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A NEW PROOF FOR GANTOS'S THEOREM ON SEMILATTICE OF BISIMPLE INVERSE SEMIGROUPS

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***ABSTRACT*.**

Gantos has shown that, if S is a semilattice of right cancellative monoids with the (LC) condition and certain further conditions, then we can associate it with a semilattice of bisimple inverse semigroups. We show that one of Gantos's conditions is equivalent to S itself having the (LC) condition. We use this equivalence to define a simple form for the multiplication which is easier to deal with than the form which Gantos used. We provide a simple proof completely independent of Gantos's result.

Keywords: *I-orders, I-quotients, right cancellative monoid, inverse hull.*

1. INTRODUCTION

An interesting concept of semigroups of left I-quotients, based on the notion of semigroups of left quotients, was developed by the author, Gould, Cegarra and Petrich, in series of papers (see [3], [7] and [8]).

Recall that a subsemigroup S of a group G is a *left order* in G or G is *a group of left quotients* of S if any element in G can be written as $a^{-1}b$ where $a, b \in S$. Ore and Dubreil [1] showed that a semigroup S has a group of left quotients if and only if S is right reversible and cancellative. By saying that a semigroup S is *right reversible* we mean for any$a, b \in S$,$Sa \cap Sb \neq \emptyset$. A different definition proposed by Fountain and Petrich in 1985 [5] was restricted to completely 0-simple semigroups of left quotients and then shortly after to that of semigroup of left quotients by Gould [10]; this idea has been extensively developed by number of authors. A subsemigroup S of a semigroup Q is a *left order* in Q if every element in Q can be written as $a^{\natural}b$ where $a, b \in S$ and $a^{\natural}$ is an inverse of a in a *subgroup* of Q. In this case we say that Q is a semigroup of *left quotients* of S. *Right orders* and *semigroup of right quotients* are defined dually. If S is both a left and right order in Q, then S is an *order* in Q and Q is a *semigroup of quotients* of S.

The author and Gould in [7] have introduced the following definition of left I-orders in inverse semigroups: A subsemigroup S of an inverse semigroup Q is a *left I-order* in Q and Q is a semigroup of *left I-quotients* of S if every element in Q can be written as $a^{-1}b$ where $a, b \in S$ and a^{-1} is the inverse of a in the sense of an inverse semigroup theory. *Right I-orders* and semigroups of *right I-quotients* are defined dually. If S is a left and right I-order in an inverse semigroup Q, we say that S is an *I-order* in Q and Q is a semigroup of *I-quotients* of S. Let S be a left I-order in Q. Then S is *straight* in Q if every $q \in Q$ can be written as $a^{-1}b$ where $a, b \in S$ and $a\,\mathcal{R}\,b$ in Q.

Clifford [1] showed that any right cancellative monoid S with the (LC) condition is the $\mathcal{R}$-class of the identity of its inverse hull $\sum(S)$. Moreover, (in our terminology) S is a left I-order in $\sum(S)$. By saying that a semigroup S has the (LC) condition we mean for any $a, b \in S$ there is an element $c \in S$ such that $Sa \cap Sb = Sc$. Clifford established that precisely bisimple inverse monoids can be regarded as inverse hulls of right cancellative monoids S satisfying the (LC) condition. The author and Gould in [7] have extended Clifford's work to a left ample semigroup with (LC). It is worth pointing out that the inverse hull of the left ample semigroup need not be bisimple.

Gantos [11] has developed a structure for semigroups Q which are semilattices Y of bisimple inverse monoids Q_{α}, such that the set of identities elements forms a subsemigroup. His structure is determined by semigroups S which are strong semilattices Y of right cancellative monoids $S_{\alpha}, \alpha \in Y$ with (LC) condition and certain morphisms satisfying two conditions.

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In this paper, we give another proof of this result. We show that one of Gantos's conditions is equivalent to S itself having the (LC) condition. We link this with Clifford's result and our definition of left I-order to introduce a new aspect for such semigroups which we can read as follows: If S is a semilattice of right cancellative monoids with (LC) and S has (LC), then S is a left I-order in a semilattice of inverse hull semigroups. Moreover, we proved that such S is a left I-order in a strong semilattice of inverse hull semigroups.

In Section 2 we give some preliminaries. Section 3 contains our new proof of Gantos's theorem.

2. PRELIMINARIES AND NOTATIONS

We begin by recalling some of the basic facts about the relations $\mathcal{R}^*$ and $\mathcal{L}^*$. Let S be a semigroup and $a, b \in S$. We call elements a and b to be related by $\mathcal{R}^*$ if and only if a and b are related by $\mathcal{R}$ in some oversemigroup of S. Dually, we can define the relation $\mathcal{L}^*$. An alternative description of $\mathcal{R}^*$ is provided by the following lemma.

Lemma 2.1 [4]: Let S be a semigroup and $a, b \in S$. Then the following are equivalent

(i) $a\,\mathcal{R}^* b$;

(ii) for all $x, y \in S^1$ $xa = ya$ if and only if $xb = yb$.

As an easy consequence of Lemma 1.1 we have:

Lemma 2.2[4]: Let S be a semigroup, $a \in S$ and e be an idempotent of S. Then the following conditions are equivalent:

(i) $a\,\mathcal{R}^*\,e$

(ii) $a = ea$ and for all $x, y \in S^1$, $xa = ya$ implies that $xe = ye$.

It is well-known that Green star relations $\mathcal{R}^*$ and $\mathcal{L}^*$ on a semigroup S are generalizations of the usual Green's relations $\mathcal{R}$ and $\mathcal{L}$ on S, respectively.

A semigroup S is *left adequate* if every $\mathcal{R}^*$-class of S contains an idempotent and the idempotents $E(S)$ of S form a semilattice. In this case every $\mathcal{R}^*$-class of S contains a unique idempotent. We denote the idempotent in the $\mathcal{R}^*$-class of a by a^+. A left adequate monoid S is *left ample* if $(ae)^+a = ae$ for each $a \in S$ and $e \in E(S)$.

