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ATTRACTIVITY RESULT FOR FRACTIONAL QUADRATIC INTEGRAL EQUATION IN BANACH SPACE

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ABSTRACT

In this paper, we proved existence the locally attractive solution for a fractional order quadratic integral equation in Banach space under lipschitz and Caratheodory conditions via a hybrid fixed point theorem.

Keywords: *Banach space, fractional order quadratic integral equation, existence results, locally attractive solution, fixed point theorem.*

Mathematics Subject Classification 2000: *26A33, 45G10, 46E99, 47H10, 47H99.*

I. INTRODUCTION

The theory of fractional calculus (that is fractional order differential and integral equation) has newly received a lot of attention and establishes a meaningful branch of nonlinear analysis. Number of research monographs and research papers has appeared devoted to integrals and differential equation of fractional order.

In this paper we study the existence of locally attractive solution of the following fractional order quadratic integral equation.

$$x(t) = g\big(t, x(t)\big) + \frac{f\big(t, x(t), x(\gamma(t))\big)}{\Gamma(\xi)} \int_0^t \frac{v(t,s,x(s))}{(t-s)^{1-\xi}}\, ds \tag{1.1}$$

Where $t \in \mathbb{R}_+ = [0,\infty)$ and $0 < \xi < 1$, $g: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $f: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $v: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are functions which satisfy special assumptions.

II. PRELIMINARIES

In this section we give the definitions, notation, hypothesis and preliminary tools, which will be used in the sequel.

Let $\mathbb{X} = \mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ be the space of absolutely continuous function on $\mathbb{R}_+$ and Ω be a subset of $\mathbb{X}$. Let a mapping $\mathbb{A}: \mathbb{X} \to \mathbb{X}$ be an operator and consider the following operator equation in $\mathbb{X}$, namely,

$$x(t) = (\mathbb{A}x)(t), \text{ for all } t \in \mathbb{R}_+ \tag{2.1}$$

Below we give some different characterization of the solutions for operator equation (2.1) on $\mathbb{R}_+$. We need the following definitions;

Definition 2.1[5]: We say that solution of the equation (2.1) are locally attractive if there exists a closed ball $\overline{B_r(0)}$ in the space $\mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ for some $x_0 \in \mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ and for some real number $r > 0$ such that for arbitrary solution $x = x(t)$ and $y = y(t)$ of equation (2.1) belonging to $\overline{B_r(0)} \cap \Omega$ we have that,

$$\lim_{t\to\infty}\big(x(t) - y(t)\big) = 0 \tag{2.2}$$

Definition 2.2[4]: Let $\mathbb{X}$ be a Banach space. A mapping $\mathbb{A}: \mathbb{X} \to \mathbb{X}$ is called Lipschitz if there is a constant $\alpha > 0$ such that, $\|\mathbb{A}x - \mathbb{A}y\| \le \alpha\|x - y\|$ for all $x, y \in \mathbb{X}$. If $\alpha < 1$, then $\mathbb{A}$ is called a contraction on $\mathbb{X}$ with the contraction constant α.

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Definition 2.3[2]: An operator $\mathbb{Q}$ on a Banach space $\mathbb{X}$ into itself is called compact if for any bounded subset S of $\mathbb{X}$, $\mathbb{Q}(S)$ is relatively compact subset of $\mathbb{X}$. If $\mathbb{Q}$ is continuous and compact, then it is called completely continuous on $\mathbb{X}$.

Definition 2.4[4]: Let $\mathbb{X}$ be a Banach space with the norm $\|\cdot\|$ and let $\mathbb{Q}: \mathbb{X} \to \mathbb{X}$ be an operator (in general nonlinear). Then $\mathbb{Q}$ is called

i. Compact if $\mathbb{Q}(\mathbb{X})$ is relatively compact subset of $\mathbb{X}$.
ii. Totally compact if $\mathbb{Q}(S)$ is totally bounded subset of $\mathbb{X}$ for any bounded subset S of $\mathbb{X}$.
iii. Completely continuous if it is continuous and totally bounded operator on $\mathbb{X}$.

It is clear that every compact operator is totally bounded but the converse need not be true.

