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GENERALIZED DIFFERENCE SEQUENCE SPACES OF MODAL INTERVAL NUMBERS DEFINED BY A SEQUENCE OF MODULUS FUNCTIONS

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ABSTRACT

In this paper we introduce and investigate the generalized difference sequence spaces of modal interval numbers $gIc(V,\lambda,f,\Delta^r,p)$, $gIc_0(V,\lambda,f,\Delta^r,p)$ *and* $gIl_\infty(V,\lambda,f,\Delta^r,p)$ *defined by a sequence of modulus functions and* $p=(p_k)$ *be any bounded sequence of positive real numbers. We also obtain some relations between these spaces as well as prove some inclusion results.*

Key words: *Difference sequence space, Modulus functions, paranorm, Modal interval numbers.*

Mathematics Subject Classification 2010: *40A05, 40C05, 46A45.*

1. INTRODUCTION

Many mathematical structures have been constructed with real or complex numbers. In recent years, these mathematical structures were replaced by fuzzy numbers or interval numbers and these mathematical structures have been very popular since 1965. Interval arithmetic is a tool in numerical computing where the rules for the arithmetic of intervals are explicitly stated. Interval arithmetic was first suggested by P.S.Dwyer [4] in 1951. Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by R.E.Moore [13], [14] in 1959 and 2009. Probably the most important paper for the development of interval arithmetic has been published by the Japanese scientist Sunaga [18].

2. PRELIMINARIES

Definition 2.1: A set consisting of a closed interval of real numbers x such that $a \le x \le b$ is called an interval number. A real interval can also be considered as a set. We denote the set of all real valued closed intervals by $I\Re$. An element of $I\Re$ is called closed interval and denoted by $\bar{x}$. That is $\bar{x} = [x_l, x_r] = \{x \in \Re : x_l \le x \le x_r\}$. An interval number $\bar{x}$ is a closed subset of real numbers. Let x_l *and* x_r be respectively referred to as the infimum (lower bound) and supremum (upper bound) of the interval number $\bar{x}$. If $\bar{x} = [0,0]$, then $\bar{x}$ is said to be a zero interval. It is denoted by $\bar{0}$. Kuo-Ping, Chiao [8] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. When $\underline{x} > \bar{x}$, $\hat{x}$ is not an interval number. But in modal analysis, $[\bar{x}, \underline{x}]$ is a valid interval. A modal $\tilde{x} = \{[\underline{x}, \bar{x}] : \underline{x}, \bar{x} \in \Re\}$ is defined by a pair of real numbers $\bar{x}, \underline{x}$. Let us denote the set of all modals by gI. If $\tilde{x} = [0,0]$, then $\tilde{x}$ is said to be a zero interval. It is denoted by $\tilde{0}$.

Definition 2.2: Let gI denote set of all modal interval numbers. Let $\tilde{x} \in gI$. Then $|\tilde{x}| = \max\{|\underline{x}|, |\bar{x}|\}$.

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Definition 2.3: For $\tilde{x}_1, \tilde{x}_2 \in gI$,

$\tilde{x}_1 = [a, a]$ (Degenerate modal interval number)

$\tilde{x}_1 = \tilde{x}_2$ if and only if $\underline{x}_1 = \underline{x}_2$ *and* $\overline{x}_1 = \overline{x}_2$.

$\tilde{x}_1 + \tilde{x}_2 = \{x \in \Re : \underline{x}_1 + \underline{x}_2 \le x \le \overline{x}_1 + \overline{x}_2\}$.

$\tilde{x}_1 \times \tilde{x}_2 = \{x \in \Re : \min(\underline{x}_1\underline{x}_2, \underline{x}_1\overline{x}_2, \overline{x}_1\underline{x}_2, \overline{x}_1\overline{x}_2) \le x \le \max(\underline{x}_1\underline{x}_2, \underline{x}_1\overline{x}_2, \overline{x}_1\underline{x}_2, \overline{x}_1\overline{x}_2)\}$

$\tilde{x}_1 / \tilde{x}_2 = \tilde{x}_1 \times \left[\frac{1}{\overline{x}_2}, \frac{1}{\underline{x}_2}\right]$

Definition 2.3: The distance between the two modal interval numbers $\tilde{x}_1, \tilde{x}_2$ is defined by

$d(\tilde{x}_1, \tilde{x}_2) = \max\{|\underline{x}_1 - \underline{x}_2|, |\overline{x}_1 - \overline{x}_2|\}$. Clearly d is a metric on gI.