We can note easily that, any right cancellative monoid is left ample. By a right cancellative semigroup we mean, a semigroup S such that for all $x, y \in S$

$$xz = yz \text{ implies } x = y.$$

Following [9], for any left ample semigroup S we can construct an embedding of S into the symmetric inverse semigroup $\mathcal{I}_S$ as follows. For each $a \in S$ we let $\rho_a \in \mathcal{I}_S$ be given by

$$\text{dom } \rho_a = Sa^+ \; and \; \text{im } \rho_a = Sa$$

and for any $x \in \text{dom } \rho_a$.

$$x\rho_a = xa.$$

Then the map $\theta_S: S \to \mathcal{I}_S$ is a (2,1)-embedding.

The inverse hull of a left ample semigroup S is the inverse subsemigroup $\sum(S)$ of $\mathcal{I}_S$ generated by im θ_S. If S is a right cancellative monoid, then for any $a \in S$ we have $a^+ = 1$. Then $\rho_a: S \to Sa$ is defined by

$$x\rho_a = xa \text{ for each } x \text{ in } S.$$

Hence dom $\rho_a = S =$ dom I_S, giving that im $\theta_S \subseteq R_1$ where R_1 is the $\mathcal{R}$ -class of I_S in $\mathcal{I}_S$.

As in [7] we say that a (2,1)-morphism $\phi: S \to T$, where S and T are left ample semigroups with Condition (LC), is *(LC)-preserving* if, for any $b, c \in S$ with $Sb \cap Sc = Sw$, we have that

$$T(b\phi) \cap S(c\phi) = S(w\phi).$$

Let S be a left I-order in an inverse semigroup Q. The Generalisation of Green's relations $\mathcal{R}^*$ and $\mathcal{L}^*$ are on S. To emphasis that $\mathcal{R}$ and L are relations Q, we may write $\mathcal{R}^Q$ and $\mathcal{L}^Q$ or $\mathcal{R}$ in Q and $\mathcal{L}$ in Q.

We will make heavy use of the following result [7, Corollary 3.10].

Lemma 2.3: [2,7] The following conditions are equivalent for a right cancellative monoid S:

(*i*) $\sum(S)$ is bisimple;

(*ii*) S has Condition (LC);

(*iii*) S is a left I-order in $\sum(S)$.

If the above conditions hold, then S is the $\mathcal{R}$-class of the identity of $\sum(S)$. Further, $\sum(S)$ is proper if and only if S is cancellative.

Conversely, the $\mathcal{R}$-class of the identity of any bisimple inverse monoid is right cancellative with Condition (LC).

To prove our main result, we will also need the following lemma.

Lemma 2.4: (*cf***.** [6]**)** Let S be a semilattice Y of right cancellative monoids $S_\alpha,\ \alpha \in Y$. Let e_α denote the identity of $S_\alpha,\ \alpha \in Y$. Then

(1) $e_\beta a_\alpha = a_\alpha e_\beta$ if $\alpha \geq \beta$;
(2) $e_\alpha e_{\alpha\beta} = e_{\alpha\beta}$ where e_α, $e_{\alpha\beta}$ are the identities of S_α and $S_{\alpha\beta}$ respectively;
(3) $E(S)$ is a semilattice;
(4) the idempotents are central;
(5) for any $a, b \in S$, $a\ \mathcal{R}^*\ b$ in S if and only if $a, b \in S_\alpha$ for some α in Y;
(6) S is a left ample semigroup.

Proof: (1) Let $e_\beta \in S_\beta$ and $a_\alpha \in S_\alpha$ for some $\alpha, \beta \in Y$, where $\alpha \geq \beta$. Then $e_\beta a_\alpha$ and $a_\alpha e_\beta$ are in $S_{\alpha\beta} = S_\beta$. Hence

$$e_\beta a_\alpha = (e_\beta a_\alpha)e_\beta = e_\beta(a_\alpha e_\beta) = a_\alpha e_\beta.$$

(2) Let $e_\alpha \in S_\alpha$ and $e_\beta \in S_\beta$ be the identities of S_α and S_β respectively. From (1) it follows that

$$e_\alpha e_{\alpha\beta} = e_\alpha e_\alpha e_{\alpha\beta} = e_\alpha e_{\alpha\beta} e_{\alpha\beta}.$$

Hence $(e_\alpha e_{\alpha\beta})e_\alpha e_{\alpha\beta} = e_\alpha e_{\alpha\beta}$, that is, $e_\alpha e_{\alpha\beta}$ is an idempotent in $S_{\alpha\beta}$. But there is only one idempotent in $S_{\alpha\beta}$, so that $e_\alpha e_{\alpha\beta} = e_{\alpha\beta} = e_{\alpha\beta} e_\alpha$.

(3) Let $e_\alpha \in S_\alpha$ and $e_\beta \in S_\beta$ for some $\alpha, \beta \in Y$. Then $e_\alpha e_\beta \in S_{\alpha\beta}$ and from (2) we have that

$$e_\alpha e_\beta = e_\alpha e_\beta e_{\alpha\beta} = e_\alpha e_{\alpha\beta} = e_{\alpha\beta}.$$

(4) Let $e_\alpha \in S_\alpha$ and $e_\beta \in S_\beta$ for some $\alpha, \beta \in Y$. Then $e_\alpha a_\beta \in S_{\alpha\beta}$ and from (1) and (2) we get

$$e_\alpha a_\beta e_{\alpha\beta} = e_\alpha e_{\alpha\beta} a_\beta = e_{\alpha\beta} a_\beta = a_\beta e_{\alpha\beta} = a_\beta e_\alpha e_{\alpha\beta}.$$

Since $e_{\alpha\beta}$ is the identity of $S_{\alpha\beta}$, we have that $e_\alpha a_\beta = a_\beta e_\alpha$.

(5) Suppose that $a\ \mathcal{R}^*\ b$ in S where $a \in S_\alpha$ and $b \in S_\beta$. Then $e_\beta a = e_\beta e_\alpha a$ and so $e_\beta b = e_\beta e_\alpha b$ which implies that $\beta \leq \alpha$. Dually, $\alpha \leq \beta$ and hence $\alpha = \beta$.