We seek the solution of (2.1) in the space $\mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ of continuous and real – valued function defined on $\mathbb{R}_+$. Define a standard norm $\|\cdot\|$ and a multiplication " $\cdot$ " in $\mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ by

$$\|x\| = sup\{|x(t)| : t \in \mathbb{R}_+\}, \;\; (xy)(t) = x(t)y(t), \;\; t \in \mathbb{R}_+ \tag{2.3}$$

Clearly, $\mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ becomes a Banach space with respect to the above norm and the multiplication in it. By $\mathcal{L}^1(\mathbb{R}_+, \mathbb{R})$ we denote the space of Lebesgue-integrable function $\mathbb{R}_+$with the norm $\|\cdot\|_{\mathcal{L}^1}$ defined by

$$\|x\|_{\mathcal{L}} = \int_0^\infty |x(t)| dt \tag{2.4}$$

Definition 2.5 [6]: Let $f \in \mathcal{L}^1[0, \mathbb{T}]$and $\alpha > 0$. The Riemann – Liouville fractional derivative of order ξ of real function f is defined as, $\mathfrak{D}^\xi f(t) = \frac{1}{\Gamma(1-\xi)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^\xi} ds, \;\; 0 < \xi < 1$
Such that $\mathfrak{D}^{-\xi} f(t) = I^\xi f(t) = \frac{1}{\Gamma(\xi)} \int_0^t \frac{f(s)}{(t-s)^{1-\xi}} ds$ respectively

Definition 2.6[5]: The Riemann-Liouville fractional integral of order $\xi \in (0,1)$ of the function $f \in \mathcal{L}^1[0, \mathbb{T}]$ is defined by the formula:

$$I^\xi f(t) = \frac{1}{\Gamma(\xi)} \int_0^t \frac{f(s)}{(t-s)^{1-\xi}} ds, \quad t \in [0, \mathbb{T}]$$

Where $\Gamma(\xi)$ denote the Euler gamma function. The Riemann-Liouville fractional derivative operator of order ξ defined by $\mathfrak{D}^\xi = \frac{d^\xi}{dt^\xi} = \frac{d}{dt} \circ I^{1-\xi}$

It may be shown that the fractional integral operator I^ξ transforms the space $\mathcal{L}^1(\mathbb{R}_+, \mathbb{R})$ into itself and has some other properties.

Theorem 2.1[4]: (Arzela-Ascoli Theorem) If every uniformly bounded and equicontinuous sequence $\{f_n\}$ of functions in $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ then it has a convergent subsequence.

Theorem 2.2[4]: A metric space X is compact iff every sequence in X has a convergent subsequence.

Theorem 2.3 (Lebesgue's dominated convergence theorem): Suppose that $\{g_n\}$ is a sequence of measurable functions, that $g_n \to g$ pointwise a.e. as $n \to \infty$, and that $|g_n| \le f, \forall\, n$, where f is integrable then g is integrable and

$$\int g d\mu = lim_{n\to\infty} \int g_n d\mu$$

III. EXISTENCE THEORY

Definition 3.1[4]: A mapping $v: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is Caratheodory if:

i) $(t, s) \to v(t, s, x)$ is measurable for each $x \in \mathbb{R}$ and
ii) $(x) \to v(t, s, x)$ is continuous almost everywhere for $t \in \mathbb{R}_+$.
Furthermore a Caratheodary function v is $\mathcal{L}^1$ −Caratheodary if:
iii) For each real number $r > 0$ there exists a function $h_r \in \mathcal{L}^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ such that $|v(t, s, x)| \le h_r(t, s)$ $a.e.$ $t \in \mathbb{R}_+$ for all $x \in \mathbb{R}$ with $|x|_r \le r$.
Finally a caratheodary function v is $\mathcal{L}^1_{\mathbb{X}}$ −caratheodary if:
iv) There exists a function $h \in \mathcal{L}^1(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$ such that $|v(t, s, x)| \le h(t, s)$, $a.e.$ $t \in \mathbb{R}_+$ for all $x \in \mathbb{R}$

For convenience, the function h is referred to as a bound function for v.

