$ must be e-dominated by some member of F_1. Thus F_1 is minimum EED set of G not containing g which is a contradiction. Thus, we conclude that there are two distinct edges in the $prn_{ee}[g,F]$ which are non-adjacent edges.

ACKNOWLEDGEMENT

The authors are thankful to the reviewer for their comments and suggestions for improving the quality of this paper.

REFERENCES

1. E. Sampathkumar and P. S. Neeralagi, The neighbourhood number of a graph, Journal of Pure and Applied Mathematics (1985), 126-136.
2. E. Sampathkumarand S. S. Kamath, Mixed Domination in Graphs, The Indian Journal of Statistics (1992).
3. R. Laskar and K. Peters, Vertex and edge domination parameters in graphs, Congressus Numerantium 48(1985), 291-261.
4. R. S. Bhat and S. S. Kamath, Strong(weak) edge-edge domination number of a graph, Applied Mathematical Sciences, Vol. 6, 2012, Pp 5525-5531.
5. T. W. Haynes, S. T. Hedetniemi and P. J. Slater, "Fundamentals of Domination in Graphs", Marcel Dekker, Inc., New-York (1998).

Source of support: Nil, Conflict of interest: None Declared.