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ITERATIVE TECHNIQUES FOR NONLINEAR SYSTEMS WITH RETARDATION AND ANTICIPATION

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ABSTRACT

In this section we give a brief summary of the work done on differential equations with retardation and anticipation and mention few results.

***Keywords**: Integro-differential equations, Monotone iterative technique, Quasilinearization.*

INTRODUCTION

Differential equations arise quite frequently as mathematical models in diverse disciplines. The study of integro differential equations [1] has been attracting the attention of many scientific researchers due to its potential as a better model to represent physical phenomena in various disciplines.

It is a natural observation that in many physical phenomenon the rate of change of the system depends both on the present state of the system as well as its past history and hence an appropriate model of the phenomena will be one that involves past history and future expectation also. This led to the study of systems involving both retardation and anticipation, for example, [2].

The theory of differential equations with finite or infinite delay has been well developed and the corresponding theory for differential equations with advanced arguments, also known as anticipative systems as well as the systems involving both anticipation and retardation simultaneously, has not progressed. It is perhaps that there were no practical applications whose modelling results such equations and also because one needs some new ideas to develop a general theory.

As most of these models are nonlinear in nature, it is important to study nonlinear differential equations with retardation and anticipation.

ITERATIVE TECHNIQUES

The fundamental question that arises in understanding the nonlinear differential equation with retardation and anticipation is the existence and uniqueness of solutions.

Using the fundamental inequality result the method of lower and upper solutions had been developed. This gave the existence of solution in a closed sector. Using the method of lower and upper solutions the monotone iterative technique [3] and the method of quasilinearization [4] have been developed to obtain existence of solutions in a closed sector.

Both monotone iterative technique and quasilinearization use the method of upper and lower solutions and develop a sequence of iterates, which are solutions of certain linear differential equations with retardation and anticipation, that converge to a solution of the original problem. Therefore these techniques are important and hence many researchers developed these techniques for different problems and studied it in various situations.

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MONOTONE ITERATIVE TECHNIQUE

In this section we give description of the above methods along with some publications below.

T. Gnana Bhaskar, V. Lakshmikantham, J. Vasundha Devi [5] developed a method called monotone iterative technique relating to coupled lower and upper solutions to obtain coupled minimal and maximal solutions for functional differential equations with retardation and anticipation in 2006.

In this paper they consider the functional differential equation with retardation and anticipation of the type

$$x' = f(t, x, x_t, x^t), \ \ t \in J = [t_0, T], \tag{3.1}$$

$$x_{t_0} = \phi_0, \ x^T = \psi_0 \tag{3.2}$$

Where

$$f \in C[J \times R \times C_1 \times C_2, R], \ C_1 = C[[-h_1, 0], R], C_2 = [[0, h_2], R] \ \ and \ \phi_0 \in C_1 \ \psi_0 \in C_2.$$

Further more, $x_t : [-h_1, 0] \to R$ *is such that* $x_t(s) = x(t+s), -h_1 \le s \le 0$ and $x^t : [0, h_2] \to R$ *is such that* $x^t(s) = x(t+s), 0 \le s \le h_2$ .The following definitions are a prerequisite to introduce the monotone method.

Definition 3.1: A function $v_0 \in C^1[[t_0 - h_1, T + h_2], R]$ is said to be coupled lower solution of (3.1) and (3.2) if $v_{0'} \le f(t, v_0, v_{0t}, w_0^t)$
where $v_{0t_0} = \phi_1, \ v_0{}^T = \psi_1, \ v_{t0} \le x_{t0}, \ v^T \le x^T;$

Definition 3.2: A function $w_0 \in C^1[[t_0 - h_1, T + h_2], R]$ is said to be coupled upper solution of (3.1) and (3.2) if $w_{0'} \ge f(t, w_0, w_{0t}, v_0^t)$
where $w_{0t_0} = \phi_2, \ w_0{}^T = \psi_2, \ \ w_{t0} \ge x_{t0}, \ w^T \ge x^T.$
Note that $\phi_1, \phi_2 \in C_1, \psi_1, \psi_2 \in C_2$ and $\phi_1 \le \phi_0 \le \phi_2, \psi_1 \le \psi_0 \le \psi_2.$

