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A FEW PROPERTIES OF FRACTIONAL HYPERBOLIC LIKE FUNCTIONS

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ABSTRACT

In this paper we consider 3^{rd} order Caputo fractional differential equation (CFDE) and obtain its solution analytically. The solutions of the 3^{rd} order CFDE are referred as fractional hyperbolic like functions. A few properties of these functions are studied. Further extended fractional hyperbolic like functions are obtained as solutions of the n^{th} order CFDE of the same family.

Keywords: *Caputo fractional differential equation, Mittag - Leffler function, fractional hyperbolic functions, fractional hyperbolic like functions, Wronskian.*

Mathematics Subject Classification: *Primary: 47G20.*

1. INTRODUCTION

Using the approach in [1] fractional hyperbolic functions have been studied in [8]. In this paper we give details of the proofs of results that were just mentioned in [8]. Thus we present results corresponding to fractional hyperbolic like functions using the theory of fractional differential equations.

2. PRELIMINARIES

In order to obtain results pertaining to fractional hyperbolic like functions we introduce definitions and concepts corresponding to fractional derivatives and fractional hyperbolic functions. We first begin with generalization of the exponential function known as the Mittag - Leffler function [5, 7].

Definition 2.1: The Mittag - Leffler function of one parameter, $E_q(z)$ is defined by

$$E_q(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(kq+1)} \quad (z\in C,\ R(q)>0)\ . \tag{2.1}$$

Definition 2.2: The Mittag - Leffler function of two parameters, $E_{q,\beta}(z)$ is defined by

$$E_{q,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(kq+\beta)} \quad (z,\beta\in C,\ R(q)>0)\ . \tag{2.2}$$

The definitions of fractional derivatives for a series by Riemann and Caputo [4] are given below.

Definition 2.3: Riemann - Liouville fractional derivative for series.
If

$$f(x)=x^{q-1}\sum_{k=0}^{\infty}a_k x^{kq}$$

then

$$D^q f(x)=\frac{d^q(f(x))}{dx^q}=x^{q-1}\sum_{k=0}^{\infty}a_{k+1}\frac{\Gamma((k+2)q)}{\Gamma((k+1)q)}x^{kq}\ . \tag{2.3}$$

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Definition 2.4: Caputo fractional derivative for series.
If

$$f(x)=\sum_{k=0}^{\infty}a_k x^{kq}$$

then

$$^{c}D^{q}f(x)=\frac{d^{q}(f(x))}{dx^{q}}=\sum_{k=0}^{\infty}a_{k+1}\frac{\Gamma(1+(k+1)q)}{\Gamma(1+kq)}x^{kq}. \tag{2.4}$$

Next we proceed to present the definitions of the fore mentioned derivatives in terms of the integrals.

Definition 2.5: Riemann - Liouville derivative of $x(t)$ is given by

$$D^{q}x(t)=\frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_{t_0}^{t}(t-s)^{-q}x(s)ds,\quad (t\in R). \tag{2.5}$$

Definition 2.6: Caputo derivative of x(t) is given by

$$^{c}D^{q}x(t)=\frac{1}{\Gamma(1-q)}\int_{t_0}^{t}(t-s)^{-q}x'(s)ds\,. \tag{2.6}$$

The initial value problem for Riemann - Liouville fractional differential equation (RLFDE) and the initial value problem for Caputo fractional differential equation (CFDE) have a basic difference. The RLFDE has a singularity at the initial point and is given by

$$D^{q}x(t)=f(t,x(t))\ ,\ x^{0}=x(t)(t-t_0)^{1-q}/t=t_0,$$

and the CFDE is given by

$$^{c}D^{q}x(t)=f(t,x(t))\ ,\ x(t_0)=x_0.$$

There exists a relation between the CFDE and RLFDE which is given by

$$^{c}D^{q}x(t)=D^{q}[x(t)-x_0].$$

It has been shown in [3, 6] that the results which hold for the initial value problem of RLFDE are also true for CFDE. On basis of this result we give the existence and uniqueness results for linear n^{th} order RLFDE and for systems and propose that they can be naturally extended for linear CFDE. We now introduce the q - exponential function [5] which is needed to define the solution of the linear Reimann - Liouville fractional differential equation.

Definition 2.7: The q - exponential function $e_q^{\lambda z}$ is defined by

$$e_q^{\lambda z}=z^{q-1}E_{q,q}(\lambda z^{q}) \tag{2.7}$$

where $(z\in C\setminus\{0\},R(q)>0)$ and $\lambda\in C.$

Definition 2.8: We define the function $e_{q,n}^{\lambda z}$ as

$$e_{q,n}^{\lambda z}=z^{q-1}\sum_{k=0}^{\infty}\frac{(k+n)!}{\Gamma[(k+n+1)q]}\frac{(\lambda z^{q})^{k}}{k!}. \tag{2.8}$$

Consider the linear fractional differential equation (LFDE)

$$[L_{nq}(y)](t):=(D_{a^+}^{nq})y(t)+\sum_{k=0}^{n-1}a_k(D_{a^+}^{kq})y(t)=0 \tag{2.9}$$

where the coefficients $\{a_j\}_{j=1}^{n-1}$ are real constants. Then we assume that the solution of the above RLFDE is of the form

$$y(t)=e_q^{\lambda(t-a)},\lambda\in C$$

and obtain the characteristic equation as

$$P_n(\lambda)=\lambda^{n}+\sum_{k=1}^{n-1}a_k\lambda^{k},\lambda\in C. \tag{2.10}$$

Please refer to [5] for lemmas and theorems that are necessary to obtain the existence and uniqueness result for LFDE (2.9).