Theorem 3.1 [8]: Let S be a non empty, convex, closed and bounded subset of the Banach space $\mathbb{X}$ and let $\mathbb{A}, \mathbb{C}: \mathbb{X} \to \mathbb{X}$ and $\mathbb{B}: S \to \mathbb{X}$ are two operators satisfying:

a) $\mathbb{A}$ and $\mathbb{C}$ are Lipschitzian with lipschitz constants ζ, η respectively.
b) $\mathbb{B}$ is completely continuous, and
c) $x = \mathbb{A}x\mathbb{B}y + \mathbb{C}, x \in S$ for all $y \in S$
d) $\zeta M + \eta < 1$ where $M = \|\mathbb{B}(s)\| = sup\{\|\mathbb{B}x\|: x \in S\}$

Then the operator equation $x = \mathbb{A}x\mathbb{B}y + \mathbb{C}x$ has a solution in S

Now for applying theorem (3.1) we study the existence of solution for the (FQIE) (1.1) under the following general assumptions:

($\mathcal{H}_1$) The function $f: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded with bound $\mathbb{F} = sup_{(t,x)}|f(t, x_1, x_2)|$, there exist a bounded function $\zeta: \mathbb{R}_+ \to \mathbb{R}$ with bound $\|\zeta\|$ such that,

$$\left|\begin{matrix} f\left(t, x(t), x\big(\gamma(t)\big)\right) \\ -f\left(t, y(t), y\big(\gamma(t)\big)\right)\end{matrix}\right| \leq \xi(t) max\{|x_1 - y_1|, |x_2 - y_2|\}$$

$$\left|\begin{matrix} f\left(t, x(t), x\big(\gamma(t)\big)\right) \\ -f\left(t, y(t), y\big(\gamma(t)\big)\right)\end{matrix}\right| \leq \zeta(t)|x(t) - y(t)|, \forall x, y \in \mathbb{R}$$

($\mathcal{H}_2$) The function $g: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\mathbb{G} = sup_{(t,x)}|g(t,x)|$, there exist a bounded function $\eta: \mathbb{R}_+ \to \mathbb{R}$ with bound $\|\eta\|$ such that

$\big|g\big(t, x(t)\big) - g\big(t, y(t)\big)\big| = \eta(t)|x(t) - y(t)|, \forall x, y \in \mathbb{R}$ and it vanish at infinity.

($\mathcal{H}_3$) The function $v: \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory condition (that is it is measurable in (t,s) for any $x \in \mathbb{R}$ and continuous in x for almost all $(t,s).$), then there exist a function $h: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that

$\big|v\big(t, s, x(s)\big)\big| \leq h(t,s)\ \forall t \in \mathbb{R}_+$, where $h(t,s) = a(s)b(t)$ and $a, b: \mathbb{R}_+ \to \mathbb{R}$

($\mathcal{H}_4$) The function $\rho: \mathbb{R}_+ \to \mathbb{R}$ defined by the formula $\rho(t) = \int_0^t \frac{h(t,s)}{(t-s)^{1-\xi}} ds$ bounded on $\mathbb{R}_+$ and $lim_{t\to\infty}\rho(t) = 0$ that is vanishing at infinity.

Remark 3.1: Note that the hypothesis $(\mathcal{H}_1)-(\mathcal{H}_4)$ hold then there exist a constant function $\mathcal{K}_1 > 0$ such that $\mathcal{K}_1 = sup_{t\geq 0}\frac{\rho(t)}{\Gamma(\xi)}$

IV. MAIN RESULT

In this section we consider the FQIE (1.1).The above hybrid fixed point theorem for three operators in Banach algebras $\mathbb{X}$, due to B.C.Dhage [8] will be used to prove existence the solution for given equation (1.1).

Theorem 4.1: Assume that hypothesis $(\mathcal{H}_1)-(\mathcal{H}_4)$ holds, further if $\|\zeta\|\mathcal{K}_1 + \|\eta\| < 1$ then FQIE (1.1) has locally attractive solution on the Banach space $\mathbb{X} = \mathcal{AC}(\mathbb{R}_+, \mathbb{R})$.

Proof: Define a non- empty, convex, closed and bounded subset S of Banach space $\mathbb{X} = \mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ as $S = \{x \in \mathbb{X}: \|x\| \leq r\}$, where r satisfies the inequality $\|\mathbb{F}\|\boldsymbol{\mathcal{K}_1} + \|\mathbb{G}\| < \mathbf{r}$.