Definition 3.3: Let r(t) be a solution of (3.1) and (3.2), then r(t) is said to be maximal solution if for every solution x(t) of (3.1) and (3.2) existing on $[t_0\text{-}h_1, T\text{+}h_2]$ the inequality $x(t) \le r(t)$ holds for $t \in [t_0 - h_1, T + h_2].$

Let $\rho(t)$ be a solution of (3.1) and (3.2), then $\rho(t)$ is said to be minimal solution if for every solution x(t) of (3.1) and (3.2) existing on $[t_0\text{-}h_1, T\text{+}h_2],$ the inequality $\rho(t) \le x(t)$ holds for $t \in [t_0 - h_1, T + h_2].$

They are using the following lemma's to develop the monotone iterative technique.

Lemma 3.4: Let $p \in C[[t_0 - h_1, T + h_2], R],$ p is continuously differentiable on $I = [t_0, T]$ and Suppose further that either

$$(A) \ \ p'(t) \le -N_1 p(t) - N_3 \int_{-h_1}^{0} p_t(s) ds \ \ on\, I \tag{3.3}$$

Or

(B) $p_{t_0}(s) \le 0, \ \ s \in [-h_1, 0], \ \ p \in C^1[[t_0 - h_1, T + h_2], R]$ and $p'(t) \le \dfrac{\lambda}{T + h_1},$

$$where \ \ t \in [t_0 - h_1, t_0], \ \min_{[t\_0\text{-}h\_1,t\_0]} \ p(s) = -\lambda \ \ , \lambda \ge 0$$

$$[N_1 + N_3 h_1](T + h_1) \le 1 \tag{3.5}$$

then $p(t) \le 0$ on I.

Lemma 3.5: Let $p \in C[[t_0 - h_1, T + h_2], R]$, p is continuously differentiable on I=[t_0,T] and $p'(t) \leq -N_1 p(t) + N_2 \int_{-h_1}^{0} p_t(s)ds + N_3 \int_{0}^{-h_2} p^t(s)ds \; on I$, where $N_1, N_2, N_3 > 0$, satisfying $N_2 h_1 + N_3 h_2 < L$. Then $P_{t_0} \leq 0, p^T \leq 0$ then, $p(t) \leq 0$ on I.

The main theorem of this paper is

Theorem 3.6: Suppose that the assumptions
$H_1 : v_0, w_0 \in C^1[I, R]$ satisfying, $v_0' \leq f(t, v_0, v_{0t}, w_0^t), v_{0t_0} = \phi_1, v_0^T = \psi_1$, and

$w_0' \geq f(t, w_0, w_{0t}, v_0^t)$, $w_{0t_0} = \phi_2, w_0^T = \psi_2$, where $\phi_1, \phi_2 \in C_1, \psi_1, \psi_2 \in C_2$ such that

$\phi_1 \leq \phi_0 \leq \phi_2$ and $v_0(t) \leq w_0(t)$.

$H_2 : f(t, x, \phi, \xi)$ is non increasing in ξ for each (t, x, ϕ);

$H_3 : f(t, x, \phi, \psi_1) - f(t, y, \xi, \psi_2) \geq -M_1(x - y) - M_3 \int_{-h_1}^{0} (\phi - \xi)(s)d$

Where $v_0(t) \leq y \leq x \leq w_0(t),\; M_1, M_3 \geq 0$

$H_4 : v_{0t_0} - \phi_0,\; \phi_0 - w_{0t_0}$ satisfying the assumptions of Lemma 3.4 are satisfied. Then there exist monotone sequences $\{v_n(t)\}$,$\{w_n(t)\}$ such that $v_n(t) \to \rho(t)$ and $w_n(t) \to r(t)$ uniformly as, $n \to \infty$ in $[t_0 - h_1, T + h_2]$ and that ρ, r re coupled minimal and maximal solutions of (3.1),(3.2).