Definition 2.4: Let us define transformation $f : N \to gI, k \to f(k) = \tilde{x}_k$, then $\tilde{x} = (\tilde{x}_k)$ is called sequence of modal interval numbers. $\tilde{x}_k$ is called the k^{th} term of sequence $\tilde{x} = (\tilde{x}_k)$, $\omega(gI)$ denote the set of all sequence of modal interval number with real terms.

Definition 2.5: Let $\tilde{x} = (\tilde{x}_k) = ([\underline{x}_k, \overline{x}_k]) \in \omega(gI)$. If $\underline{x}_k = \overline{x}_k$, for all $k \in N$, then the sequence $\tilde{x} = (\tilde{x}_k)$ is called degenerate sequence of modal interval numbers.

Definition 2.6: A sequence $\tilde{x} = (\tilde{x}_k)$ of modal interval numbers is said to be convergent to a modal $\tilde{x}_0$ if for each $\varepsilon > 0$ there exists a positive integer k_0 such that $d(\tilde{x}_k, \tilde{x}_0) < \varepsilon$ for all $k \ge k_0$ and we denote it by $\lim_k \tilde{x}_k = \tilde{x}_0$. Equivalently $\lim_k \tilde{x}_k = \tilde{x}_0$ if and only if $\lim_k \underline{x}_k = \underline{x}_0$ *and* $\lim_k \overline{x}_k = \overline{x}_0$.

Definition 2.7: A sequence of modal interval numbers $\tilde{x} = (\tilde{x}_k)$ is said to be bounded if there exist a positive number A such that $d(\tilde{x}_k, \tilde{0}) \le A$ for all $k \in N$.

Definition 2.8: A sequence of modal interval numbers $\tilde{x} = (\tilde{x}_k)$ is said to be Cauchy sequence of modal interval numbers if for every $\varepsilon > 0$ there exists a $k_0 \in N$ such that $d(\tilde{x}_n, \tilde{x}_m) < \varepsilon$ whenever $n, m \ge k_0$.

Definition 2.9: Let $X(gI)$ is the subset of $\omega(gI)$. A norm on $X(gI)$ is a non negative function $\|\cdot\|_{X(gI)} : X(gI) \to R^+ \cup \{0\}$ that satisfies the following properties: For all $\tilde{x}, \tilde{y} \in X(gI)$ *and* $\alpha \in R$, for all $\tilde{x} \in X(gI) - \{0\}$,

i. $\|\tilde{x}\|_{X(gI)} > 0$
ii. $\|\tilde{x}\|_{X(gI)} = 0 \Leftrightarrow \tilde{x} = \tilde{0}$
iii. $\|\tilde{x} + \tilde{y}\|_{X(gI)} \le \|\tilde{x}\|_{X(gI)} + \|\tilde{y}\|_{X(gI)}$
iv. $\|\alpha\tilde{x}\|_{X(gI)} = |\alpha|\|\tilde{x}\|_{X(gI)}$.

Definition 2.10: A function f: $[0,\infty) \to [0,\infty)$ is called a modulus if

i) $f(x) = 0$ if and only if $x = 0$,
ii) $f(x + y) \le f(x) + f(y)$ for $x,y \ge 0$,
iii) f is increasing,
iv) f is continuous from the right at 0.

Remark 2.11:. If f_1 and f_2 are modulus functions, then f_1+f_2 is a modulus function and that the function f^i (I is a positive integer), the composition of a modulus function f with itself *i* times is also a modulus function.

Definition 2.12: Let p=(p_k) be a positive sequence of real numbers. If $0 \le p_k \le \sup p_k = H$, $D = \max(1, 2^{H-1})$, then $|a_k + b_k|^{p_k} \le D\{|a_k|^{p_k} + |b_k|^{p_k}\}$ for all k and $a_k, b_k \in \mathbb{R}$.

Definition 2.13: Let $= (\lambda_n)$. be a non-decreasing sequence of positive reals tending to infinity and $\lambda_1 = 1$ and $\lambda_{n+1} \le \lambda_n + 1$. The generalized de la Vallee-Poussin means is defined by

$$t_n(\tilde{x}) = \frac{1}{\lambda_n}\sum_{k\in I_n} \tilde{x}_k,$$

Where $I_n = [n - \lambda_n + 1, n]$. A sequence $\tilde{x}$ =($\tilde{x}_k$) is said to be (V, λ)- summable to a number $\tilde{L}$ if $t_n(\tilde{x}) \to \tilde{L}$ as n→ ∞.