Conversely, suppose that $b \in S_\alpha$ and $xb = yb$ for some $x, y \in S$ where $x \in S_\beta$ and $y \in S_\gamma$. Then $\beta\alpha = \alpha\gamma$ as $xb, yb \in S_{\alpha\beta} = S_{\alpha\gamma}$. Thus $xbe_{\alpha\beta} = ybe_{\alpha\beta}$ so that from (1) we get $xe_{\alpha\beta}b = ye_{\alpha\beta}b$, and so $xe_{\alpha\beta}(be_{\alpha\beta}) = ye_{\alpha\beta}(be_{\alpha\beta})$. Now $xe_{\alpha\beta}, ye_{\alpha\beta}, be_{\alpha\beta}$ all lie in $S_{\alpha\beta}$ which is right cancellative, so that $xe_{\alpha\beta} = ye_{\alpha\beta}$. As in the proof of (3) we have that $e_\alpha e_\beta = e_\beta e_\alpha = e_{\alpha\beta}$. Hence $xe_\beta e_\alpha = ye_\beta e_\alpha = ye_\gamma e_\alpha$ and then $xe_\alpha = ye_\alpha$. Also, if $xb = b$, that is, $xb = e_\alpha b$, then $xe_\alpha = e_\alpha e_\alpha = e_\alpha$. Thus $b\ \mathcal{R}^* e_\alpha$ in S. Hence for any $a \in S_\alpha$ we have that $a\ \mathcal{R}^* b$ in S as required.

(6) From (3) we have that $E(S)$ is a semilattice. By (5) we deduce that each $\mathcal{R}^*$-class contains an idempotent which must be unique as $E(S)$ is a semilattice. Notice that if $a \in S_\alpha$, then $a^+ = e_\alpha$. To see that S is left ample, let $a \in S_\alpha$ and $e_\beta \in S_\beta$. We have to show that $ae_\beta = (ae_\beta)^+ a$. Using (1) and the fact that $e_\alpha e_\beta = e_\beta e_\alpha = e_{\alpha\beta}$ as in the proof of (3) we get

$$(ae_\beta)^+ a = e_{\alpha\beta} a = ae_{\alpha\beta} = ae_\alpha e_\beta = ae_\beta$$

as required.

3. PROOF OF THE THEOREM

Gantos's main theorem states: Let S be a strong semilattice Y of right cancellative monoids $S_\alpha,\ \alpha \in Y$ with (LC) condition and connecting morphisms $\varphi_{\alpha,\beta}, \alpha \geq \beta$. Suppose in addition that (C_2) holds, where (C_2): if $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$ for all $a_\alpha, b_\alpha, c_\alpha \in S_\alpha$, then

$$S_\beta(a_\alpha\varphi_{\alpha,\beta}) \cap S_\beta(b_\alpha\varphi_{\alpha,\beta}) = S_\beta(c_\alpha\varphi_{\alpha,\beta})$$

for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$. In the terminology of Section 2 (C_2) says that the connecting morphisms are (LC)-preserving. He obtained a semigroup Q which is a semilattice Y of bisimple inverse semigroup Q_α, with identity $e_\alpha, \alpha \in Y$ such that $\{e_\alpha : \alpha \in Y\}$ is a subsemigroup of Q. In fact, Q_α is the inverse hull of S_α for each $\alpha \in Y$. We show that (C_2) is equivalent to S having the (LC) condition. We then reprove Gantos's result. In Theorems 3.13 and 3.15, we provide a simple proof completely independent of [11].

Let $\sum(S)$ be the inverse hull of left I-quotents of a right cancellative monoid S with (LC). In the rest of this section we identify S with $S\theta_S$, where θ_S is the embedding of S into $\mathcal{I}_S$. We write $a^{-1}b$ short for the element $\rho_a{}^{-1}\rho_b$ of $\sum(S)$ where $a, b \in S$.

Theorem 3.1: Let $Q = [Y; S_\alpha]$ be a semilattice of right cancellative monoids S_α with identity e_α, $\alpha \in Y$. Suppose that S, and each S_α, has (LC). Then $Q = [Y; \sum_\alpha]$ is a semilattice of bisimple inverse monoids (where $\sum_\alpha$ is the inverse hull of S_α) and the multiplication in Q is defined by: for $a^{-1}b \in \sum_\alpha$, $c^{-1}d \in \sum_\beta$,

$$a^{-1}bc^{-1}d = (ta)^{-1}(rd)$$

where $S_{\alpha\beta}b \cap S_{\alpha\beta}c = S_{\alpha\beta}w$ and $tb = rc = w$ for some $t, r \in S_{\alpha\beta}$.

Proof: By Lemma 2.3, each S_α is a left I-order in $\sum_\alpha$ where S_α is the $\mathcal{R}$ -class of the identity of $\sum_\alpha$. We prove the theorem by means of a sequence of lemmas. We begin by the following lemma due to Clifford.

Lemma 3.2: (cf. [2, Lemma 4.1]) Let T be a right cancellative monoid. Then for $a, b \in T$ we have

$$a\, \mathcal{L}\, b \textit{ if and only if } a = ub,$$

for some unit u of T.

Lemma 3.3: Let Q be an inverse monoid. Let $a, b, c, d \in R_1$. Then

$$a^{-1}b = c^{-1}d \textit{ if and only if } a = uc \textit{ and } b = ud,$$

for some unit u.

Proof: Suppose that $a^{-1}b = c^{-1}d$ where $a, b, c, d \in R_1$. Since $a, b, c, d \in R_1$we have that

$$a^{-1}\, \mathcal{R}\, a^{-1}b = c^{-1}d\, \mathcal{R}\, c^{-1} \textit{ in } Q.$$

Then $a\, \mathcal{L}\, c$ in Q. Since $a\, \mathcal{R}\, b$, it follows that $b = aa^{-1}b = ac^{-1}d$. We claim that $ac^{-1}$is a unit. As $a\, \mathcal{L}\, c$, it follows that $ac^{-1}\mathcal{L}\, cc^{-1} = 1$. Since $c^{-1}\, \mathcal{R}\, c^{-1}$ we have that $1 = ac^{-1}\, \mathcal{R}\, ac^{-1}$ and hence $u = ac^{-1}$ is a unit, and we obtain $b = ud$. Since $u = ac^{-1}$ and $a\, \mathcal{L}\, c$ we have that $uc = ac^{-1}c = a$. The converse is clear.

Lemma 3.4: The multiplication is well-defined.