We denote $\mathbb{R}_+$ as the set of all non-negative real numbers.

Next we state the theorem in which fractional hyperbolic functions are obtained as solutions of 2^{nd} order Caputo fractional differential equation.

Theorem 2.1: Consider the Initial value problem (IVP) of α^{th} order homogeneous fractional differential equation with Caputo derivative given by

$$^{c}D^{\alpha}x(t)-x(t)=0,\ 1<\alpha\le 2,\ t\in\mathbb{R}_+, \tag{2.11}$$

with initial conditions

$$x(0)=1,\ ^{c}D^{q}x(0)=0, \qquad \text{where} \qquad \alpha=2q,\quad \frac{1}{2}<q\le 1. \tag{2.12}$$

Then the general solution of (2.11) - (2.12) is given by $c_1\ x(t)+c_2\ y(t)$ (c_1, c_2 being arbitrary constants) where $x(t)$ and $y(t)$ are infinite series solutions of the form

$$x(t)=\sum_{k=0}^{\infty}\frac{t^{2kq}}{\Gamma(1+2kq)}=\cosh_q t \qquad \text{(Say)}$$

$$y(t)=\sum_{k=0}^{\infty}\frac{t^{(2k+1)q}}{\Gamma(1+(2k+1)q)}=\sinh_q t\ \ \text{(say)},\ \ t\in\mathbb{R}_+,\ \ \frac{1}{2}<q\le 1.$$

The definition of Wronskian corresponding to Caputo fractional differential equation of order α is as follows:

This definition is parallel to the definition of Wronskian in Ordinary differential equations [2].

Definition 2.9 (Wronskian): Let $\phi_1,\phi_2,\dots,\phi_n$ be n real or complex valued functions defined on some nonempty interval I in $\mathbb{R}_+$ each having derivatives of order $\alpha=nq,\ n\in N.$ Then the fractional Wronskian of these n functions is the determinant of the matrix W of order n defined on I and whose value at $t\in I$ is

$$W(t)=W(\phi_1,\phi_2,\phi_3,\dots,\phi_n)(t)=\begin{vmatrix}\phi_1(t) & \phi_2(t) & \cdots & \phi_n(t)\\ ^{c}D^{q}\phi_1(t) & ^{c}D^{q}\phi_2(t) & \cdots & ^{c}D^{q}\phi_n(t)\\ \vdots & \vdots & \vdots & \vdots\\ ^{c}D^{(n-1)q}\phi_1(t) & ^{c}D^{(n-1)q}\phi_2(t) & \cdots & ^{c}D^{(n-1)q}\phi_n(t)\end{vmatrix} \tag{2.13}$$

3. FRACTIONAL HYPERBOLIC LIKE FUNCTIONS THROUGH THIRD ORDER CFDE

In [8], 2^{nd} order CFDE of the form $^{c}D^{\alpha}x(t)-x(t)=0, 1<\alpha\le 2,$ with initial conditions $x(0)=1,$ $^{c}D^{q}x(0)=0,$ where $\alpha=2q, \frac{1}{2}<q\le 1, t\in\mathbb{R}_+$ is considered and results pertaining to this equation were stated and proved.

In this section 3^{rd} order CFDE of the same family is considered and some important results corresponding to this equation are obtained.

We now state and prove the theorem in which the solutions of 3^{rd} order CFDE are obtained.

Consider the α^{th} order $(2<\alpha\le 3)$ homogeneous Caputo fractional initial value problem,

$$^{c}D^{\alpha}x(t)-x(t)=0 \tag{3.1}$$

$$x(0)=1,\ ^{c}D^{q}x(0)=0,\ ^{c}D^{2q}x(0)=0 \tag{3.2}$$

where $\quad \alpha=3q,\ \ \frac{2}{3}<q\le 1,\quad t\in\mathbb{R}_+.$

Theorem 3.1: The general solution of the CFDE (3.1) is given by $c_1 x(t) + c_2 y(t) + c_3 z(t)$ ($c_{1,} c_2$ and c_3 being arbitrary constants) where $x(t)$, $y(t)$ and $z(t)$ are infinite series solutions of the form

$$x(t) = \sum_{k=0}^{\infty} \frac{t^{3kq}}{\Gamma(1+3kq)}, \tag{3.3}$$

$$y(t) = \sum_{k=0}^{\infty} \frac{t^{(3k+1)q}}{\Gamma(1+(3k+1)q)}, \tag{3.4}$$

$$z(t) = \sum_{k=0}^{\infty} \frac{t^{(3k+2)q}}{\Gamma(1+(3k+2)q)}, \ t \in \mathbb{R}_+. \tag{3.5}$$