3. MAIN RESULTS

Definition 3.1: Let f be a modulus function and p= (p_k) be any sequence of strictly positive real numbers. We define the following sequence of modal interval numbers.

$$gIc(V,\lambda,f,\Delta^r,p) = \left\{\tilde{x} \in w(gI): \lim_n \frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\| \Delta^r \tilde{x}_k - \tilde{L}\right\|\right)\right]^{p_k} = 0, for\ some\ \tilde{L} \in gI\right\}$$

$$gIc_0(V,\lambda,f,\Delta^r,p) = \left\{\tilde{x} \in w(gI): \lim_n \frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\| \Delta^r \tilde{x}_k \right\|\right)\right]^{p_k} = 0\right\}$$

$$gIl_\infty(V,\lambda,f,\Delta^r,p) = \left\{\tilde{x} \in w(gI): \sup_n \frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\| \Delta^r \tilde{x}_k \right\|\right)\right]^{p_k} < \infty\right\}$$

where $\Delta^0(x) = (\tilde{x}_k)$, $\Delta^r x = (\Delta^r \tilde{x}_k) = (\Delta^{r-1}\tilde{x}_k - \Delta^{r-1}\tilde{x}_{k+1})$ for all $k \in N$ and the binomial representation, $\Delta^r \tilde{x}_k = \sum_{v=0}^{r}(-1)^v \binom{r}{v} \tilde{x}_{k+v}$ for all $k \in N$.

Throughout the paper Z will denote any one of the notation *gIc*, *gIc₀* and *gIl*∞.

In the case *f(x) = x*, *p_k = 1* for all *k* ∈ *N* and *p_k = 1* for all *k* ∈ *N*, we shall write Z(V, λ, Δ^r) and Z$(V, \lambda, f, \Delta^r)$ instead of Z$(V, \lambda, f, \Delta^r, p)$ respectively.

Theorem 3.2: *Let the sequence (p_k) be bounded. Then the sequence spaces of modal interval numbers $Z(V,\lambda,f,\Delta^r,p)$ are linear spaces of modal interval numbers.*

Theorem 3.3: *Let f be a modulus function. Then*

$$gIc_0(V,\lambda,f,\Delta^r,p) \subset gIc(V,\lambda,f,\Delta^r,p) \subset gIl_\infty(V,\lambda,f,\Delta^r,p).$$

Proof: The first inclusion is obvious. We establish the second inclusion.

Let $\tilde{x} \in gIc(V,\lambda,f,\Delta^r,p)$.Then, by definition of f we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^r \tilde{x}_k\|)]^{p_k} \le \frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\Delta^r \tilde{x}_k - \tilde{L}\right\|\right) + f\left(\left\|\tilde{L}\right\|\right)\right]^{p_k}$$

$$\le D\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\Delta^r \tilde{x}_k - \tilde{L}\right\|\right)\right]^{p_k} + D\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\tilde{L}\right\|\right)\right]^{p_k}.$$

There exists a positive integer K_L such that $\left\|\tilde{L}\right\| \le K_L$. Hence, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^r \tilde{x}_k\|)]^{p_k} \le \frac{D}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\|\Delta^r \tilde{x}_k - \tilde{L}\right\|\right)\right]^{p_k} + \frac{D}{\lambda_n}[K_L f(1)]^{p_k}\lambda_n.$$

Since $\tilde{x} \in gIc(V,\lambda,f,\Delta^r,p)$, we have $\tilde{x} \in gIl_\infty(V,\lambda,f,\Delta^r,p)$ and this completes the proof.

Theorem 3.4: $gIc_0(V,\lambda,f,\Delta^r,p)$ *is a paranormed (need not total paranorm) space of modal interval numbers with*

$$g_\Delta(\tilde{x}) = \sup_n \left(\frac{1}{\lambda_n}\sum_{k\in I_n}\left[f\left(\left\| \Delta^r \tilde{x}_k \right\|\right)\right]^{p_k}\right)^{\frac{1}{M}}$$

Where M = max (1, sup p_k).

Theorem 3.5: *If* $r \geq 1$, *then the inclusion* $Z(V,\lambda,f,\Delta^{r-1},p) \subset Z(V,\lambda,f,\Delta^{r},p)$ *is strict. In general* $Z(V,\lambda,f,\Delta^{i}) \subset Z(V,\lambda,f,\Delta^{r})$ *for all i= 1, 2, …, r-1 and the inclusion is strict.*

Proof: We give the proof for Z= gIl_∞ only. It can be proved in a similar way for Z=gIc_0 and Z=gIc.