Now we define the operators $\mathbb{A}: \mathbb{X} \to \mathbb{X}$ and $\mathbb{B}: S \to \mathbb{X}$ and $\mathbb{C}: \mathbb{X} \to \mathbb{X}$ by,

$$\mathbb{A}x(t) = f\left(t, x(t), x\big(\gamma(t)\big)\right), t \in \mathbb{R}_+ \tag{4.1}$$

$$\mathbb{B}x(t) = \frac{1}{\Gamma(\xi)}\int_0^t \frac{v(t,s,x(s))}{(t-s)^{1-\xi}} ds, t \in \mathbb{R}_+ \tag{4.2}$$

$$\mathbb{C}x(t) = g\big(t, x(t)\big), t \in \mathbb{R}_+ \tag{4.3}$$

The FQIE is equivalent to the operator equation

$$x(t) = \mathbb{A}x(t)\mathbb{B}x(t) + \mathbb{C}x(t), \forall t \in \mathbb{R}_+ \tag{4.4}$$

We shall show that, the operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ satisfy all the conditions of theorem (3.1)

This will be achieved in the following series of steps.

Step-I: First to show that $\mathbb{A}$ and $\mathbb{C}$ are Lipschitzian on $\mathbb{X}$.

Let $x, y \in \mathbb{X}$, then by $(\mathcal{H}_1)$ for $t \in \mathbb{R}_+$ we have,

$$\begin{aligned}|\mathbb{A}x(t) - \mathbb{A}y(t)| &= \left|f\left(t, x(t), x(\gamma(t))\right) - f\left(t, y(t), y(\gamma(t))\right)\right| \\ &\leq \zeta(t) max\{|x_1 - y_1|, |x_2 - y_2|\} \leq \zeta(t)|x(t) - y(t)|\end{aligned}$$

Taking supremum over t, we obtain

$$\|\mathbb{A}x - \mathbb{A}y\| \leq \|\zeta\|\|x - y\| \text{ for all } x, y \in \mathbb{R}.$$

There fore $\mathbb{A}$ is lipschitzian with lipschitz constant $\|\zeta\|$.

Now to show $\mathbb{C}$ is Lipschitzian on $\mathbb{X}$ for any $x, y \in \mathbb{X}$we have

$$\begin{aligned}|\mathbb{C}x(t) - \mathbb{C}y(t)| &= \left|g\left(t, x(t)\right) - g\left(t, y(t)\right)\right| \\ &\leq \eta(t)|x(t) - y(t)|\end{aligned}$$

Taking supremum over t, we obtain

$$\|\mathbb{C}x - \mathbb{C}y\| \leq \|\eta\|\|x - y\|, \text{ for all } x, y \in \mathbb{R}$$

There fore $\mathbb{C}$ is lipschitzian with lipschitz constant $\|\eta\|$.

Step-II: To show $\mathbb{B}$ is completely continuous on $\mathbb{X}$.

That is to show that $\mathbb{B}$ is continuous, uniformly bounded and equicontinuous.

Let by dominated convergence theorem, let $\{x_n\}$ be a sequence in S such that $\{x_n\} \to x$. Then,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{B}x_n(t) &= \lim_{n\to\infty} \left\{\frac{1}{\Gamma(\xi)} \int_0^t \frac{v\left(t, s, x_n(s)\right)}{(t - s)^{1-\xi}} ds\right\} x_n \\ &= \frac{1}{\Gamma(\xi)} \int_0^t \frac{v\left(t, s, x(s)\right)}{(t - s)^{1-\xi}} ds \\ &= \mathbb{B}x(t),, \forall t \in \mathbb{R}_+\end{aligned}$$

This shows that $\mathbb{B}x_n$ converges to $\mathbb{B}x$ pointwise on S.

Next to show that sequence $\{\mathbb{B}x_n\}$ is an equicontinuous sequence in S.