J. Vasundhara Devi *et al.* [6] extend the fruitful method of monotone iterative technique to integro differential equations with retardation and anticipation in 2010. In this paper they consider the integro differential equations with retardation and anticipation of the type

$$x' = f(t, x, Sx, x_t, x^t),\; t \in J = [t_0, T], \tag{3.6}$$

$$x_{t_0} = \phi_0,\; x^T = \psi_0 \tag{3.7}$$

where $\phi_0 \in C_1$, $\psi_0 \in C_2$ and $f \in C[J \times R \times R \times C_1 \times C_2, R]$ the integral operator $S \in C[C[J, R], C[J, R]]$ by $Sx(t) = \int_{t_0}^{t} K(t, s)x(s)ds, t, s \in J$ where $K(t, s) \in C[J \times J, R_+]$, and $\max_{t \in J} K(t,s) = k_1$.

They gave the definitions of coupled minimal and maximal solutions in a suitable form.

Definition 3.7: A function $v_0 \in C^1[[t_0 - h_1, T + h_2], R]$ is said to be coupled upper solution of (3.1) and (3.2) if

$$v_{0'} \leq f(t, v_0, Sv_0, v_{0t}, w_0^t)$$

where $$v_{0t_0} = \phi_1,\; v_0^T = \psi_1,\; v_{t0} \leq x_{t0},\; v^T \leq x^T;$$

Definition 3.8: A function $w_0 \in C^1[[t_0 - h_1, T + h_2], R]$ is said to be coupled upper solution of (3.1) and (3.2) if

$$w_{0'} \geq f(t, w_0, Sw_0,\; w_{0t}, v_0^t)$$

where $$w_{0t_0} = \phi_2,\; w_0^T = \psi_2,\quad w_{t0} \geq x_{t0},\; w^T \geq x^T.$$

Note that $\phi_1, \phi_2 \in C_1, \psi_1, \psi_2 \in C_2$ and $\phi_1 \leq \phi_0 \leq \phi_2, \psi_1 \leq \psi_0 \leq \psi_2$.

The main result of this paper is

Theorem 3.9: Suppose that the assumptions
$H_1 : v_0, w_0 \in C^1[I, R]$ satisfying, $v_0' \leq f(t, v_0, Sv_0, v_{0t}, w_0^t), v_{0t_0} = \phi_1, v_0^T = \psi_1$, and

$w_0' \geq f(t, w_0,\; Sw_0, w_{0t}, v_0^t)$, $w_{0t_0} = \phi_2, w_0^T = \psi_2$ where $\phi_1, \phi_2 \in C_1, \psi_1, \psi_2 \in C_2$ such that

$\phi_1 \leq \phi_0 \leq \phi_2$ and $v_0(t) \leq w_0(t)$ *where* I=[t_0-h_1,T+h_2]

$H_2: f(t,x,\overline{x},\phi,\xi)$ is non increasing in ξ for each $(t,x,\overline{x},\phi)$;

$$H_3: f(t,x,\overline{x},\phi,\psi_1)-f(t,y,\overline{y},\xi,\psi_2)\geq -M_1(x-y)-M_2(\overline{x}-\overline{y})-M_3\int_{-h_1}^{0}(\phi-\xi)(s)d$$

Where $v_0(t)\leq y\leq x\leq w_0(t), Sv_0(t)\leq \overline{y}\leq \overline{x}\leq Sw_0(t)$ $M_1,M_2,M_3\geq 0$

with $M_1+M_2k_1(T-t_0)+M_3h_1](T-t_0)\leq 1$

and $M(T-t_0)>\dfrac{1}{2},\ v_{0t}\leq\xi\leq\phi\leq w_{0t},\ \phi,\xi\in C_1,\ \psi\in C_2$

$H_4: v_{0t_0}-\phi_0,\ \phi_0-w_{0t_0}$ satisfying the assumptions of Lemma 3.4 are satisfied. Then there exist monotone sequences $\{v_n(t)\}$,$\{w_n(t)\}$ such that $v_n(t)\to\rho(t)$ and $w_n(t)\to r(t)$ uniformly as, $n\to\infty$ in $[t_0\text{-}h_1,T+h_2]$ and that ρ,r re coupled minimal and maximal solutions of (3.1),(3.2).