Let $\tilde{x} \in gIl_\infty(V,\lambda,f,\Delta^{r-1})$. Then, we have

$$\sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} \left[f\left(\left\|\Delta^r \tilde{x}_k\right\|\right)\right] < \infty. \tag{3}$$

By definition, we have

$$\frac{1}{\lambda_n} \sum_{k\in I_n} \left[f\left(\left\|\Delta^r \tilde{x}_k\right\|\right)\right] \leq \frac{1}{\lambda_n} \sum_{k\in I_n} \left[f\left(\left\|\Delta^{r-1} \tilde{x}_k\right\|\right)\right] + \frac{1}{\lambda_n} \sum_{k\in I_n} \left[f\left(\left\|\Delta^{r-1} \tilde{x}_{k+1}\right\|\right)\right] < \infty.$$

Thus, $gIl_\infty(V,\lambda,f,\Delta^{r-1}) \subset gIl_\infty(V,\lambda,f,\Delta^{r})$. Proceeding in this way one will have $gIl_\infty(V,\lambda,f,\Delta^{i}) \subset gIl_\infty(V,\lambda,f,\Delta^{r})$ for i= 1, 2, …, r-1. Let $\lambda_n = n$ for each n $\in$N.

Let $\tilde{x} = ([k^r, k^r])$ and f(x) = x. Then $\tilde{x}$ belongs to g$Il_\infty(V,\lambda,\mathrm{f},\Delta^{\mathrm{r}})$ but $\tilde{x}$ does not belongs to $gIl_\infty(V,\lambda,f,\Delta^{r-1})$. Hence the inclusion is strict.

Proposition 3.6: $gIc(V,\lambda,f,\Delta^{r-1},p) \subset gIc_0(V,\lambda,f,\Delta^{r},p)$.

Theorem 3.7: *Let f, f_1, f_2 be modulus functions. Then we have*

(i). $Z(V,\lambda,f_1,\Delta^r,p) \subset Z(V,\lambda,f o f_1,\Delta^r,p)$.
(ii). $Z(V,\lambda,f_1,\Delta^r,p) \cap Z(V,\lambda,f_2,\Delta^r,p) \subset Z(V,\lambda,f_1+f_2,\Delta^r,p)$.

Proof: (i).Let $(\tilde{x}_k) \in \mathrm{gIc}_0(V,\lambda,\mathrm{f}_1,\Delta^{\mathrm{r}},\mathrm{p})$.

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that f(t)$< \epsilon$ for $0 \leq t \leq \delta$.

Take $y_k = \mathrm{f}_1(\|\Delta^{\mathrm{r}}\tilde{x}_{\mathrm{k}}\|)$ and consider

$$\sum_{k\in I_n} [f(y_k)]^{p_k} = \sum_1 [f(y_k)]^{p_k} + \sum_2 [f(y_k)]^{p_k} \tag{4}$$

Where the first summation is over $y_k \leq \delta$ and s second summation is over $y_k > \delta$.

Since f is continuous, we have

$$\sum_1 [f(y_k)]^{p_k} < \lambda_n \epsilon^H. \tag{5}$$

For $y_k > \delta$, we get $y_k < \frac{y_k}{\delta} \leq 1 + \frac{y_k}{\delta}$.

By definition of modulus function we have $y_k > \delta$,

$$f(y_k) \leq 2f(1)\frac{y_k}{\delta}.$$

Hence,

$$\frac{1}{\lambda_n} \sum_2 [f(y_k)]^{p_k} \leq max(1,(2f(1)\delta^{-1})^H) \frac{1}{\lambda_n} y_k. \tag{6}$$

From (5) and (6), we get $gIc_0(V,\lambda,f_1,\Delta^r,p) \subset gIc_0(V,\lambda,f o f_1,\Delta^r,p)$.

The proof of (ii) follows from the following inequality.

$$\left[(f_1+f_2)\left(\left\|\Delta^r \tilde{x}_k\right\|\right)\right]^{p_k} \leq D\left[f_1\left(\left\|\Delta^r \tilde{x}_k\right\|\right)\right]^{p_k} + D\left[f_2\left(\left\|\Delta^r \tilde{x}_k\right\|\right)\right]^{p_k}.$$

Proposition 3.8: *Let f be a modulus function. Then* $Z(V,\lambda,\Delta^r,p) \subset Z(V,\lambda,f,\Delta^r,p)$.

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