Proof: We transform the CFDE (3.1) to a system of equations of q^{th} order, $\frac{2}{3} < q \le 1$ by taking $\alpha = 3q$ and setting

$$^{c}D^{q} x(t) = z(t), \ ^{c}D^{q} y(t) = x(t), \ ^{c}D^{q} z(t) = y(t) \tag{3.6}$$

with initial conditions

$$x(0) = 1, \ y(0) = 0, \ z(0) = 0. \tag{3.7}$$

Let

$$x(t) = \sum_{k=0}^{\infty} a_k t^{kq}, \ y(t) = \sum_{k=0}^{\infty} b_k t^{kq}, \ z(t) = \sum_{k=0}^{\infty} c_k t^{kq} \tag{3.8}$$

be the solutions of the IVP(3.1) - (3.2) where a_k, b_k and $c_k{}'s$ are unknown constants and $t \in \mathbb{R}_+$.

Using the initial conditions (3.7) in (3.8) we obtain

$$a_0 = 1, \ b_o = 0, \ c_0 = 0.$$

Consider the equation $^{c}D^{q} x(t) = z(t).$

Substituting (3.8) in the above equation we get

$$^{c}D^{q}\left[\sum_{k=0}^{\infty} a_k t^{kq} \right] = \sum_{k=0}^{\infty} c_k t^{kq}.$$

This yields

$$\sum_{k=0}^{\infty} a_{k+1} \frac{\Gamma(1+(k+1)q)}{\Gamma(1+kq)} t^{kq} = \sum_{k=0}^{\infty} c_k t^{kq}.$$

Further, comparing the coefficients of the same power we get

$$a_{k+1} = \frac{\Gamma(1+kq)}{\Gamma(1+(k+1)q)} c_k \quad \text{for} \quad k = 0,1,2,\ldots.$$

Similarly by using $^{c}D^{q} y(t) = x(t)$ and $^{c}D^{q} z(t) = y(t), \ t \in \mathbb{R}_+$, we obtain

$$b_{k+1} = \frac{\Gamma(1+kq)}{\Gamma(1+(k+1)q)} a_k, \ c_{k+1} = \frac{\Gamma(1+kq)}{\Gamma(1+(k+1)q)} b_k, \quad \text{for} \quad k = 0,1,2\ldots.$$

By substituting successively, we obtain the values of all coefficients $a_1, a_2, \ldots, b_1, b_2, \ldots$ and $c_1, c_2, \ldots$ and finally we get the solutions of CFDE (3.1) - (3.2) as

$$x(t) = \sum_{k=0}^{\infty} \frac{t^{3kq}}{\Gamma(1+3kq)}, \ t \in \mathbb{R}_+,$$

$$y(t) = \sum_{k=0}^{\infty} \frac{t^{(3k+1)q}}{\Gamma(1+(3k+1)q)}, \ t \in \mathbb{R}_+,$$

and

$$z(t) = \sum_{k=0}^{\infty} \frac{t^{(3k+2)q}}{\Gamma(1+(3k+2)q)}, \ t \in \mathbb{R}_+.$$

This completes the proof.

Remark 3.2: We denote these series by $N_{3,0}^q, N_{3,1}^q$ and $N_{3,2}^q$ respectively.

Hence

$$N_{3,0}^q = \sum_{k=0}^{\infty} \frac{t^{3kq}}{\Gamma(1+3kq)}, \ t \in \mathbb{R}_+, \tag{3.9}$$

$$N_{3,1}^q = \sum_{k=0}^{\infty} \frac{t^{(3k+1)q}}{\Gamma(1+(3k+1)q)}, \ t \in \mathbb{R}_+, \tag{3.10}$$

$$N_{3,2}^q = \sum_{k=0}^{\infty} \frac{t^{(3k+2)q}}{\Gamma(1+(3k+2)q)}, \ t \in \mathbb{R}_+. \tag{3.11}$$

To verify the result in the Theorem 3.1 we provide another method below.

Verification:

Consider the IVP (3.1) - (3.2).

Let the solutions of the IVP (3.1) - (3.2) be given by

$$x(t) = \sum_{k=0}^{\infty} \frac{t^{3kq}}{\Gamma(1+3kq)}, \ t \in \mathbb{R}_+,$$

$$y(t) = \sum_{k=0}^{\infty} \frac{t^{(3k+1)q}}{\Gamma(1+(3k+1)q)}, \ t \in \mathbb{R}_+,$$

and

$$z(t) = \sum_{k=0}^{\infty} \frac{t^{(3k+2)q}}{\Gamma(1+(3k+2)q)}, \ t \in \mathbb{R}_+ .$$

To verify them, we consider

$$\begin{aligned} {}^cD^q x(t) &= {}^cD^q \left[\sum_{k=0}^{\infty} \frac{t^{3kq}}{\Gamma(1+3kq)} \right], \\ &= {}^cD^q \left[1 + \frac{t^{3q}}{\Gamma(1+3q)} + \frac{t^{6q}}{\Gamma(1+6q)} + \frac{t^{9q}}{\Gamma(1+9q)} + \cdots \right] \\ &= \frac{t^{2q}}{\Gamma(1+2q)} + \frac{t^{5q}}{\Gamma(1+5q)} + \frac{t^{8q}}{\Gamma(1+8q)} + \cdots \end{aligned}$$