The conclusion of this theorem is obtained by using the following lemma.

Lemma 3.10: Let $p\in C[[t_0-h_1,T+h_2],R],$ p is continuously differentiable on $I=[t_0,T]$ and Suppose further that either

$$(A)\ p'(t)\leq -N_1p(t)-N_2Sp(t)-N_3\int_{-h_1}^{0}p_t(s)ds\ on\,I \tag{3.3}$$

where $Sp(t)=\int_{t_0}^{t}K(t,s)p(s)ds$

and,

$$[N_1+N_2k_1(T-t_0)+N_3h_1](T-t_0)\leq 1 \tag{3.4}$$

$$\max_{t\in J}\ K(t,s)=k_1$$

Or

(B) $p_{t_0}(s)\leq 0,\ \ s\in[-h_1,0],\ \ p\in C^1[[t_0-h_1,T+h_2],R]$ and $p'(t)\leq\dfrac{\lambda}{T-t_0-h_1},$

where $t\in[t_0-h_1,t_0],\ \min_{[t\_0\text{-}h\_1,t\_0]}\ p(s)=-\lambda\ \ ,\lambda\geq 0$

$$[N_1+N_2k_1(T\text{-}t_0)+N_3h_1](T+h_1)\leq 1 \tag{3.5}$$

then $p(t)\leq 0$ on I.

QUASILINEARIZATION

Even though the monotone iterative technique gives monotone iterates that converge to a solution of the given system, the rate of convergence has attracted the attention of researchers.

The basic idea of the original method of quasilinearization developed by Bellman and Kalaba is to provide an explicit analytic representation for the solution of nonlinear differential equations. This method has been developed for all types of differential and integral equations.

The importance of quasilinearization is that these iterates converge quadratically.

This quadratic convergence is obtained by putting additional assumptions on the given function f than in case of the monotone method.

The method of quasilinearization for differential equations with retardation and anticipation has been studied.

In 2009, Z. Drici, F. A. McRae and J. Vasundhara devi [7] extend the method of quasilinearization for the functional differential equations with retardation and anticipation. In this paper the anticipation term was tackled by using a decision function, so that the theory of delay differential equations could be utilized and then fixed point theorems were applied to obtain the existence of solutions of functional differential equations with retardation and then develop the method of generalized quasilinearization.

Before proceeding in this direction they use the Theorem 3.6 to get the unique solutions for the considered linear functional differential equations with retardation and anticipation. These solutions form monotone sequences which converge uniformly to a unique solution of the IVP. To show that this convergence is quadratic they use the following Quasilinearization theorem.

Theorem 4.1: Suppose

H_1: (i) All second order frechet derivatives of $f(t,x,\phi,\psi)$ exist and are bounded;

(ii) $f(t,x,\phi,\psi)$ is convex in x, ϕ and is concave in ψ;

(iii) $f_x(t,x,\phi,\psi)$ is nondecreasing in ϕ for each (t,x,ψ) and nondecreasing in ψ for each (t,x,ϕ)

(iv) $f_\phi(t,x,\phi,\psi)$ is nondecreasing in x for each (t,ϕ,ψ) and is independent in ψ for each (t,x,ϕ)

(v) $f_\psi(t,x,\phi,\psi)$ is nondecreasing in x for each (t,ϕ,ψ) and is independent of ϕ for each (t,x,ψ)

H_2: (i) $-M_1 \le f_x(t,x,\phi,\psi) \le -N_1, \quad 0 < N_1 < M_1;$

(ii) $-M_3 \int_{-h_1}^{0} \eta_1(s)ds \le f_\phi(t,x,\phi,\psi)\eta_1 \le N_3 \int_{-h_1}^{0} \eta_1(s)ds, \quad 0 < N_3 < M_3,$

(iii) $0 \le f_\psi(t,x,\phi,\psi)\eta_2 \le N_4 \int_{0}^{h_2} \eta_2(s)ds$, *where* $\eta_2 \in C_2,\ N_4 > 0;$

(iv) $[N_2h_1 + N_3h_2] < 1;$

H_3: α_0, β_0 are natural lower and upper solutions of (3.1) and (3.2),

H_4: $v_{0t_0} - \phi_0,\ \phi_0 - w_{0t_0}$ satisfying the assumptions of Lemma 3.4.