Let $t_1, t_2 \in \mathbb{R}_+$be arbitrary with $t_1 < t_2$ then

$$\begin{aligned}|\mathbb{B}x_n(t_2) - \mathbb{B}x_n(t_1)| &= \left|\begin{array}{l}\frac{1}{\Gamma(\xi)} \int_0^{t_2} \frac{v\left(t_2, s, x_n(s)\right)}{(t_2 - s)^{1-\xi}} ds \\ - \frac{1}{\Gamma(\xi)} \int_0^{t_1} \frac{v\left(t_1, s, x_n(s)\right)}{(t_1 - s)^{1-\xi}} ds\end{array}\right| \\ &\leq \frac{1}{\Gamma(\xi)} \left|\begin{array}{l}\int_0^{t_2} (t_2 - s)^{\xi-1} v\left(t_2, s, x_n(s)\right) ds \\ - \int_0^{t_2} (t_1 - s)^{\xi-1} v\left(t_1, s, x_n(s)\right) ds\end{array}\right| + \frac{1}{\Gamma(\xi)} \left|\begin{array}{l}\int_0^{t_2} (t_1 - s)^{\xi-1} v\left(t_1, s, x_n(s)\right) ds \\ - \int_0^{t_1} (t_1 - s)^{\xi-1} v\left(t_1, s, x_n(s)\right) ds\end{array}\right| \\ &\leq \frac{1}{\Gamma(\xi)} \left|\int_0^{t_2} (t_2 - s)^{\xi-1} h(s) ds - \int_0^{t_2} (t_1 - s)^{\xi-1} h(s) ds\right| \\ &\quad + \frac{1}{\Gamma(\xi)} \left|\int_0^{t_2} (t_1 - s)^{\xi-1} h(s) ds - \int_0^{t_1} (t_1 - s)^{\xi-1} h(s) ds\right| \\ &\leq \frac{\|h\|_{\mathcal{L}^1}}{\Gamma(\xi)} \left\{\begin{array}{c}\left|\int_0^{t_2} \left[(t_2 - s)^{\xi-1} - (t_1 - s)^{\xi-1}\right] ds\right| \\ + \left|\int_{t_1}^{t_2} (t_1 - s)^{\xi-1} ds\right|\end{array}\right\} \\ &\leq \frac{\|h\|_{\mathcal{L}^1}}{\Gamma(\xi)} \left\{\left|\left[\frac{(t_2 - s)^{\xi}}{-\xi}\right]_0^{t_2} - \left[\frac{(t_1 - s)^{\xi}}{-\xi}\right]_0^{t_2}\right| + \left|\left[\frac{(t_1 - s)^{\xi}}{-\xi}\right]_{t_1}^{t_2}\right|\right\} \\ &\leq \frac{\|h\|_{\mathcal{L}^1}}{\xi} \left\{\begin{array}{c}\left|\begin{array}{c}-\left[(t_2 - t_2)^{\xi} - (t_2 - 0)^{\xi}\right] + \\ \left[(t_1 - t_2)^{\xi} - (t_1 - 0)^{\xi}\right]\end{array}\right| + \\ \left|-\left[(t_1 - t_2)^{\xi} - (t_1 - t_1)^{\xi}\right]\right|\end{array}\right\} \\ &\leq \frac{\|h\|_{\mathcal{L}^1}}{\xi} \left\{\left|(t_2)^{\xi} + (t_1 - t_2)^{\xi} - (t_1)^{\xi}\right| + \left|-(t_1 - t_2)^{\xi}\right|\right\} \\ &\to 0 \text{ as } t_1 \to t_2, \forall n \in \mathcal{N}\end{aligned}$$

This shows that the sequence converges uniformly, and by using property of uniform convergence that is uniform convergence imply continuity

Hence $\mathbb{B}$ is continuous on S.

Step-III: To show $\mathbb{B}$ is compact operator on S, for this to show that $\mathbb{B}$ is uniformly bounded and equicontinuous in S. First we show that $\mathbb{B}$ is uniformly bounded. Let $x \in S$ be arbitrary then

$$\begin{aligned}
|\mathbb{B}x(t)| &= \left|\frac{1}{\Gamma(\xi)}\int_0^t \frac{v\big(t,s,x(s)\big)}{(t-s)^{1-\xi}}ds\right| \\
&\leq \frac{1}{\Gamma(\xi)}\int_0^t \frac{\left|v\big(t,s,x(s)\big)\right|}{(t-s)^{1-\xi}}ds \\
&\leq \frac{1}{\Gamma(\xi)}\int_0^t \frac{h(t,s)}{(t-s)^{1-\xi}}ds \\
&\leq \frac{1}{\Gamma(\xi)}\rho(t)
\end{aligned}$$

Taking supremum over t, we obtain $\|\mathbb{B}x\| \leq \frac{\rho(t)}{\Gamma(\xi)} = \mathcal{K}_1, \forall\, t \in \mathbb{R}_+$

Hence $\mathbb{B}$ is uniformly bounded subset of S.

Now to show $\mathbb{B}$ is equicontinuous on S.