Proof: Suppose that we have elements a_1, b_1, a_2, b_2 of S_α, c_1, d_1, c_2, d_2 of S_β such that

$$a_1{}^{-1}b_1 = a_2{}^{-1}b_2 \text{ in } \textstyle\sum_\alpha \text{ and } c_1{}^{-1}d_1 = c_2{}^{-1}d_2 \text{ in } \textstyle\sum_\beta.$$

By Lemma 3.3,

$$a_1 = u_1a_2,\ b_1 = u_1b_2$$

for some unit $u_1 \in S_\alpha$ and

$$c_1 = v_1c_2,\ d_1 = v_1d_2$$

for some unit $v_1 \in S_\beta$. By definition,

$$a_1{}^{-1}b_1c_1{}^{-1}d_1 = (t_1a_1)^{-1}(r_1d_1)$$

Where

$$S_{\alpha\beta}b_1 \cap S_{\alpha\beta}c_1 = S_{\alpha\beta}w_1 \text{ and } t_1b_1 = r_1c_1 = w_1$$

for some $t_1, r_1, w_1 \in S_{\alpha\beta}$. Also,

$$a_2{}^{-1}b_2c_2{}^{-1}d_2 = (t_2a_2)^{-1}(r_2d_2)$$

Where

$$S_{\alpha\beta}b_2 \cap S_{\alpha\beta}c_2 = S_{\alpha\beta}w_2 \text{ and } t_2b_2 = r_2c_2 = w_2$$

for some $t_2, r_2, w_2 \in S_{\alpha\beta}$.

We have to show that $a_1{}^{-1}b_1c_1{}^{-1}d_1 = a_2{}^{-1}b_2c_2{}^{-1}d_2$, that is,

$$(t_1a_1)^{-1}(r_1d_1) = (t_2a_2)^{-1}(r_2d_2)$$

and to do this we need to prove that

$$t_1a_1 = ut_2a_2 \text{ and } r_1d_1 = ur_2d_2$$

for some unit u in $S_{\alpha\beta}$, using Lemma 3.3. We aim to prove that $S_{\alpha\beta}w_1 = S_{\alpha\beta}w_2$. We get this if we prove that $S_{\alpha\beta}b_1 = S_{\alpha\beta}b_2$ and $S_{\alpha\beta}c_1 = S_{\alpha\beta}c_2$.

Since $b_1 = u_1b_2$, using Lemma 2.4, we have that

$$e_{\alpha\beta}b_1 = e_{\alpha\beta}u_1b_2 = (u_1e_{\alpha\beta})b_2 = (u_1e_{\alpha\beta})(e_{\alpha\beta}b_2)$$

and as $b_2 = u_1{}^{-1}b_1$, we have

$$S_{\alpha\beta}b_1 = S_{\alpha\beta}e_{\alpha\beta}b_1 = S_{\alpha\beta}e_{\alpha\beta}b_2 = S_{\alpha\beta}b_2.$$

Similarly, $S_{\alpha\beta}e_{\alpha\beta}c_1 = S_{\alpha\beta}e_{\alpha\beta}c_2$. Hence $S_{\alpha\beta}w_1 = S_{\alpha\beta}w_2$ so that $w_1\, \mathcal{L}\, w_2$ in $S_{\alpha\beta}$. By Lemma 3.2, $w_1 = lw_2$ for some unit l in $S_{\alpha\beta}$. Then

$$w_1 = t_1b_1 = lw_2 = l(t_2b_2) = lt_2(u_1{}^{-1}b_1).$$

But, by Lemma 2.4 $a_1\, \mathcal{R}^*b_1$ in S, it follows that $t_1a_1 = lt_2u_1{}^{-1}a_1 = lt_2a_2$. Since

$$w_1 = r_1c_1 = lw_2 = lr_2c_2 = lr_2v_1{}^{-1}c_1$$

and $c_1\, \mathcal{R}^*d_1$ in S, again using Lemma 2.4 we have

$$r_1d_1 = lr_2v_1{}^{-1}d_1 = lr_2v_1{}^{-1}v_1d_2 = lr_2d_2$$

as required.

In order to prove the associative law we need to introduce subsidiary lemmas. The proof of the next lemma is depends only on the fact that S_α is right cancellative and the proof can be found in [11].

Lemma 3.5: $(S_\alpha a_\alpha \cap S_\alpha b_\alpha)c_\alpha = S_\alpha a_\alpha c_\alpha \cap S_\alpha b_\alpha c_\alpha \; for\ all\ a_\alpha, b_\alpha, c_\alpha \in S_\alpha$.

In the following lemma we prove the equivalence between S having the (LC) condition and (C_2) mentioned in the introduction.

Lemma 3.6: Let $S = [Y; S_\alpha]$ be a semilattice Y of right cancellative monoids S_α with the (LC) condition. Then S has (LC) if and only if whenever $\beta \leq \alpha$, if $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$ $(a_\alpha, b_\alpha, c_\alpha \in S_\alpha)$, then if

$$S_\beta(a_\alpha e_\beta) \cap S_\beta(b_\alpha e_\beta) = S_\beta(c_\alpha e_\beta).$$

Proof: Suppose that $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$ implies $S_\beta a_\alpha \cap S_\beta b_\alpha = S_\beta c_\alpha$ for all $\beta \leq \alpha$. Let $a \in S_\alpha$ and $b \in S_\beta$ for some $\alpha, \beta \in Y$. Then $ae_{\alpha\beta}, e_{\alpha\beta}b \in S_{\alpha\beta}$ so that as $S_{\alpha\beta}$ has (LC) we know that

$$S_{\alpha\beta}(e_{\alpha\beta}a) \cap S_{\alpha\beta}(e_{\alpha\beta}b) = S_{\alpha\beta}c$$

for some $c \in S_{\alpha\beta}$. Now, let $d \in Sa \cap Sb$, say $d \in S_\gamma$ so that $\gamma \leq \alpha\beta$ and $d = ua = vb$ for some $u, v \in S$. By assumption,

$$S_\gamma(e_{\alpha\beta}a)e_\gamma \cap S_\gamma\big(e_{\alpha\beta}b\big)e_\gamma = S_\gamma c e_\gamma.$$

Then $S_\gamma a e_\gamma \cap S_\gamma b e_\gamma = S_\gamma c e_\gamma$. Now,

$$d = ua = vb = \big(e_\gamma u\big)a = \big(e_\gamma v\big)b \in S_\gamma a \cap S_\gamma b = S_\gamma c$$

as $e_\gamma u, e_\gamma v \in S_\gamma$. Then $d \in S_\gamma c$ and so $Sd \subseteq Sc$. Thus $Sa \cap Sb \subseteq Sc$. Also, $c \in S_{\alpha\beta}a \subseteq Sa$ and $c \in S_{\alpha\beta}b \subseteq Sb$.