Differentiating using Caputo derivative both sides we get

$${}^cD^{2q} x(t) = \frac{t^{q}}{\Gamma(1+q)} + \frac{t^{4q}}{\Gamma(1+4q)} + \frac{t^{7q}}{\Gamma(1+7q)} + \cdots$$

Differentiating once again using Caputo derivative both sides we obtain

$$\begin{aligned} {}^cD^{3q} x(t) &= 1 + \frac{t^{3q}}{\Gamma(1+3q)} + \frac{t^{6q}}{\Gamma(1+6q)} + \frac{t^{9q}}{\Gamma(1+9q)} + \cdots \\ &= x(t) \end{aligned}$$

or $\quad {}^cD^{3q} x(t) - x(t) = 0.$

Also the initial condition $x(0) = 1$ is satisfied.

Hence $x(t)=\sum_{k=0}^{\infty}\frac{t^{3kq}}{\Gamma(1+3kq)}, t\in\mathbb{R}_+$ is the solution of the IVP (3.1) – (3.2).

Similarly we can verify that y(t) and z(t) are also the solutions of the IVP (3.1) - (3.2) .

This completes the verification.

We now state and prove a theorem that relates the Wronskian and the solutions of the CFDE (3.1).

Theorem 3.3 (Wronskian Property): Let $x(t), y(t)$ and $z(t)$ be three solutions of the CFDE (3.1). These three solutions are linearly independent on R_+ if and only if the Wronskian $W(t)\neq 0$ for every $t\in\mathbb{R}_+$.

Proof: Suppose the Wronskian of the solutions $x(t)$, $y(t)$ and $z(t)$ of CFDE (3.1) be such that $W(t)\neq 0$. we show that $x(t)$, $y(t)$ and $z(t)$ are linearly independent solutions. If possible assume that $x(t)$, $y(t)$ and $z(t)$ are linearly dependent solutions. Then there exist constants a,b,c where a,b,c are not simultaneously zero such that

$$ax(t) + by(t) + cz(t) = 0$$

Without loss of generality suppose $a\neq 0$.
Then

$$x(t) = -\frac{b}{a}y(t) - \frac{c}{a}z(t).$$

By setting we get

$$h = -\frac{b}{a} \quad \text{and} \quad l = -\frac{c}{a},$$

We get

$$x(t) = hy(t) + lz(t).$$

Now consider the Wronskian

$$W(t)=\begin{vmatrix} x(t) & y(t) & z(t) \\ {}^cD^q(x(t)) & {}^cD^q(y(t)) & {}^cD^q(z(t)) \\ {}^cD^{2q}(x(t)) & {}^cD^{2q}(y(t)) & {}^cD^{2q}(z(t)) \end{vmatrix}$$

$$=\begin{vmatrix} h\,y(t)+lz(t) & y(t) & z(t) \\ h\,{}^cD^q y(t)+l\,{}^cD^q z(t) & {}^cD^q y(t) & {}^cD^q z(t) \\ h\,{}^cD^{2q} y(t)+l\,{}^cD^{2q} z(t) & {}^cD^{2q} y(t) & {}^cD^{2q} z(t) \end{vmatrix}$$

$$=\begin{vmatrix} h\,y(t) & y(t) & z(t) \\ h\,{}^cD^q y(t) & {}^cD^q y(t) & {}^cD^q z(t) \\ h\,{}^cD^{2q} y(t) & {}^cD^{2q} y(t) & {}^cD^{2q} z(t) \end{vmatrix} + \begin{vmatrix} l\,z(t) & y(t) & z(t) \\ l\,{}^cD^q z(t) & {}^cD^q y(t) & {}^cD^q z(t) \\ l\,{}^cD^{2q} z(t) & {}^cD^{2q} y(t) & {}^cD^{2q} z(t) \end{vmatrix}$$

$$= 0.$$

Hence $W(t)=0$, which is a contradiction. Therefore the solutions $x(t)$, $y(t)$ and $z(t)$ are linearly independent solutions.

Now to show that $W(t)\neq 0$, when $x(t)$, $y(t)$ and $z(t)$ are linearly independent solutions.

Suppose if possible W(t) = 0 for some $t\in\mathbb{R}_+$.
Then

$$\begin{vmatrix} x(t) & y(t) & z(t) \\ {}^cD^q x(t) & {}^cD^q y(t) & {}^cD^q z(t) \\ {}^cD^{2q} x(t) & {}^cD^{2q} y(t) & {}^cD^{2q} z(t) \end{vmatrix} = 0.$$

This implies that there exists a linear combination of columns as

$$a\begin{bmatrix} x(t) \\ {}^{c}D^{q}x(t) \\ {}^{c}D^{2q}x(t) \end{bmatrix} + b\begin{bmatrix} y(t) \\ {}^{c}D^{q}y(t) \\ {}^{c}D^{2q}y(t) \end{bmatrix} + c\begin{bmatrix} z(t) \\ {}^{c}D^{q}z(t) \\ {}^{c}D^{2q}z(t) \end{bmatrix} = 0$$

where at least one of a,b and c is different from zero.