Let $t_1, t_2 \in \mathbb{R}_+$, then

$$\begin{aligned}
|\mathbb{B}x(t_2) - \mathbb{B}x(t_1)| &= \left|\begin{array}{l} \frac{1}{\Gamma(\xi)}\int_0^{t_2} \frac{v\big(t_2,s,x(s)\big)}{(t_2-s)^{1-\xi}}ds \\ -\frac{1}{\Gamma(\xi)}\int_0^{t_1} \frac{v\big(t_1,s,x(s)\big)}{(t_1-s)^{1-\xi}}ds \end{array}\right| \\
&\leq \frac{1}{\Gamma(\xi)}\left|\begin{array}{l} \int_0^{t_2}(t_2-s)^{\xi-1}v\big(t_2,s,x(s)\big)ds \\ -\int_0^{t_2}(t_1-s)^{\xi-1}v\big(t_1,s,x(s)\big)ds \end{array}\right| + \frac{1}{\Gamma(\xi)}\left|\begin{array}{l} \int_0^{t_2}(t_1-s)^{\xi-1}v\big(t_1,s,x(s)\big)ds \\ -\int_0^{t_1}(t_1-s)^{\xi-1}v\big(t_1,s,x(s)\big)ds \end{array}\right| \\
&\leq \frac{1}{\Gamma(\xi)}\left|\int_0^{t_2}(t_2-s)^{\xi-1}h(s)ds - \int_0^{t_2}(t_1-s)^{\xi-1}h(s)ds\right| \\
&\quad + \frac{1}{\Gamma(\xi)}\left|\int_0^{t_2}(t_1-s)^{\xi-1}h(s)ds - \int_0^{t_1}(t_1-s)^{\xi-1}h(s)ds\right| \\
&\leq \frac{\|h\|_{\mathcal{L}^1}}{\Gamma(\xi)}\left\{\begin{array}{c} \left|\int_0^{t_2}\big[(t_2-s)^{\xi-1} - (t_1-s)^{\xi-1}\big]ds\right| \\ +\left|\int_{t_1}^{t_2}(t_1-s)^{\xi-1}ds\right| \end{array}\right\} \\
&\leq \frac{\|h\|_{\mathcal{L}^1}}{\Gamma(\xi)}\left\{\left|\left[\frac{(t_2-s)^{\xi}}{-\xi}\right]_0^{t_2} - \left[\frac{(t_1-s)^{\xi}}{-\xi}\right]_0^{t_2}\right| + \left|\left[\frac{(t_1-s)^{\xi}}{-\xi}\right]_{t_1}^{t_2}\right|\right\} \\
&\leq \frac{\|h\|_{\mathcal{L}^1}}{\xi}\left\{\begin{array}{c} \left|\begin{array}{c} -\big[(t_2-t_2)^{\xi} - (t_2-0)^{\xi}\big] + \\ \big[(t_1-t_2)^{\xi} - (t_1-0)^{\xi}\big] \end{array}\right| + \\ \left|-\big[(t_1-t_2)^{\xi} - (t_1-t_1)^{\xi}\big]\right| \end{array}\right\} \\
&\leq \frac{\|h\|_{\mathcal{L}^1}}{\xi}\left\{\left|(t_2)^{\xi} + (t_1-t_2)^{\xi} - (t_1)^{\xi}\right| + \left|-(t_1-t_2)^{\xi}\right|\right\} \\
&\to 0 \text{ as } t_1 \to t_2, \forall n \in \mathcal{N}
\end{aligned}$$

Implies that $\mathbb{B}(S)$ is equicontinuous.

Hence $\mathbb{B}$ is compact subset of S.

Implies that $\mathbb{B}$ is completely continuous on S.

Step-IV: To show $x = \mathbb{A}x\mathbb{B}y + \mathbb{C}x \Rightarrow x \in S, \forall y \in S$.

Let $x \in \mathbb{X}$, and $y \in S$ such that $x = \mathbb{A}x\mathbb{B}y + \mathbb{C}x$

By assumptions ($\mathcal{H}_1$-$\mathcal{H}_4$)

$$\begin{aligned}|x(t)| &= |\mathbb{A}x(t)\mathbb{B}x(t) + \mathbb{C}x(t)|\\ &\le |\mathbb{A}x(t)||\mathbb{B}x(t)| + |\mathbb{C}x(t)|\\ &\le \left|f\left(t, x(t), x(\gamma(t))\right)\right| \frac{1}{\Gamma(\xi)} \int_0^t \frac{|v(t,s,x(s))|}{(t-s)^{1-\xi}} ds + |g(t,x(t))|\\ &\le \mathbb{F}\frac{1}{\Gamma(\xi)} \int_0^t \frac{|v(t,s,x(s))|}{(t-s)^{1-\xi}} ds + \mathbb{G}\\ &\le \mathbb{F}\frac{\rho(t)}{\Gamma(\xi)} + \mathbb{G}\end{aligned}$$

Taking supremum over t on $\mathbb{R}_+$, we obtain $\|\mathbb{F}\|\mathcal{K}_1 + \|\mathbb{G}\|, \ \forall x \in S$

That is we have, $\|x\| = \|\mathbb{A}x(t)\mathbb{B}x(t) + \mathbb{C}x(t)\| \le r, \forall x \in S$.