Then there exist monotone sequences $\{\alpha_n\}, \{\beta_n\}$ which converge uniformly on J to a unique solution of (3.1) and (3.2) and the convergence is quadratic.

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In 2012 J. Vasundhara Devi developed the method of quasilinearization [8] for the integro differential equations involving both retardation and anticipation. Due to the anticipation term the problem becomes complex even to find the existence and uniqueness of solutions of the linear integro differential equation with anticipation and retardation. Hence we use the monotone iterative technique Theorem 3.9 developed for integro differential equations with retardation and anticipation in to obtain the solution for the linear integro differential equation with retardation and anticipation and hence develop the method of quasilinearization.

They develop the method of quasilinearization for the integro differential equation with retardation and anticipation for the (3.6) and (3.7).

To prove the main Theorem they list the following assumptions relative to (3.6) and (3.7) for convenience.

Theorem 4.2: Suppose

H_1: (i) All second order frechet derivatives of $f(t,x,\xi,\phi,\psi)$ exist and are bounded;

(ii) $f(t,x,\xi,\phi,\psi)$ is convex in x, ξ,ϕ and is concave in ψ ;

(iii) $f_x(t,x,\xi,\phi,\psi)$ is nondecreasing in x for each (t,ξ,ϕ,ψ), nondecreasing in ϕ for each (t,x,ξ,ψ) nondecreasing in ξ for each (t,x,ϕ,ψ) and nondecreasing in ψ for each (t,x,ϕ)

(iv) $f_\phi(t,x,\xi,\phi,\psi)$ is nondecreasing in x for each (t,ξ,ϕ,ψ), is nondecreasing in ξ for each (t,x,ϕ,ψ) and is independent in ψ for each (t,x,ξ,ϕ)

(v) $f_\psi(t,x,\xi,\phi,\psi)$ is nondecreasing in x for each (t,ξ,ϕ,ψ), nondecreasing in ξ for each (t,x,ϕ,ψ) and is independent of ϕ for each (t,x,ϕ,ψ)

H_2: (i) $-M_1 \le f_x(t,x,\xi,\phi,\psi) \le -N_1, \quad 0 < N_1 < M_1;$

(ii) $M_2 \le f_\xi(t,x,\xi,\phi,\psi) \le -N_2, \quad 0 < N_2 < M_2;$

(iii) $-M_3 \int_{-h_1}^{0} \eta_1(s)ds \le f_\phi(t,x,\xi,\phi,\psi)\eta_1 \le N_3 \int_{-h_1}^{0} \eta_1(s)ds, \quad 0 < N_3 < M_3,$

(iv) $0 \le f_\psi(t,x,\xi,\phi,\psi)\eta_2 \le N_4 \int_0^{h_2} \eta_2(s)ds, where\ \eta_2 \in C_2,\ N_4 > 0;$

(v) $\left[N_2 k_1 T + N_3 h_1 + N_4 h_2\right] < N_1;$

$H_3 : \alpha_0, \beta_0$ are natural lower and upper solutions of (3.6) and (3.7),

$H_4 : v_{0t_0} - \phi_0,\ \phi_0 - w_{0t_0}$ satisfying the assumptions of Lemma 3.4.

Then there exist monotone sequences $\{\alpha_n\}, \{\beta_n\}$ which converge uniformly on J to a unique solution of (3.6) and (3.7) and the convergence is quadratic.

EULER SOLUTION

All these results are abstract in the sense that there is no specific procedure to obtain a solution of the considered equations, so the Euler solutions for integro differential equations are studied in 2012 by J. Vasundhara Devi *et al*. [9]. In this paper they developed the Euler solutions for (3.6) and (3.7).