Without loss of generality suppose $a \neq 0$.

Then $x(t) = -\dfrac{b}{a}y(t) - \dfrac{c}{a}z(t).$

This implies that solutions $x(t), y(t)$ and $z(t)$ are linearly dependent, which is a contradiction to the assumption that these solutions are linearly independent.

Hence $W(t) \neq 0.$

Thus the proof is complete.

We now present below the addition formulae for solutions of third order CFDE. The proofs can be obtained by following the technique used for solutions of the 2^{nd} order CFDE in [8].

Addition Formulae:

Let $\eta \in \mathbb{R}_+$ Then the solution $\left(x(t),\ y(t),\ z(t)\right)$ of the CFDS (3.6) possesses the properties

$$x(t+\eta) = x(t)x(\eta) + y(t)z(\eta) + z(t)y(\eta), \tag{3.12}$$

$$y(t+\eta) = x(t)y(\eta) + y(t)x(\eta) + z(t)z(\eta), \tag{3.13}$$

$$z(t+\eta) = x(t)z(\eta) + y(t)y(\eta) + z(t)x(\eta). \tag{3.14}$$

We use the method of linear algebra to prove these properties. Let $\eta \in \mathbb{R}_+$ be arbitrary. If $\left(x(t), y(t), z(t) \right)$ is a solution of the CFDS (3.6) then $(x(t+\eta), y(t+\eta), z(t+\eta))$ is also a solution of the CFDS (3.6) .

Now $x(t+\eta)$ can be expressed as a linear combination of $x(t),\ y(t)$ and $z(t)$.

Hence $x(t+\eta) = c_1x(t) + c_2y(t) + c_3z(t),\ \ t \in \mathbb{R}_+.$ (3.15)

For a given $\eta \in \mathbb{R}_+,\ c_1,\ c_2$ and c_3 need to be uniquely determined.

For $t = 0$, in view of the initial conditions (3.7) we get $c_1 = x(\eta)$.

Further, we have

$$z(t+\eta) = {}^{c}D^{q}x(t+\eta) = c_1\,{}^{c}D^{q}x(t) + c_2\,{}^{c}D^{q}y(t) + c_3\,{}^{c}D^{q}z(t)$$
$$= c_1\, z(t) + c_2\, x(t) + c_3\, y(t).$$

$$y(t+\eta) = {}^{c}D^{2q}x(t+\eta) = c_1\,{}^{c}D^{q}z(t) + c_2\,{}^{c}D^{q}x(t) + c_3\,{}^{c}D^{q}y(t)$$
$$= c_1\, y(t) + c_2\, z(t) + c_3\, x(t).$$

For $t = 0$, we get $c_2 = z(\eta),\ c_3 = y(\eta).$ Now we substitute the values of c_1, c_2 and c_3 in (3.15) to get the following relation.

$$x(t+\eta) = x(t)x(\eta) + y(t)z(\eta) + z(t)y(\eta).$$

Similarly we can obtain the relations

$$y(t+\eta) = x(t)y(\eta) + y(t)x(\eta) + z(t)z(\eta)$$

And $z(t+\eta) = x(t)z(\eta) + y(t)y(\eta) + z(t)x(\eta).$

These are addition formulae for the solutions $x(t),\ y(t)$ and $z(t)$ of CFDE (3.1).

From these relations we derive, for $\eta = t$,

$$x(2t) = x^2(t) + 2y(t)z(t) \tag{3.16}$$

$$y(2t) = 2x(t)y(t) + z^2(t) \tag{3.17}$$

$$z(2t) = 2z(t)x(t) + y^2(t). \tag{3.18}$$

These results may be easily used to obtain the values of $x(3t)$, $y(3t)$ and $z(3t)$ and many similar relations.

Similar to the Euler's formulae for the 2^{nd} order CFDE in [8], we can obtain the Euler's formulae for the 3^{rd} order CFDE (3.1).

Euler's Formulae: The solutions of the CFDE (3.1) are $E_q(t^q)$, $E_q(\omega t^q)$ and $E_q(\omega^2 t^q)$ where $1, \omega, \omega^2 \left(\omega = \frac{-1+\sqrt{3}i}{2}\right)$ are the roots of the characteristic equation $\lambda^3 - 1 = 0.$ Now we express $E_q(t^q)$, $E_q(\omega t^q)$ and $E_q(\omega^2 t^q)$ in terms of $N_{3,0}^q(t)$, $N_{3,1}^q(t)$ and $N_{3,2}^q(t)$ respectively as follows:

$$
\begin{aligned}
(i) E_q(t^q) &= \sum_{k=0}^{\infty} \frac{t^{kq}}{\Gamma(1+kq)}, \quad t \in \mathbb{R}_+ \\
&= 1 + \frac{t^q}{\Gamma(1+q)} + \frac{t^{2q}}{\Gamma(1+2q)} + \frac{t^{3q}}{\Gamma(1+3q)} + \frac{t^{4q}}{\Gamma(1+4q)} + \cdots \\
&= 1 + \frac{t^{3q}}{\Gamma(1+3q)} + \frac{t^{6q}}{\Gamma(1+6q)} + \cdots + \left(\frac{t^q}{\Gamma(1+q)} + \frac{t^{4q}}{\Gamma(1+4q)} + \cdots\right) + \left(\frac{t^{2q}}{\Gamma(1+2q)} + \frac{t^{5q}}{\Gamma(1+5q)} + \cdots\right) \\
&= N_{3,0}^q(t) + N_{3,1}^q(t) + N_{3,2}^q(t).
\end{aligned}
$$

$$
\begin{aligned}
(ii) E_q(\omega t^q) &= \sum_{k=0}^{\infty} \frac{\omega^k t^{kq}}{\Gamma(1+kq)}, \quad t \in \mathbb{R}_+ \\
&= 1 + \frac{\omega t^q}{\Gamma(1+kq)} + \frac{\omega^2 t^{2q}}{\Gamma(1+2q)} + \frac{t^{3q}}{\Gamma(1+3q)} + \frac{\omega t^{4q}}{\Gamma(1+4q)} + \cdots \\
&= 1 + \frac{t^{3q}}{\Gamma(1+3q)} + \frac{t^{6q}}{\Gamma(1+6q)} + \cdots + \omega\left(\frac{t^q}{\Gamma(1+q)} + \frac{t^{4q}}{\Gamma(1+4q)} + \cdots\right) + \omega^2\left(\frac{t^{2q}}{\Gamma(1+2q)} + \frac{t^{5q}}{\Gamma(1+5q)} + \cdots\right) \\
&= N_{3,0}^q(t) + \omega N_{3,1}^q(t) + \omega^2 N_{3,2}^q(t).
\end{aligned}
$$

$$
\begin{aligned}
(iii) E_q(\omega^2 t^q) &= \sum_{k=0}^{\infty} \frac{\omega^{2k} t^{kq}}{\Gamma(1+kq)}, \quad t \in \mathbb{R}_+ \\
&= 1 + \frac{\omega^2 t^q}{\Gamma(1+q)} + \frac{\omega t^{2q}}{\Gamma(1+2q)} + \frac{t^{3q}}{\Gamma(1+3q)} + \frac{\omega^2 t^{4q}}{\Gamma(1+4q)} + \cdots \\
&= 1 + \frac{t^{3q}}{\Gamma(1+3q)} + \frac{t^{6q}}{\Gamma(1+6q)} + \cdots + \omega^2\left(\frac{t^q}{\Gamma(1+q)} + \frac{t^{4q}}{\Gamma(1+4q)} + \cdots\right) + \omega\left(\frac{t^{2q}}{\Gamma(1+2q)} + \frac{t^{5q}}{\Gamma(1+5q)} + \cdots\right) \\
&= N_{3,0}^q(t) + \omega^2 N_{3,1}^q(t) + \omega N_{3,2}^q(t).
\end{aligned}
$$

Thus we obtain the following relations:

$$E_q(t^q) = N_{3,0}^q(t) + N_{3,1}^q(t) + N_{3,2}^q(t), \tag{3.19}$$

$$E_q(\omega t^q) = N_{3,0}^q(t) + \omega N_{3,1}^q(t) + \omega^2 N_{3,2}^q(t), \tag{3.20}$$

$$E_q(\omega^2 t^q) = N_{3,0}^q(t) + \omega^2 N_{3,1}^q(t) + \omega N_{3,2}^q(t), \quad t \in \mathbb{R}_+. \tag{3.21}$$

We can also express $N^q_{3,0}(t), N^q_{3,1}(t)$ and $N^q_{3,2}(t)$ in terms of $E_q(t^q),\ E_q(\omega t^q)$ and $E_q(\omega^2 t^q).$ By solving (3.19) , (3.20) and (3.21) we get

$$N^q_{3,0}(t) = \frac{1}{3}E_q(t^q) + \frac{1}{3}E_q(\omega t^q) + \frac{1}{3}E_q(\omega^2 t^q), \tag{3.22}$$

$$N^q_{3,1}(t) = \frac{1}{3}E_q(t^q) + \frac{\omega^2}{3}E_q(\omega t^q) + \frac{\omega}{3}E_q(\omega^2 t^q), \tag{3.23}$$

$$N^q_{3,2}(t) = \frac{1}{3}E_q(t^q) + \frac{\omega}{3}E_q(\omega t^q) + \frac{\omega^2}{3}E_q(\omega^2 t^q),\ \ t \in \mathbb{R}_+. \tag{3.24}$$

4. EXTENSION OF FRACTIONAL HYPERBOLIC LIKE FUNCTIONS TO n^{th} ORDER CFDE.

It is observed that, the results obtained in Section 3 can be generalized to the CFDE of order $\alpha,\ n-1 < \alpha \le n.$ We proceed to do so in this section.