Step-V: Finally we show that $\zeta M + \eta < 1$

Since

$$\begin{aligned}M = \|\mathbb{B}(S)\| &= sup_{x\in S}\{sup_{t\in \mathbb{J}}|\mathbb{B}x(t)|\}\\ &= sup_{x\in S}\left\{sup_{t\in\mathbb{J}}\left|\frac{1}{\Gamma(\xi)}\int_0^t \frac{v(t,s,x(s))}{(t-s)^{1-\xi}}ds\right|\right\}\\ &\le sup_{x\in S}\left\{\frac{1}{\Gamma(\xi)}\int_0^t \frac{|v(t,s,x(s))|}{(t-s)^{1-\xi}}ds\right\}\\ &\le sup_{x\in S}\left\{\frac{1}{\Gamma(\xi)}\int_0^t \frac{h(t,s)}{(t-s)^{1-\xi}}ds\right\}\\ &\le sup_{x\in S}\left\{\frac{\rho(t)}{\Gamma(\xi)}\right\} = \mathcal{K}_1\end{aligned}$$

and therefore $\zeta M + \eta$, we have $(\|\zeta\|\mathcal{K}_1 + \|\eta\|) < 1$.

Hence assumption(c) of theorem (3.1) is proved

Step-VI: Now to show the solution is locally attractive on $\mathbb{R}_+$. Then we have

$$\begin{aligned}|x(t) - y(t)| &= \left|\begin{array}{l}\left\{\left[f\left(t, x(t), x(\gamma(t))\right)\right]\frac{1}{\Gamma(\xi)}\int_0^t \frac{v(t,s,x(s))}{(t-s)^{1-\xi}}ds + g(t,x(t))\right\} - \\ \left\{\left[f\left(t, y(t), y(\gamma(t))\right)\right]\frac{1}{\Gamma(\xi)}\int_0^t \frac{v(t,s,y(s))}{(t-s)^{1-\xi}}ds + g(t,y(t))\right\}\end{array}\right|\\ &\le \left|\left[f\left(t, x(t), x(\gamma(t))\right)\right]\right|\frac{1}{\Gamma(\xi)}\int_0^t \frac{|v(t,s,x(s))|}{(t-s)^{1-\xi}}ds + |g(t,x(t))|\\ &+ \left|\left[f\left(t, y(t), y(\gamma(t))\right)\right]\right|\frac{1}{\Gamma(\xi)}\int_0^t \frac{|v(t,s,y(s))|}{(t-s)^{1-\xi}}ds + |g(t,y(t))|\\ &\le 2\mathbb{F}\frac{1}{\Gamma(\xi)}\int_0^t \frac{h(t,s)}{(t-s)^{1-\xi}}ds + 2\mathbb{G}\\ &\le 2\mathbb{F}\frac{\rho(t)}{\Gamma(\xi)} + 2\mathbb{G}, \forall t \in \mathbb{R}_+\end{aligned}$$

Since $\lim_{t\to\infty}\rho(t) = 0$, $\lim_{t\to\infty} g(t, x(t)) = 0$

For $\epsilon > 0$, there is real number $\mathbb{T}' > 0, \mathbb{T}'' > 0$ such that $\rho(t) \le \frac{\Gamma(\xi)\epsilon}{4F}$ for all $t \ge \mathbb{T}'$ and $|g(t, x(t))| < \frac{\epsilon}{4}$, if we choose $\mathbb{T}^* = max\{\mathbb{T}', \mathbb{T}''\}$

Then from above inequality it follows that $|x(t) - y(t)| < \epsilon$ for all $t \ge \mathbb{T}^*$.

Hence FQIE (1.1) has a locally attractive solution on $\mathbb{R}_+$.

This completes the proof.

