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ON AN UNDIRECTED GRAPH STRUCTURE OF A COMMUTATIVE RING

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ABSTRACT

Let R be a commutative ring with unity and Z(R) be the set of all zero-divisors of R. For $x \in Z(R)$, the annihilator of x is the set $ann_R(x) = \{y \in R | yx = 0\}$. The new annihilator graph of R, denoted by $ANN_G(R)$, is the undirected graph whose set of vertices is $Z(R)^* = Z(R) - \{0\}$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cap ann_R(y)$. In this paper, we investigate the relationship among the new annihilator graph $ANN_G(R)$, the annihilator graph AG(R) and the zero-divisor graph Γ(R).

Keywords: *Annihilator graph, New Annihilator graph, Commutative ring, Zero-divisor graph.*

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1. INTRODUCTION

Let R be a commutative ring with unity and $Z(R)$ be the set of all zero-divisors of R. For every $X \subseteq R$, we denote $X - 0\}$ by X^*. The concept of a zero-divisor graph of a commutative ring R was first introduced by I. Beck in [5], where all the elements of the ring R were taken as the vertices of the graph. D. F. Anderson and P. S. Livingston [1] modified the concept and defined the zero-divisor graph $\Gamma(R)$, as the undirected graph whose vertex set is $Z(R)^*$ and two distinct vertices x and y are adjacent if and only if $xy = 0$. For $x \in Z(R)$, the annihilator of x is the set $ann_R(x) = \{y \in R | yx = 0\}$. A. Badawi [4] defined the annihilator graph $AG(R)$, as the undirected graph whose vertex set is $Z(R)^*$ and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cup ann_R(y)$. A new annihilator graph of R, denoted by $ANN_G(R)$, is defined by P. P. Baruah and K. Patra [10], as the undirected graph whose set of vertices is $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if $ann_R(xy) \neq ann_R(x) \cap ann_R(y)$. In this paper, we investigate the relationship among the graphs $ANN_G(R)$, $AG(R)$ and $\Gamma(R)$. In [1], it was shown that $\Gamma(R)$ is connected with $diam(\Gamma(R)) \leq 3$ If $\Gamma(R)$ contains a cycle it was shown that $gr(\Gamma(R)) \leq 4$ in [9] and a simple proof is given in [3]. Thus $diam(\Gamma(R)) \in \{0, 1, 2, 3\}$ and $gr(\Gamma(R)) \in \{3, 4, \infty\}$. In [4], it was shown that $diam(AG(R)) \in \{0, 1, 2\}$ and $gr(AG(R)) \in \{3, 4, \infty\}$. In [10], it was shown that $diam(ANN_G(R)) \in \{0, 1, 2\}$ and $gr(ANN_G(R)) \in \{3, 4, \infty\}$.

Now we state some definitions and notations used throughout this paper. Let G be an undirected graph. We say that G is *connected* if there exists a path between any two distinct vertices. The *distance* between two vertices x and y of G, denoted by $d(x, y)$, is the length of a shortest path connecting them ($d(x, x) = 0$ and if such a path does not exist, then $d(x, y) = \infty$). The *diameter* of G is $diam(G) = sup\{d(x, y) \mid x$ and y are vertices of $G\}$. The *girth* of G, denoted by $gr(G)$, is the length of a shortest cycle in G (if G contains no cycle, then $gr(G) = \infty$). We denote by C^n the graph consisting of a *cycle* with n vertices. A graph G is *complete* if any two distinct vertices are adjacent. The complete graph with n vertices will be denoted by K^n (we allow n to be an infinite cardinal). A *complete bipartite* graph is a graph G which may be partitioned into two disjoint nonempty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. If one of the vertex set is singleton, we call G is a *star graph*. We denote the complete bipartite graph by $K^{m,n}$, where $|A| = m$ and $|B| = n$ (we allow m and n to be an infinite cardinal); hence a star graph is a $K^{1,n}$.

Throughout this paper, R is a commutative ring with unity, $Z(R)$ is the set of all zero-divisors of R, $N(R)$ is the set of all nilpotent elements of R, $U(R)$ is the group of units of R. For any two graphs G and H, if G is identical to H, then we write $G = H$; otherwise, we write $G \neq H$. The distance between two distinct vertices x and y of the zero-divisor graph $\Gamma(R)$ will be denoted by $d_{\Gamma(R)}(x, y)$. Any undefined terminology is as standard as in [6] or [7].