Thus $c \in Sa \cap Sb$. Hence $Sc \subseteq Sa \cap Sb$ and we get $Sc = Sa \cap Sb$.

On the other hand, suppose that S has (LC) and let $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$, so that $c_\alpha = u_\alpha b_\alpha = v_\alpha b_\alpha$ for some $u_\alpha, v_\alpha \in S_\alpha$. We claim that

$$Sa_\alpha \cap Sb_\alpha = Sc_\alpha.$$

As S has the (LC) condition there exists $d \in S_\xi$ such that $Sa_\alpha \cap Sb_\alpha = Sd.$ Then $d = ka_\alpha = hb_\alpha$ for some $k, h \in S$ and so $\xi \leq \alpha$. Since $c_\alpha \in Sa_\alpha \cap Sb_\alpha$ we have that $c_\alpha = rd$ for some $r \in S$ so that $\alpha \leq \xi$. Hence $\alpha = \xi$, that is, $d \in S_\alpha$ and we can write $d = d_\alpha$.

From $c_\alpha = rd$ we have that $c_\alpha = (e_\alpha r)d_\alpha \in S_\alpha d_\alpha$ so that $S_\alpha c_\alpha \subseteq S_\alpha d_\alpha$.

Since $d_\alpha = ka_\alpha = hb_\alpha = (e_\alpha k)a_\alpha = (e_\alpha h)b_\alpha$, we have that $d_\alpha \in S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$, and so $S_\alpha d_\alpha \subseteq S_\alpha c_\alpha$. Thus $S_\alpha d_\alpha = S_\alpha c_\alpha$. Hence $d_\alpha \,\mathcal{L}\, c_\alpha$ in S_α, so that $d_\alpha \,\mathcal{L}\, c_\alpha$ in S. We have

$$Sa_\alpha \cap Sb_\alpha = Sc_\alpha.$$

Hence our claim is established.

Now let $\beta \leq \alpha$. Since S_β has the (LC) condition and $e_\beta a_\alpha, e_\beta b_\alpha \in S_\beta$ we have that

$$S_\beta(e_\beta a_\alpha) \cap S_\beta(e_\beta b_\alpha) = S_\beta w_\beta$$

for some $w_\beta \in S_\beta$. We aim to show that $S_\beta(e_\beta c_\alpha) = S_\beta w_\beta$.

Since $w_\beta \in S_\beta a_\alpha \cap S_\beta b_\alpha \subseteq Sa_\alpha \cap Sb_\alpha$ we have that $w_\beta \in Sc_\alpha$ and so $w_\beta = lc_\alpha$ for some $l \in S$, say $l \in S_\eta$ so that $\eta \geq \beta$. Since $w_\beta = e_\beta w_\beta = e_\beta l c_\alpha$ and $\eta \geq \beta$, it follows that $w_\beta = e_\beta w_\beta = le_\beta c_\alpha$, by Lemma 2.4. Then $w_\beta = (le_\beta)(e_\beta c_\alpha) \in S_\beta c_\alpha$ so that $S_\beta w_\beta \subseteq \; S(e_\beta c_\alpha)$.

Conversely, since $c_\alpha = u_\alpha c_\alpha = v_\alpha b_\alpha$ and $\beta \leq \alpha,$ it follows that $e_\beta c_\alpha = e_\beta u_\alpha e_\beta a_\alpha = e_\beta v_\alpha e_\beta b_\alpha$, by Lemma 2.4. It follows that $e_\beta c_\alpha \in S_\beta a_\alpha \cap S_\beta b_\alpha = S_\beta w_\beta$. Hence $S_\beta(e_\beta c_\alpha) \subseteq S_\beta w_\beta$. Thus $S_\beta(e_\beta c_\alpha) = S_\beta w_\beta$ as required.

Lemma 3.7: Let $a^{-1}b, a^{-1}e_\alpha \in \sum_\alpha$ and $c^{-1}d, e_\beta d \in \sum_\beta$ where $a, b \in S_\alpha$, $c, d \in S_\beta$ and e_α, e_β are the identities elements in S_α and S_β respectively. Then

(i) $a^{-1}be_\beta d = \big(ae_\beta\big)^{-1}(bd)$,

(ii) $(a^{-1}e_\alpha)(c^{-1}d) = (ca)^{-1}\big(de_{\alpha\beta}\big)$.

Proof: (*i*) We have that $S_{\alpha\beta}e_\beta \cap S_{\alpha\beta}b = S_{\alpha\beta} \cap S_{\alpha\beta}b = S_{\alpha\beta}b$ and

$$e_{\alpha\beta}b = \big(be_{\alpha\beta}\big)e_\beta = \big(e_{\alpha\beta}b\big)e_\beta = b\, e_{\alpha\beta},$$

Using Lemma 2.4. We have

$$\begin{aligned}(a^{-1}b)(e_\beta d) &= (a^{-1}b)({e_\beta}^{-1}d)\\ &= (e_{\alpha\beta}a)^{-1}(e_{\alpha\beta}bd)\\ &= (e_{\alpha\beta}a)^{-1}(bd).\end{aligned}$$

(*ii*) We have that $S_{\alpha\beta}c \cap S_{\alpha\beta}e_\alpha = S_{\alpha\beta}c \cap S_{\alpha\beta} = S_{\alpha\beta}c$ and

$$e_{\alpha\beta}c = (ce_{\alpha\beta})e_\alpha = (e_{\alpha\beta}c)e_\alpha = c\,e_{\alpha\beta},$$

Using Lemma 2.4. We have

$$(a^{-1}e_\alpha)(c^{-1}d) = (ca)^{-1}(de_{\alpha\beta})$$

as required.

Lemma 3.8: Let $a^{-1}b \in \sum_\alpha$, $e_\beta d, d^{-1}e_\beta \in \sum_\beta$ and $x^{-1}y \in \sum_\gamma$ where e_β is the identity element in S_β where $a, b \in S_\alpha$, $e_\beta, d \in S_\beta$ and $x, y \in S_\gamma$. Then

(*i*) $(a^{-1}be_\beta d)x^{-1}y = a^{-1}b(e_\beta dx^{-1}y)$;

(*ii*) $(a^{-1}bde_\beta)x^{-1}y = a^{-1}b(d^{-1}e_\beta x^{-1}y)$.