V. Example

Example 5.1: Consider the following fractional quadratic integral equation of type (1.1)

$$x(t) = \frac{e^{-t}}{t}x(t) + \frac{cos2tx(t).}{\Gamma(\xi)}\int_0^t \frac{st^2}{t^{\xi+5}\left[t^2+s^2+(x(t))^2\right]}(t-s)^{\xi-1}ds \tag{5.1}$$

Where $\xi = \frac{1}{2}$

Solution: Here $g(t, x(t)) = \frac{e^{-t}}{t} x(t)$ and $f\left(t, x(t), x(\gamma(t))\right) = cos2tx(t)$,

$v(t, s, x(s)) = \frac{st^2}{t^{\xi+5}\left[t^2+s^2+(x(t))^2\right]}$ and $h(t,s) = \frac{st^2}{t^{\xi+5}}$

a) First to show the hypothesis ($\mathcal{H}_1$) is satisfied:

$$\left| \begin{matrix} f\left(t, x(t), x(\gamma(t))\right) \\ -f\left(t, y(t), y(\gamma(t))\right) \end{matrix} \right| \leq cos2t|x(t) - y(t)|, \forall x, y \in \mathbb{R}$$

$$\left| \begin{matrix} f\left(t, x(t), x(\gamma(t))\right) \\ -f\left(t, y(t), y(\gamma(t))\right) \end{matrix} \right| \leq \zeta(t)|x(t) - y(t)|, \forall x, y \in \mathbb{R}$$

Since $\zeta(t) = cos2t$ which is bounded and maps from $\mathbb{R}_+$ to $\mathbb{R}$.
Implies that the hypothesis ($\mathcal{H}_1$) satiesfied.

b) To show the hypothesis ($\mathcal{H}_2$) is satisfied:

$$\left|g(t, x(t)) - g(t, y(t))\right| \leq \left|\frac{e^{-t}}{t}\right| |x(t) - y(t)|$$
$$\leq \eta(t)|x(t) - y(t)|$$

Since $\eta(t) = \frac{e^{-t}}{t}$ which is bounded and maps from $\mathbb{R}_+$ to $\mathbb{R}$.
And $lim_{t\to\infty} g(t, x(t)) = lim_{t\to\infty} \frac{e^{-t}}{t} x(t) = 0$
That is $g(t, x(t))$ vanishing at infinity.
Implies that the hypothesis ($\mathcal{H}_2$) satiesfied.

c) To show the hypothesis ($\mathcal{H}_3$) is satisfied:

$\left|v(t, s, x(s))\right| \leq h(t,s)\ \forall t, s \in \mathbb{R}_+$
i.e $\frac{st^2}{t^{\xi+5}\left[t^2+s^2+(x(t))^2\right]} \leq \frac{st^2}{t^{\xi+5}}$
Implies that the hypothesis ($\mathcal{H}_3$) satisfied.

d) To show the hypothesis ($\mathcal{H}_4$) is satisfied: Now

$$\rho(t) = \int_0^t \frac{h(t,s)}{(t-s)^{1-\xi}} ds = \int_0^t \frac{\frac{st^2}{t^{\xi+5}}}{(t-s)^{1-\xi}} ds = \frac{1}{t^{\xi+3}} \int_0^t \frac{s}{(t-s)^{1-\xi}} ds$$
$$= \frac{1}{t^{\xi+3}} \int_0^t s(t-s)^{\xi-1} ds = \frac{1}{t^{\xi+2}} \left\{ \left[\frac{s(t-s)^{\xi}}{\xi}\right]_0^t - \int_0^t 1. \frac{(t-s)^{\xi}}{\xi} ds \right\}$$
$$= \frac{1}{t^{\xi+3}} \left\{ \left[\frac{t(t-t)}{\xi} - \frac{0(t-0)}{\xi}\right] - \left[\frac{(t-s)^{\xi+1}}{\xi(\xi+1)}\right]_0^t \right\}$$
$$= \frac{1}{t^{\xi+3}} \left[\frac{(t)^{\xi+1}}{\xi^2+\xi}\right] = \frac{1}{t^2(\xi^2+\xi)}, \xi = \frac{1}{2}$$

Implies $\rho(t)$ is continuous and bounded on $\mathbb{R}_+$ and vanish at infinity.

It follows that conditions ($\mathcal{H}_1$)-($\mathcal{H}_4$) are satisfied. Thus by theorem (4.1), above problem has a locally attractive solution $\mathbb{R}_+$.

CONCLUSION

In this paper we have studied the existence of solution for fractional quadratic integral equation. The result has been obtained by using hybrid fixed point theorem for three operators in Banach space. The main result is well illustrated with the help of example.

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