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2. MAIN RESULTS

This section provides the study of some basic properties of $ANN_G(R)$. If $|Z(R)^*| = 1$ for a commutative ring R, then R is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/\langle X^2\rangle$. In this case $ANN_G(R) = AG(R) = \Gamma(R)$. Hence throughout this paper, we consider commutative rings with $|Z(R)^*| \geq 2$.

Theorem 2.1: Let R be a commutative ring. Suppose that $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$. If $d_{\Gamma(R)}(x, y) = 3$, then $ANN_G(R)$ contains a cycle of length 3 and $gr(ANN_G(R)) = 3$.

Proof: Suppose that $x - y$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$ for some distinct $x, y \in Z(R)^*$. Suppose that $d_{\Gamma(R)}(x, y) = 3$. So assume $x - a - b - y$ is a shortest path connecting x and y in $\Gamma(R)$, where $a, b \in Z(R)^*$ and $a \neq b$. This implies $xa = 0$, $ab = 0$, $by = 0$, $xb \neq 0$ and $ay \neq 0$. This implies $y \in ann_R(xb)$. Since $y \notin ann_R(x)$, we have $ann_R(xb) \neq ann_R(x)$. Thus $x — b$ is an edge of $ANN_G(R)$ by [Lemma 2.1(1), 10].We have $x — a — b$ is a path in $ANN_G(R)$ by [Lemma 2.1 (2), 10]. Thus $x — a — b — x$ is a cycle of length 3 in $ANN_G(R)$, and hence $gr(ANN_G(R)) = 3$.

Theorem 2.2: Let R be a commutative ring and suppose that $ANN_G(R) \neq \Gamma(R)$. Then $gr(ANN_G(R)) = 3$.

Proof: Since $ANN_G(R) \neq \Gamma(R)$, there are some distinct $x, y \in Z(R)^*$ such that $x — y$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$. Since $\Gamma(R)$ is connected, we have $|Z(R)^*| \geq 3$. Again, since $diam(\Gamma(R)) \in \{0, 1, 2, 3\}$, we have $d_{\Gamma(R)}(x, y) \in \{2, 3\}$.

Case-1: Let $d_{\Gamma(R)}(x, y) = 2$. So assume $x — a — y$ is a shortest path connecting x and y in $\Gamma(R)$. Then $x — a — y$ is a path of length 2 from x to y in $ANN_G(R)$ by [Lemma 2.1(2), 10]. Since $x — y$ is an edge of $ANN_G(R)$, we have $ANN_G(R)$ contains a cycle of length 3. Hence $gr(ANN_G(R)) = 3$.

Case-2: Let $d_{\Gamma(R)}(x, y) = 3$. Then $gr(ANN_G(R)) = 3$ by Theorem 2.1.

Thus combining both the cases, we have $gr(ANN_G(R)) = 3$.

Theorem 2.3: Let R be a non-reduced commutative ring with $|N(R)^*| \geq 2$ and suppose that $ANN_{NG}(R)$ is the (induced) subgraph of $ANN_G(R)$ with vertices $N(R)^*$. Then $ANN_{NG}(R)$ is complete.

Proof: Suppose that x and y are two distinct elements of $N(R)^*$ such that $xy \neq 0$. Assume that $x - y$ is not an edge of $ANN_{NG}(R)$. Then $ann_R(xy) = ann_R(x) \cap ann_R(y)$ by [Lemma 2.1(1), 10]. Hence we have
$ann_R(x) = ann_R(xy) = ann_R(y)$.

Let n be the least positive integer such that $y^n = 0$. Suppose that $xy^m \neq 0$ for each m, $1 \leq m < n$. Then $y^{n-1} \in ann_R(xy) - ann_R(x)$, which is a contradiction. So assume that m, $1 \leq m < n$ is the least positive integer such that $xy^m = 0$. Since $xy \neq 0$, we have $1 < m < n$. Hence $y^{m-1} \in ann_R(xy) - ann_R(x)$, which is a contradiction. Thus $x — y$ is an edge of $ANN_{NG}(R)$.