Proof: (*i*) Let $a^{-1}b, e_\beta d, x^{-1}y$ be as in the hypothesis. Then

$$\begin{aligned}(a^{-1}be_\beta d)x^{-1}y &= (ae_{\alpha\beta})^{-1}(bd)x^{-1}y \text{ by Lemma 3.7 } (i),\\ &= (t_1a)^{-1}(r_1y)\end{aligned}$$

where $t_1bd = r_1x = w_1$ and

$$S_{\alpha\beta\gamma}(bde_{\alpha\beta\gamma}) \cap S_{\alpha\beta\gamma}(xe_{\alpha\beta\gamma}) = S_{\alpha\beta\gamma}w_1$$

for some $t_1, r_1, w_1 \in S_{\alpha\beta\gamma}$.

On the other hand, by definition of multiplication,

$$\begin{aligned}a^{-1}b(e_\beta dx^{-1}y) &= a^{-1}b((t_2e_\beta)^{-1}r_2y)\\ &= (t_3a)^{-1}(r_3r_2y)\end{aligned}$$

where $t_2d = r_2x = w_2$ with

$$S_{\beta\gamma}(de_{\beta\gamma}) \cap S_{\beta\gamma}(xe_{\beta\gamma}) = S_{\beta\gamma}w_2 \quad (1)$$

for some $t_2, r_2, w_2 \in S_{\beta\gamma}$ and $t_3b = r_3t_2e_{\alpha\beta\gamma} = w_3$ with

$$S_{\alpha\beta\gamma}be_{\alpha\beta\gamma} \cap S_{\alpha\beta\gamma}t_2e_{\alpha\beta\gamma} = S_{\alpha\beta\gamma}w_3 \quad (2)$$

for some $t_3, r_3, w_3 \in S_{\alpha\beta\gamma}$. Using (1) and Lemma 3.6 gives

$$S_{\alpha\beta\gamma}d \cap S_{\alpha\beta\gamma}x = S_{\alpha\beta\gamma}w_2 \quad (3)$$

We must show that $(t_1a)^{-1}(r_1y) = (t_3a)^{-1}(r_3r_2y)$. By using Lemma 3.3, we have to show that $t_1a = ut_3a$ and $r_1y = ur_3r_2y$ for some unit u in $S_{\alpha\beta\gamma}$.

Once we know $w_1\, \mathcal{L}\, w_3\, d$ in $S_{\alpha\beta\gamma}$, we have that $w_1 = hw_3d$ for some unit h in $S_{\alpha\beta\gamma}$ by Lemma 3.2. Hence $t_1bd = ht_3bd$ so that $t_1e_{\alpha\beta\gamma}bd = ht_3e_{\alpha\beta\gamma}bd$. Since t_1, ht_3 and $e_{\alpha\beta\gamma}bd$ are in $S_{\alpha\beta\gamma}$, which is right cancellative we obtain $t_1 = ht_3$ so that $t_1a = ht_3a$.

Now,

$$w_1 = r_1x = t_1bd = ht_3bd = hr_3t_2d = hr_3r_2x.$$

As r_1, hr_3r_2 and $e_{\alpha\beta\gamma}x$ are in $S_{\alpha\beta\gamma}$ again by right cancellativity in $S_{\alpha\beta\gamma}$ we have that $r_1 = hr_3r_2$ and so $r_1y = hr_3r_2y$.

Now, as S has (LC)

$$\begin{aligned}S_{\alpha\beta\gamma}w_1 &= S_{\alpha\beta\gamma}bd \cap S_{\alpha\beta\gamma}x\\ &= S_{\alpha\beta\gamma}bd \cap S_{\alpha\beta\gamma}d \cap S_{\alpha\beta\gamma}x\\ &= S_{\alpha\beta\gamma}bd \cap S_{\alpha\beta\gamma}w_2 && \text{by (3)}\\ &= S_{\alpha\beta\gamma}bd \cap S_{\alpha\beta\gamma}t_2d\\ &= S_{\alpha\beta\gamma}bde_{\alpha\beta\gamma} \cap S_{\alpha\beta\gamma}t_2de_{\alpha\beta\gamma}\\ &= (S_{\alpha\beta\gamma}b \cap S_{\alpha\beta\gamma}t_2)de_{\alpha\beta\gamma} && \text{by Lemma 3.5}\\ &= S_{\alpha\beta\gamma}w_3d && \text{by (2).}\end{aligned}$$

(*ii*) Let $a^{-1}b, d^{-1}e_\beta, x^{-1}y$ be as in the hypothesis. Then,

$$\begin{aligned}(a^{-1}bd^{-1}e_\beta)x^{-1}y &= (t_1a)^{-1}(r_1e_\beta)x^{-1}y\\ &= (t_2t_1a)^{-1}(r_2y)\end{aligned}$$

where $t_1b = r_1d = w_1$ with

$$S_{\alpha\beta}(be_{\alpha\beta}) \cap S_{\alpha\beta}(de_{\alpha\beta}) = S_{\alpha\beta}w_1 \quad (4)$$

for some $t_1, r_1, w_1 \in S_{\alpha\beta}$ and $t_2 r_1 = r_2 x = w_2$ with

$$S_{\alpha\beta\gamma} r_1 \cap S_{\alpha\beta\gamma} x = S_{\alpha\beta\gamma} w_2 \qquad (5)$$

for some $t_2, r_2, w_2 \in S_{\alpha\beta\gamma}$. By (4) and Lemma 3.6 we have

$$S_{\alpha\beta\gamma} b \cap S_{\alpha\beta\gamma} d = S_{\alpha\beta\gamma} w_1. \qquad (6)$$

On the other hand, by Lemma 3.7 (*ii*),

$$\begin{aligned} a^{-1}b\big(d^{-1}e_\beta x^{-1}y\big) &= a^{-1}b(xd)^{-1}(ye_{\beta\gamma}) \\ &= (t_3 a)^{-1}(r_3 y e_{\beta\gamma}) \end{aligned}$$

where

$$t_3 b = r_3 xd = w_3,\, S_{\alpha\beta\gamma}(xd) \cap S_{\alpha\beta\gamma}(be_{\alpha\beta\gamma}) = S_{\alpha\beta\gamma} w_3$$

for some $t_3, r_3, w_3 \in S_{\alpha\beta\gamma}$.