Definition 5.1: An Euler solution for the integro differential equation with retardation and anticipation (3.6) and (3.7) is any arc x=x(t) which is the uniform limit of Euler polygonal arcs x_{π_j}, corresponding to some sequence π_j such that $\pi_j \to 0,$ as the diameter $\mu_{\pi_j} \to 0,$ as $j \to \infty$.

The following result give guarantee for the existence of an Euler solution.

Theorem 5.2: Assume that

$$\left| f(t,x,Sx,x_t,z^t) \right| \le g(t,|x|_0(t),|z(t)|) + \int_{t_0}^{t} H(t,s,|x(s)|)ds, \tag{5.1}$$

where $f : I \times \mathbb{R} \times \mathbb{R} \times C_0 \times C_1 \to \mathbb{R}, K : I^2 \times \mathbb{R} \to \mathbb{R}_+, g \in C[I \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$ is non decreasing in t for each (u, v), is non decreasing in u for each (t, v) , is non decreasing in v for each (t, u) $H \in C[I^2 \times \mathbb{R}_+, \mathbb{R}_+]$ is non decreasing in t for each (s, u), is non decreasing in s for each (t, u) , is non decreasing in u for each (t, s), $|x|_0(t) = max_{t-h_1 \le t+s \le t} |x(t+s)|$ and $\mathrm{r}(\mathrm{t}, \mathrm{t}_0, \mathrm{u}_0)$ $ is the maximal solution of the scalar integro differential equation

$$u' = g(t,u,u) + \int_{t_0}^{t} H(t,s,u)ds \tag{5.2}$$

$$u(t_0) = u_0,\ u(T) = \psi_0(0), \tag{5.3}$$

existing on $[\mathrm{t}_0, \mathrm{T}]$ and $|z(t)| \le r(t),$ and z^t is the reasonable estimate of x^t. Then,

(a) there exists at least one Euler solution $x(t) = x(t,t_0,\phi_0(0))$ of the IVP (3.6) and (3.7) which satisfies the Lipschitz condition;

(b) any Euler solution x(t) of (3.6) and (3.7) satisfies the

$$\left| x(t) - \phi_0(0) \right| \le r(t,t_0,u_0) - u_0,\ t \in [t_0, T], \tag{5.4}$$

where $u_0 = |\phi_0|$.

REFERENCES

1. V.Lakshmikantham, M.Rama Mohan Rao, Theory of integro differential Equations, Gordon and Beeach Science Publishers, S.A.,1995
2. T. G. Bhaskar, V. Lakshmikantham, Functional differential systems with retardation and anticipation, Nonlinear anal. Real. World Appl., 8 (2007), 865-871.
3. Bashir Ahmad, S. Sivasundaram, Existence and monotone iterative technique for impulsive hybrid functional differential systems with anticipation and retardation, Applied Mathematics and Computation, 197 (2008), 515-524.
4. A. S. Vatsala, J.Vasundhara Devi, Quasilinearization for Second order Singular Boundary Value Problems with solutions in Weighted Spaces, Journal of the Korean Mathematical Society, Vol. 37, (2002), No. 5, 823-833.

5. V. Lakshmikantham, T.Gnana Bhaskar and J.Vasundhara Devi, Monotone iterative technique for functional differential equations with retardation and anticipation, Non Linear Analysis 66(2007), 2237-2242.
6. J.Vasundhara Devi, Ch.V.Sreedhar, S.Nagamani, Monotone iterative technique for integro di_erential equations with retardation and anticipation, Communications and Applied Analysis 14(2010), no. 4, 325-336.
7. Z. Drici, F.A.McRae, and J.Vasundhara Devi, Quasilinearization for functional differential equations with retardation and anticipation, Non-linear Analysis (2008), 70 (2009), 1763-1775.
8. J.Vasundhara Devi, Ch. V. Sreedhar, Quasilinearization for integro differential equations with retardation and anticipation, Nonlinear Studies, Vol. 19, (2012), no. 1, 99-122.
9. J.Vasundhara Devi, Ch. V. Sreedhar, Euler Solutions for integro differential equations with retardation and anticipation, Nonlinear Dynamics and Systems Theory 12(3) (2012), 237-250.

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