Example 2.1: Consider the non-reduced commutative ring $R = \mathbb{Z}_2 \times \mathbb{Z}_8$.. Then $N(R) = \{(0, 0), (0, 2), (0, 4), (0, 6)\}$. Then $ANN_{NG}(R) = K^3$ and hence $ANN_{NG}(R)$ is complete.

Theorem 2.4: Let R be a non-reduced commutative ring, and suppose that $N(R)^2 \neq \{0\}$. Then $ANN_G(R) \neq \Gamma(R)$ and $gr(ANN_G(R)) = 3$.

Proof: Since $N(R)^2 \neq \{0\}$, we have $ANN_G(R) \neq \Gamma(R)$ by [Theorem 3.13, 4] and Theorem 2.3. Hence $gr(ANN_G(R)) = 3$ by Theorem 2.2.

Theorem 2.5: Let R be a non-reduced commutative ring such that $Z(R)$ is not an ideal of R. Then $ANN_G(R) \neq \Gamma(R)$ and $gr(ANN_G(R)) = 3$.

Proof: Since $Z(R)$ is not an ideal of R, we have $diam(\Gamma(R)) = 3$ by [Corollary 2.5, 8]. Thus $ANN_G(R) \neq \Gamma(R)$ by [Theorem 2.1, 10]. Hence $gr(ANN_G(R)) = 3$ by Theorem 2.2..

Now we observe the following Example 2.2 and then we have the Theorem 2.6.

Example 2.2:

(1) Consider the non-reduced commutative ring $R = \mathbb{Z}_9$. Then $ANN_G(R) = K^{1,1}$ and hence $gr(ANN_G(R)) = \infty$.

(2) Consider the non-reduced commutative ring $R = \mathbb{Z}_2[X] / \langle X^3 \rangle$. Then $ANN_G(R) = K^3$ and hence $gr(ANN_G(R)) = 3$.

Theorem 2.6: Let R be a non-reduced commutative ring with $|Z(R)^*| \geq 2$. Then $gr(ANN_G(R)) \in \{3, \infty\}$.

Proof: We have $gr(ANN_G(R)) \in \{3, 4, \infty\}$ by [Corollary 2.4.1, 10]. We have to show that $gr(ANN_G(R)) \neq 4$. If possible suppose that $gr(ANN_G(R)) = 4$. Then we have $ANN_G(R) = AG(R)$ and $gr(AG(R)) = 4$ by [Corollary 2.3.2, 10]. Since $gr(AG(R)) = 4$, we have $AG(R) \neq \Gamma(R)$ by [Theorem 3.16, 4]. Thus $ANN_G(R) \neq \Gamma(R)$ and hence $gr(ANN_G(R)) = 3$ by Theorem 2.2, a contradiction. Hence $gr(ANN_G(R)) \in \{3, \infty\}$.

Remark 2.1: For a non-reduced commutative ring R, if $ANN_G(R)$ contains a cycle then $gr(ANN_G(R)) = 3$ by Theorem 2.6.

Theorem 2.7: Let R be a commutative ring such that $ANN_G(R) \neq \Gamma(R)$. Then the following statements are equivalent:

(1) $\Gamma(R)$ is a star graph;

(2) $\Gamma(R) = K^{1,2}$;

(3) $ANN_G(R) = K^3$.

Proof:

(1) ⇒ (2): Suppose that $\Gamma(R)$ is a star graph. Then $gr(\Gamma(R)) = \infty$. Since $ANN_G(R) \neq \Gamma(R)$, we have R is non-reduced by [Theorem 3.7, 10] and $|Z(R)^*| \geq 3$. Since $\Gamma(R)$ is a star graph, there are two nonempty sets U and V such that $Z(R)^* = U \cup V$ with $|U| = 1$, $U \cap V = \emptyset$, $UV = \{0\}$ and $v_1 v_2 \neq 0$ for every $v_1, v_2 \in V$. We assume $U = \{u\}$ for some $u \in Z(R)^*$. Since $ANN_G(R) \neq \Gamma(R)$, there are some $v, w \in V$ such that $v — w$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$. Since $ann_R(v) = \{0, u\}$ for each $v \in V$ and $ann_R(vw) \neq ann_R(v) \cap ann_R(w)$, we have $ann_R(vw) \neq \{0, u\}$. Thus $ann_R(vw) = \{0\} \cup V$ and $vw = u$. Since $U = \{vw\}$ and $UV = \{0\}$, we have $v(vw) = v^2 w = 0$ and $w(vw) = w^2 v = 0$. We need to show that $V = \{v, w\}$. Suppose that there is a $z \in V$ such that $z \notin \{v, w\}$. Then $uz = vwz = 0$. Assume that $(vz + vw) = v$ .Then $w(vz + vw)) = wv$. But $w(vz + vw) = wvz + w^2 v = 0 + 0 = 0$. Thus we have $wv = 0$, a contradiction. Thus $(vz + vw) \neq v$. Since $v, z \in V$, we have $vz \neq 0$ and thus $(vz + vw) \neq vw$. Thus $v, (vz + vw), vw$ are distinct elements of $Z(R)^*$. Since $v^2 w = 0$ and $w \in V$, we have either $v^2 = 0$ or $v^2 = vw$ or $v^2 = w$. Suppose that $v^2 = w$. Since $vw = u \neq 0$, we have $vw = v(v^2) = v^3 = u \neq 0$. Since $v^2 w = 0$, we have $v^4 = v^2 w = 0$. Thus we have $v^2, v^3, v^2 + v^3$ are distinct elements of $Z(R)^*$, and hence $v^2 — v^3 — (v^2 + v^3) — v^2$ is a cycle of length 3 in $\Gamma(R)$, a contradiction. Thus we assume either $v^2 = 0$ or $v^2 = u$. In both the cases, we have $v^2 z = 0$. Since v, $(vz + vw)$, vw are distinct elements of $Z(R)^*$ and $v^2 w = w^2 v = v^2 z = 0$, we have $v — (vz + vw) — vw — v$ is a cycle of length 3 in $\Gamma(R)$, a contradiction. Thus we have $V = \{v, w\}$ and hence $|V| = 2$. Therefore $\Gamma(R) = K^{1,2}$.

(2) ⇒ (3): Since $ANN_G(R) \neq \Gamma(R)$ and $\Gamma(R) = K^{1,2}$, we conclude that $ANN_G(R) = K^3$.

(3) ⇒ (1): Since $ANN_G(R) = K^3$, we have $|Z(R)^*| = 3$. Since $\Gamma(R)$ is connected and $ANN_G(R) \neq \Gamma(R)$, we have exactly one edge of $ANN_G(R)$ is not an edge of $\Gamma(R)$. Thus $\Gamma(R)$ is a star graph.

Example 2.3: Consider the non-reduced commutative ring $R = \mathbb{Z}_2[X] / \langle X^3 \rangle$. Then $X + \langle X^3 \rangle — X + X^2 + \langle X^3 \rangle$ is an edge of $ANN_G(R)$ that is not an edge of $\Gamma(R)$. Now $X + \langle X^3 \rangle — X^2 + \langle X^3 \rangle — X + X^2 + \langle X^3 \rangle$ is the only path in $ANN_G(R)$ of length 2 from $X + \langle X^3 \rangle$ to $X + X^2 + \langle X^3 \rangle$ and it is also a path in $\Gamma(R)$. Here $ANN_G(R) = K^3$, $\Gamma(R) = K^{1,2}$, $gr(\Gamma(R)) = \infty$ and $gr(ANN_G(R)) = 3$.

Theorem 2.8: Let R be a non-reduced commutative ring with $|Z(R)^*| \geq 2$. Then the following statements are equivalent:

(1) $ANN_G(R)$ is a star graph;

(2) $gr(ANN_G(R)) = \infty$;

(3) $ANN_G(R) = \Gamma(R)$ and $gr(\Gamma(R)) = \infty$;

(4) $ANN_G(R) = AG(R)$ and $gr(AG(R)) = \infty$;

(5) $gr(AG(R)) = \infty$;

(6) $N(R)$ is a prime ideal of R and either $Z(R) = N(R) = \{0, -w, w\}$ $(-w \neq w)$ for some nonzero $w \in R$ or $Z(R) \neq N(R)$ and $N(R) = \{0, w\}$ for some nonzero $w \in R$ (and hence $wZ(R) = \{0\}$);

(7) Either $ANN_G(R) = K^{1,1}$ or $ANN_G(R) = K^{1,\infty}$;

(8) Either $AG(R) = K^{1,1}$ or $AG(R) = K^{1,\infty}$;

(9) Either $\Gamma(R) = K^{1,1}$ or $\Gamma(R) = K^{1,\infty}$.