We have to show that $(t_2 t_1 a)^{-1}(r_2 y) = (t_3 a)^{-1}\big(r_3 y e_{\beta\gamma}\big)$. By using Lemma 3.3, we have to show that $t_3 a = v t_2 t_1 a$ and $r_3 y = v r_2 y$ for some unit v in $S_{\alpha\beta\gamma}$.

Once we know $w_3 \,\mathcal{L}\, w_2\, d$ in $S_{\alpha\beta\gamma}$, we have $w_3 = k w_2 d$ for some unit k in $S_{\alpha\beta\gamma}$, by Lemma 3.2. Hence $r_3 xd = k r_2 xd$ so that $r_3 e_{\alpha\beta\gamma} xd = k r_2 e_{\alpha\beta\gamma} xd$. Since $r_3, e_{\alpha\beta\gamma} xd$ and $k r_2$ are in $S_{\alpha\beta\gamma}$ which is right cancellative we obtain $r_3 = k r_2$ so that $r_3 y = k r_2 y$. Now,

$$w_3 = t_3 b = r_3 xd = k r_2 xd = k t_2 r_1 d = k t_2 t_1 b.$$

Hence $t_3 e_{\alpha\beta\gamma} b = k t_2 t_1 e_{\alpha\beta\gamma} b$ where $t_3, e_{\alpha\beta\gamma} b$ and $k t_2 t_1$ are in $S_{\alpha\beta\gamma}$ again by right cancellativity in $S_{\alpha\beta\gamma}$. we have that $t_3 = k t_2 t_1$ and so $t_3 a = k t_2 t_1 a$.

Now,

$$\begin{aligned} S_{\alpha\beta\gamma} w_3 &= S_{\alpha\beta\gamma} b \cap S_{\alpha\beta\gamma} xd \\ &= S_{\alpha\beta\gamma} b \cap S_{\alpha\beta\gamma} xd \cap S_{\alpha\beta\gamma} d \\ &= S_{\alpha\beta\gamma} xd \cap S_{\alpha\beta\gamma} w_1 && \text{by (6)} \\ &= S_{\alpha\beta\gamma} xd \cap S_{\alpha\beta\gamma} r_1 d \\ &= S_{\alpha\beta\gamma} xde_{\alpha\beta\gamma} \cap S_{\alpha\beta\gamma} r_1 de_{\alpha\beta\gamma} \\ &= (S_{\alpha\beta\gamma} x \cap S_{\alpha\beta\gamma} r_1) de_{\alpha\beta\gamma} && \text{by Lemma 3.5} \\ &= S_{\alpha\beta\gamma} w_2 d && \text{by (5)} \end{aligned}$$

as required.

Lemma 3.9: The associative law holds in Q.

Proof: Suppose that $a^{-1}b \in \sum_\alpha$, $c^{-1}d \in \sum_\beta$ and $s^{-1}t \in \sum_\gamma$ where $a, b \in S_\alpha$, $c, d \in S_\beta$ and $s, t \in S_\gamma$. From Lemma 3.8, we have that

$$\begin{aligned} a^{-1}b(c^{-1}ds^{-1}t) &= a^{-1}b\big(c^{-1}e_\beta e_\beta d. s^{-1}t\big) \\ &= a^{-1}b\big(c^{-1}e_\beta . e_\beta d s^{-1}t\big) \\ &= (a^{-1}bc^{-1}e_\beta)\big(e_\beta d. s^{-1}t\big) \\ &= (a^{-1}bc^{-1}e_\beta . e_\beta d)s^{-1}t \\ &= (a^{-1}b(c^{-1}e_\beta . e_\beta d))s^{-1}t \\ &= (a^{-1}bc^{-1}d)s^{-1}t. \end{aligned}$$

From Lemmas 3.9 and 3.4 we get the proof of Theorem 3.1.

Let $a \in S_\alpha$ and $b \in S_\beta$ for some $\alpha, \beta \in Y$. By Lemmas 3.7 and 2.4,

$$e_\alpha a e_\beta b = {e_\alpha}^{-1} a {e_\beta}^{-1} b = (e_\alpha e_{\alpha\beta})^{-1}(ab) = e_{\alpha\beta}(ab) = ab$$

and we get the following lemma;

Lemma 3.10: The multiplication on Q extends the multiplication on S.

The next corollary now is clear.

Corollary 3.11: The semigroup S defined as above is a left I-order in $Q = \bigcup_{\alpha\in Y} \sum_\alpha$.

The following lemma shows that the `strong' in Gantos's result is automatic.

Lemma 3.12: [7] Let $P = [Y; S_\alpha]$ where each S_α is a monoid with identity e_α, such that $E = \{e_\alpha : \alpha \in Y\}$ is a subsemigroup of P. Then E is a semilattice isomorphic to Y and E is central in P.
If we define $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ by $a_\alpha \phi_{\alpha,\beta} = a_\alpha e_\beta$ where $\alpha \geq \beta$, then each $\phi_{\alpha,\beta}$ is a monoid morphism, and $P = [Y; S_\alpha; \phi_{\alpha,\beta}]$.

Let $S = [Y; S_\alpha]$ be a semilattice Y of of right cancellative monoids S_α with identity e_α, $\alpha \in Y$ such that each S_α, $\alpha \in Y$ has the (LC) condition. By Lemma 2.4, $E = \{e_\alpha : \alpha \in Y\}$ is a subsemigroup of S. Hence S is a strong semilattice Y with connecting morphisms $\varphi_{\alpha,\beta} : S_\alpha \to S_\beta$ givening by $a_\alpha \varphi_{\alpha,\beta} = a_\alpha e_\beta$ where $\alpha \geq \beta$ for any $a_\alpha \in S_\alpha$, by Lemma 3.12. In fact, every semilattice of right cancellative monoids is a strong semilattice of cancellative monoids (see [13, Exercises III.7.12]). If S has the (LC) condition, then by Corollary 3.11, S has a semigroup of left I-quotients $Q = \bigcup_{\alpha \in Y} \sum_\alpha$ where $\sum_\alpha$ is the inverse hull of S_α, $\alpha \in Y$. It is easy to see that e_α is the identity of $\sum_\alpha$. From Lemma 3.6 and Theorem 3.11 of [7], the $\varphi_{\alpha,\beta}$'s lift to morphisms $\phi_{\alpha,\beta} : \sum_\alpha \to \sum_\beta$ and $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ for all $\alpha \geq \beta \geq \gamma$, and $\phi_{\alpha,\alpha}$ is the identity on $\sum_\alpha$. Hence Q is a strong semilattice of bisimple inverse monoids $\sum_\alpha$'s, $\alpha \in Y$, by Lemma 3.12. The following theorem is now clear.