Consider the n^{th} order fractional IVP of the form

$${}^cD^\alpha x(t) - x(t) = 0 \tag{4.1}$$

$$x(0) = 1,\ {}^cD^q x(0) = 0,\ \dots,\ {}^cD^{(n-1)q}x(0) = 0 \tag{4.2}$$

where $n-1 < \alpha \le n,$ with $\alpha = nq,\ \dfrac{n-1}{n} < q \le 1,\ \ n \in N$ fixed.

Theorem 4.1: The general solution of the CFDE (4.1) is given by $c_1x_1(t) + c_2x_2(t) + \dots + c_nx_n(t)$ where $c_1,\ c_2, \dots,\ c_n$ are arbitrary constants and $x_1(t),\ x_2(t),\ \dots,\ x_n(t)$ are series solutions of the form

$$\begin{aligned}
x_1(t) &= \sum_{k=0}^{\infty}\frac{t^{nkq}}{\Gamma(1+nkq)},\\
x_2(t) &= \sum_{k=0}^{\infty}\frac{t^{(nk+1)q}}{\Gamma(1+(nk+1)q)},\\
&\ \ \vdots \qquad\qquad \vdots\\
x_n(t) &= \sum_{k=0}^{n}\frac{t^{(nk+(n-1))q}}{\Gamma(1+(nk+(n-1))q)},\ \ t \in \mathbb{R}_+.
\end{aligned} \tag{4.3}$$

Proof: Let $x_1(t),\ x_2(t),\ \dots,\ x_n(t)$ be 'n' solutions of the CFDE (4.1) - (4.2) such that

$${}^cD^q x_1(t) = x_n(t),\ {}^cD^q x_2(t) = x_1(t), \dots,\ {}^cD^q x_n(t) = x_{n-1}(t) \tag{4.4}$$

with initial conditions

$$x_1(0) = 1,\ x_2(0) = 0, \dots,\ x_n(0) = 0. \tag{4.5}$$

Let

$$x_1(t) = \sum_{k=0}^{\infty}a_{1k}t^{kq},\ x_2(t) = \sum_{k=0}^{\infty}a_{2k}t^{kq},\ \dots,\ x_n(t) = \sum_{k=0}^{\infty}a_{nk}t^{kq} \tag{4.6}$$

where $a_{ik}{}'s,\ i = 1,\ 2,\ \dots,\ n,\ k = 0,\ 1,\ \dots,\ \infty$ are unknown constants and $t \in \mathbb{R}_+.$

From the initial conditions (4.5), we obtain

$$a_{10} = 1,\ a_{20} = 0,\ \dots,\ a_{n0} = 0.$$

Now consider the equation

$${}^cD^q x_1(t) = x_n(t).$$

Substituting (4.6) in the above equation we get

$${}^cD^q\left[\sum_{k=0}^{\infty}a_{1k}t^{kq}\right] = \sum_{k=0}^{\infty}a_{nk}t^{kq}$$

which gives

$$\sum_{k=0}^{\infty} a_{1(k+1)} \frac{\Gamma(1+(k+1)q)}{\Gamma(1+kq)} t^{kq} = \sum_{k=0}^{\infty} a_{nk} t^{kq}.$$

Further, comparison of the coefficients of the same power yields

$$a_{1(k+1)} = \frac{\Gamma(1+kq)}{\Gamma(1+(k+1)q)} a_{nk} \quad \text{for} \quad k = 0, 1, 2, \ldots .$$

Similarly by using

$${}^{c}D^{q} x_2(t) = x_1(t), \ldots, \; {}^{c}D^{q} x_n(t) = x_{n-1}(t) \text{ we get}$$

$$a_{2(k+1)} = \frac{\Gamma(1+kq)}{\Gamma(1+(k+1)q)} a_{1k}, \; a_{3(k+1)} = \frac{\Gamma(1+kq)}{\Gamma(1+(k+1)q)} a_{2k}, \; \ldots .$$

Using the above recursive relations, we obtain the values of

$$a_{11}, a_{12}, \ldots, a_{21}, a_{22}, \ldots, a_{n1}, a_{n2}, \ldots$$

and finally the solutions are given by

$$x_1(t) = \sum_{k=0}^{\infty} \frac{t^{nkq}}{\Gamma(1+nkq)},$$

$$x_2(t) = \sum_{k=0}^{\infty} \frac{t^{(nk+1)q}}{\Gamma(1+(nk+1)q)},$$

$$\vdots \qquad \vdots$$

$$x_n(t) = \sum_{k=0}^{n} \frac{t^{(nk+(n-1))q}}{\Gamma(1+(nk+(n-1))q)}, \; t \in \mathbb{R}_+.$$

The proof is complete.