Proof:

(1) ⇒ (2): Since $ANN_G(R)$ is a star graph, we have $gr(ANN_G(R)) = \infty$.

(2) ⇒ (3): Since $gr(ANN_G(R)) = \infty$, we have $ANN_G(R) = \Gamma(R)$ by Theorem 2.2 and hence $gr(\Gamma(R)) = \infty$.

(2) ⇒ (4): Since $gr(ANN_G(R)) = \infty$, we have $ANN_G(R) = AG(R)$ by [Corollary 2.3.2, 10] and hence $gr(AG(R)) = \infty$.

(3) ⇒ (4): Since $ANN_G(R) = \Gamma(R)$ and $gr(\Gamma(R)) = \infty$, we have $ANN_G(R) = AG(R)$ by [Theorem 3.6, 4] and hence $gr(AG(R)) = \infty$.

(4) ⇒ (5): It is obvious.

(5) ⇔ (6): It follows from [Theorem 3.18, 4]

(6) ⇒ (7): First suppose that $N(R)$ is a prime ideal of R and $Z(R) = N(R) = \{0, -w, w\}$ $(-w \neq w)$ for some nonzero $w \in R$. Since $ANN_G(R)$ is connected, we have $ANN_G(R) = K^{1,1}$. Next assume that $N(R)$ is a prime ideal of R with $Z(R) \neq N(R)$ and $N(R) = \{0, w\}$ for some nonzero $w \in R$. We need to show $Z(R)$ is an infinite set. Let $u \in Z(R) - N(R)$ and assume $r > s \geq 1$. To show $Z(R)$ is infinite we have to prove $u^s \neq u^r$. Suppose that $u^s = u^r$. This implies $u^s(1 - u^{r-s}) = 0$. Since $N(R) = \{0, w\}$ is prime ideal, we have $(1 - u^{r-s}) = w$. Now $1 - w \in U(R)$. This implies $u^{r-s} \in U(R)$, which is a contradiction. Thus $u^s \neq u^r$ and hence $Z(R)$ is an infinite set. Again since $N(R) = \{0, w\}$ is prime and $wZ(R) = \{0\}$, we conclude that $ANN_G(R) = K^{1,\infty}$. Thus we have either $ANN_G(R) = K^{1,1}$ or $ANN_G(R) = K^{1,\infty}$.

(6) ⇔ (8): It follows from [Theorem 3.18, 4]

(7) ⇒ (8): We have $AG(R)$ is connected by [Theorem 2.1, 4]. Also every edge of $AG(R)$ is an edge of $ANN_G(R)$ by [Lemma 2.1(5), 10]. Thus we have either $AG(R) = K^{1,1}$ or $AG(R) = K^{1,\infty}$.

(8) ⇔ (9): It follows from [Theorem 3.18, 4]

(9) ⇒ (1): In both the cases of (9) we have $\Gamma(R)$ is a star graph and $\Gamma(R) \neq K^{1,2}$. Thus $ANN_G(R) = \Gamma(R)$ by Theorem 2.7 and hence $ANN_G(R)$ is a star graph.

Example 2.4:

(1) Let $R = \mathbb{Z}_9$. Then $Z(R) = N(R) = \{0, -3, 3\}$ and $ANN_G(R) = AG(R) = \Gamma(R) = K^{1,1}$.

(2) Let $R = \mathbb{Z}_2[X, Y] / \langle X^2, XY \rangle$ and suppose that $u = X + \langle X^2 + XY \rangle$ and $v = Y + \langle X^2 + XY \rangle$ belongs to R. Then $Z(R) = (u, v)R$, $N(R) = \{0, u\}$ and $Z(R) \neq N(R)$. Thus we have $ANN_G(R) = AG(R) = \Gamma(R) = K^{1,\infty}$.

3. CONCLUSION

Let R be a commutative ring with unity. In this paper, we have discussed some basic properties of $ANN_G(R)$. We have also examined some properties of when R is non-reduced commutative ring R.

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