Theorem 3.13: Let $S = [Y; S_\alpha; \varphi_{\alpha,\beta}]$ and for each α, let S_α be a right cancellative monoid with Condition (LC) and $\sum_\alpha$ as its inverse hull of left I-quotients. Suppose that S has the (LC) condition. Then S is a left I-order in a strong semilattice of monoids $Q = [Y; \sum_\alpha; \phi_{\alpha,\beta}]$ where $\varphi_{\alpha,\beta}$'s lift to $\phi_{\alpha,\beta}$'s, $\alpha \geq \beta$.

We aim now to prove the converse of Theorem 3.1. Let Q be a semilattice Y of bisimple inverse monoids Q_α, (with identity e_α) such that $E = \{e_\alpha : \alpha \in Y\}$ is a subsemigroup of Q. By Lemma 3.12, E is central in Q. Further if we define $\phi_{\alpha,\beta} : Q_\alpha \to Q_\beta$ by $q_\alpha \phi_{\alpha,\beta} = q_\alpha e_\beta$ ($\alpha \geq \beta$), then each $\phi_{\alpha,\beta}$ is a monoid morphism and $Q = [Y; Q_\alpha; \phi_{\alpha,\beta}]$. Let S_α be the $\mathcal{R}$-class of the identity e_α in Q_α. Clearly, $\phi_{\alpha,\beta}|_{S_\alpha} : S_\alpha \to S_\beta$ and $S = [Y; S_\alpha; \phi_{\alpha,\beta}|_{S_\alpha}]$ is a strong semilattice Y of right cancellative monoids S_α. We wish to show that S has the (LC) condition. By Lemma 3.6, to show that S has (LC) condition we have to show that $\phi_{\alpha,\beta}|_{S_\alpha}$is (LC)-preserving ($\alpha \geq \beta$). We need the following technical lemma from [12] (see, Lemma 3.2 of [2]).

Lemma 3.14: (*cf.* [12, Lemma X.1.5]) Let Q be a bisimple inverse monoid and let R be the $\mathcal{R}$-class of the identity. For any $a, b, c \in R$,

$$Ra \cap Rb = Rc \text{ if and only if } a^{-1}ab^{-1}b = c^{-1}c\,.$$

Returning to our argument before Lemma 3.14. Let $S_\alpha a \cap S_\alpha b = S_\alpha c$ where $a, b, c \in S_\alpha$. Then, we have that $a^{-1}ab^{-1}b = c^{-1}c$. We claim that

$$(e_\beta a)^{-1}(e_\beta a)(e_\beta b)^{-1}(e_\beta b) = (e_\beta c)^{-1}(e_\beta c)$$

where $\alpha \geq \beta$.

Since E is central in Q we have

$$\begin{aligned}
(e_\beta a)^{-1}(e_\beta a)(e_\beta b)^{-1}(e_\beta b) &= a^{-1}e_\beta e_\beta ab^{-1}e_\beta b \\
&= a^{-1}e_\beta ab^{-1}e_\beta b \\
&= a^{-1}ae_\beta b^{-1}b \\
&= e_\beta a^{-1}ab^{-1}b \\
&= e_\beta c^{-1}c \\
&= e_\beta c^{-1}e_\beta c \\
&= (e_\beta c)^{-1}(e_\beta c).
\end{aligned}$$

Hence our claim is established. By the above lemma $S_\beta e_\beta a \cap S_\beta e_\beta b = S_\beta e_\beta c$ where $\alpha \geq \beta$. Thus by Lemma 3.6, S has the (LC) condition and the following theorem is clear.

Theorem 3.15: Let Q be a semilattice Y of bisimple inverse monoids Q_α, (with identity e_α) such that $E = \{e_\alpha : \alpha \in Y\}$ is a subsemigroup of Q. Then there is a subsemigroup S of Q with the (LC) condition which is a strong semilattice of right cancellative monoids S_α where S_α is the $\mathcal{R}^{Q_\alpha}$-class of e_α. Moreover, S is a left I-order in Q.

Combining Theorem 3.1 and Theorem 3.15, we get the following corollary.

Corollary 3.16: (*cf.* [11, Main Theorem]) Let $S = [Y; S_\alpha]$ be a semilattice Y of right cancellative monoids S_α with identity e_α, such that each S_α has (LC). Suppose in addition that for any $\alpha \geq \beta$, if $S_\alpha a_\alpha \cap S_\alpha b_\alpha = S_\alpha c_\alpha$, then $S_\beta a_\alpha \cap S_\beta b_\alpha = S_\beta c_\alpha$. For each $\alpha \in Y$, let Q_α be the inverse hull of S_α, so that Q_α is a bisimple inverse monoid, and S_α is the $\mathcal{R}^{Q_\alpha}$-class of e_α. Then $Q = [Y; Q_\alpha]$ is a semigroup of left I-quotients of S, such that $E = \{e_\alpha : \alpha \in Y\}$ is a subsemigroup.

Conversely, let $Q = [Y; Q_\alpha]$ be a semilattice Y of bisimple inverse monoids Q_α, with identity e_α, such that $E = \{e_\alpha : \alpha \in Y\}$ is a subsemigroup. Then $S = [Y; R_{e_\alpha}]$ is a semilattice of right cancellative monoids R_{e_α}, such that each R_{e_α} has (LC) and for any $\alpha \geq \beta$, if $R_{e_\alpha} a_\alpha \cap S_{e_\alpha} R_\alpha = R_{e_\alpha} c_\alpha$, then $R_{e_\beta} a_\alpha \cap S_{e_\beta} R_\alpha = R_{e_\beta} c_\alpha$.

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