Remark 4.2: For sake of parallel notation to the work in earlier section, the solutions $x_1(t), x_2(t). ., x_n(t)$ can be represented by $N_{n,0}^{q}, N_{n,1}^{q}, \ldots, N_{n,n-1}^{q}$ respectively. Thus

$$N_{n,0}^{q} = \sum_{k=0}^{\infty} \frac{t^{nkq}}{\Gamma(1+nkq)},$$

$$N_{n,1}^{q} = \sum_{k=0}^{\infty} \frac{t^{(nk+1)q}}{\Gamma(1+(nk+1)q)}, \tag{4.7}$$

$$\vdots \qquad \vdots$$

$$N_{n,n-1}^{q} = \sum_{k=0}^{n} \frac{t^{(nk+(n-1))q}}{\Gamma(1+(nk+(n-1))q)}, \; t \in \mathbb{R}_+.$$

A suitable notation to represent the above solutions conveniently is as follows:

$$N_{n,r}^{q}(t) = \sum_{k=0}^{\infty} \frac{t^{(nk+r)q}}{\Gamma(1+(nk+r)q)}, \; r = 0,1,2,\ldots,(n-1), \; n \in N, \; t \in \mathbb{R}_+. \tag{4.8}$$

Functions $N_{n,r}^{q}(t)$ in (4.8) are extended fractional hyperbolic like functions.

Now we state and prove a theorem which relates the Wronskian and the solutions of the CFDE (4.1).

Theorem 4.3: Let $x_1(t), x_2(t), \ldots, x_n(t)$ be n solutions of the CFDE (4.1). These n solutions are linearly independent on $\mathbb{R}_+$ if and only if the Wronskian $W(t) \neq 0$ for every $t \in \mathbb{R}_+$.

Proof: Let there be a point t_1 in $\mathbb{R}_+$ such that $W(t_1) \neq 0.$ Assume that there are n constants $c_1, c_2, \ldots c_n$ such that $c_1 x_1(t) + c_2 x_2(t) + \ldots + c_n x_n(t) = 0, t \in \mathbb{R}_+.$ To show that $x_1(t), x_2(t), \ldots, x_n(t)$ are linearly independent, we must arrive at $c_1 = c_2 = \ldots c_n = 0.$ At $t = t_1$ in $\mathbb{R}_+$ we have

$$
\begin{aligned}
&c_1 x_1(t_1) + c_2 x_2(t_1) + \ldots + c_n x_n(t_1) = 0 \\
&c_1 {}^c D^q x_1(t_1) + c_2 {}^c D^q x_2(t_1) + \ldots + c_n {}^c D^q x_n(t_1) = 0 \\
&\quad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
&c_1 {}^c D^{(n-1)q} x_1(t_1) + c_2 {}^c D^{(n-1)q} x_2(t_1) + \ldots + c_n {}^c D^{(n-1)q} x_n(t_1) = 0
\end{aligned}
$$

These are n simultaneous homogeneous equations in $c_1, c_2, \ldots c_n$ as unknown coefficients. Observe that the determinant formed by the coefficients of the n equations, $W(t_1) \neq 0$, hence clearly, $c_1 = c_2 = \ldots c_n = 0.$ Therefore the solutions are linearly independent.

To obtain a sufficient condition assume that the solutions $x_1(t), x_2(t), \ldots, x_n(t)$ are linearly independent. We show that Wronskian $W(t) \neq 0.$

If possible suppose that $\mathrm{W}(\mathrm{t}) = 0$ for some $t \in \mathbb{R}_+.$ Then

$$
\begin{vmatrix}
x_1(\mathrm{t}) & x_2(t) & \cdots & x_n(t) \\
{}^c D^q x_1(t) & {}^c D^q x_2(t) & \cdots & {}^c D^q x_n(t) \\
\vdots & \vdots & \vdots & \vdots \\
{}^c D^{(n-1)q} x_1(t) & {}^c D^{(n-1)q} x_2(t) & \cdots & {}^c D^{(n-1)q} x_n(t)
\end{vmatrix} = 0.
$$

This implies that there exists a linear combination of columns as

$$
\mathrm{c}_1 \begin{bmatrix} x_1(t) \\ {}^c D^q x_1(t) \\ \vdots \\ {}^c D^{(n-1)q} x_1(t) \end{bmatrix} + c_2 \begin{bmatrix} x_2(t) \\ {}^c D^q x_2(t) \\ \vdots \\ {}^c D^{(n-1)q} x_2(t) \end{bmatrix} + \cdots + c_n \begin{bmatrix} x_n(t) \\ {}^c D^q x_n(t) \\ \vdots \\ {}^c D^{(n-1)q} x_n(t) \end{bmatrix} = 0
$$

where $c_1, c_2, \ldots, c_n$ are not simultaneously zero. If $c_1 \neq 0$ then $x_1(t) = -\frac{c_2}{c_1} x_2(t) - \frac{c_3}{c_1} x_3(t) - \cdots - \frac{c_n}{c_1} x_n(t).$

This implies that $x_1(t), x_2(t), \ldots, x_n(t)$ are linearly dependent solutions,

which is a contradiction to the assumption that these solutions are linearly independent.

Hence $W(t) \neq 0.$

The proof is complete.

5. CONCLUSION

In this paper we have obtained the solutions of the 3^{rd} order CFDE in the form fractional hyperbolic like functions and their properties are studied. Further solutions of n^{th} order CFDE are also obtained.

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