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# A COMMON FIXED POINT THEOREM FOR SIX MAPPINGS

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*ABSTRACT*

*Using notion of compatibility, weak compatibility and commutatively we have generalized fixed point theorem for six mappings satisfying rational inequality.*

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## 1. INTRODUCTION AND PRELIMINARIES

The concept of common fixed point theorem for commuting mapping was given by Jungck [4]. The notion of weak commutativity was introduced by Sessa [6]. Imdad and Khan [5] has proved a common fixed point theorem for six mappings which was extension of Fisher [1] and Jeong- Rhoades [3].

**Definition: 1.1 [6]:** A pair of self-mapping *(A, B)* on a metric space *(X, d)* is said to be weakly commuting if $d(ABx, BAx) \leq d(Bx, Ax)$ for all $x$ in X. Obviously, commuting mappings are weakly commuting but the converse is not necessarily true.

**Definition 1.2[4]:** A pair of self mappings *(A, B)* of a metric space *(X, d)* is said to be compatible if $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t \in X$ . Obviously, weakly commuting mappings are compatible but the converse is not necessarily true.

The following theorem is given by Fisher [1]

**Theorem 1.1:** Let S and T be tow self mappings of a complete metric space *(X, d)* such that for all *x, y* in *X* either

(a) $d(Sx,Ty) \leq \dfrac{b[d(x,Ty)]^2 + c[d(y,Sx)]^2}{d(x,Ty)+d(y,Sx)}$

If $d(x,Ty)+d(y,Sx) \neq 0$, $0 \leq b, c, b + c < 1$ or

(b) $d(Sx,Ty) = 0 \; if \; d(x,Ty)+d(y,Sx) = 0$

If one of S or T is continuous than S and T have a unique common fixed point.

Motivated by Fisher [2] and imdad and khan [5], in the present paper, an extension of theorem 1.1 is generalized for power n by improving the contraction condition and choosing suitable weak commutativity conditions.

## 2 MAIN RESULT

We prove the following

**Theorem 2.1:** Let A, B; S, T, I and J be self mappings of a complete metric space (X, d) satisfying AB(X) ⊂ J(X), ST(X) ⊂ I(X) and for each *x*, y Є X either

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$$d(ABx, STy) \le \alpha_1 \left[ \frac{[d(ABx, Jy)]^n + [d(STy, Ix)]^n}{[d(ABx, Jy)]^{n-1} + [d(STy, Ix)]^{n-1}} \right] + \alpha_2 [d(ABx, Ix) + d(STy, Jy)] + \alpha_3 d(Ix, Jy) \tag{2.1}$$

$if d(ABx, Jy)^{n-1} + d(STy, Ix)^{n-1} \neq 0$ , $\alpha_i \geq, 0$ (i = 1, 2, 3) with atleast one $\alpha_i$ non zero and

$$2\alpha_1 + 2\alpha_2 + \alpha_3 < 1 \text{ or } d(ABx, STy) = 0 if [d(ABx, Jy)]^{n-1} + [d(STy, Ix)]^{n-1} = 0 \tag{2.2}$$

If either

(i) {AB, I} are compatible, I or AB is continuous and (ST, J) are weakly compatible,
or
(ii) {ST, J} are compatible, J or ST is continuous and (AB, I) are weakly compatible then AB, ST, I and J have a unique common fixed point. Furthermore if the pairs (A, B), (A, I), (B, I), (S, T), (S, J), (T, J) are commuting mappings then A, B, S, T, I and J have a unique common fixed point.''

**Proof:** Let $x_0$ be an arbitrary point in X. Since AB(X) $\subset$ J(X) we can find a point $x_1$ in X such that AB $x_0$ = J$x_1$. Also since ST(X) subset of I(X) we can choose a point $x_2$ with ST$x_1$ = I $x_2$. Using this argument repeatedly one can construct a sequence $\{z_n\}$ such that $z_{2n} = ABx_{2n} = J x_{2n+1}$, $z_{2n+1} = STx_{2n+1} = I x_{2n+2}$ for n = 0, 1, 2…..

For brevity let us put

$u_{2n} = d(ABx_{2n}, STx_{2n+1})$ and $u_{2n+1} = d(STx_{2n+1}, ABx_{2n+2})$ for n = 0, 1, 2…… Now we distinguish two cases:

(i) Suppose that $u_{2n} + u_{2n+1} \neq 0$ for n = 0, 1, 2…..

Then using the inequality (2.1.1), we have

$$u_{2n+1} = d(z_{2n+1}, z_{2n+2}) = d(STx_{2n+1}, ABx_{2n+2})$$

$$\le \alpha_1 \left[ \frac{[d(ABx_{2n+1}, Jx_{2n+1})]^n + [d(STx_{2n+1}, Ix_{2n+1})]^n}{[d(ABx_{2n+2}, Jx_{2n+1})]^{n-1} + [d(STx_{2n+1}, Ix_{2n+2})]^{n-1}} \right] + \alpha_2 [d(ABx_{2n+1}, Ix_{2n+2}) + d(STx_{2n+1}, Jy_{2n+2})] + \alpha_3 d(Ix_{2n+2}, Jy_{2n+1})$$

or

$$d(z_{2n+1}, z_{2n+2}) \le \frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_1 - \alpha_2} d(z_{2n}, z_{2n+1})$$

Similarly we can show that

$$d(z_{2n}, z_{2n+1}) \le \frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_1 - \alpha_2} d(z_{2n-1}, z_{2n})$$

thus for every $n$ we have $d(z_n, z_{n+1}) \le kd(z_{n-1}, z_n)$ (2.3)

which shows that $\{z_n\}$ is a cauchy sequence in the complete metric space (X, d) and so has a limit point $z$ in X. hence the sequence $ABx_{2n} = Jx_{2n+1}$ and $STx_{2n+1} = Ix_{2n+2}$ which are subsequences also converge to the point $z$.

let us now assume that $I$ is continuous so that the sequence $\{I^2 x_{2n}\}$ and $\{IABx_{2n}\}$ converges to $Iz$. Also in view of compatibility of $\{I, AB\}$, $\{ABIx_{2n}\}$ converges to $Iz$.

$$d\left(ABIx_{2n}, STx_{2n+1}\right) \le \alpha_1 \left[ \frac{\left[d\left(ABIx_{2n}, Jx_{2n+1}\right)\right]^n + \left[d\left(STx_{2n+1}, I^2 x_{2n}\right)\right]^n}{\left[d\left(ABIx_{2n}, Jx_{2n+1}\right)\right]^{n-1} + \left[d\left(STx_{2n+1}, I^2 x_{2n}\right)\right]^{n-1}} \right. \\ \left. + \alpha_2 \left[d\left(ABz, Iz\right) + d\left(STx_{2n+1}, Jx_{2n+1}\right) + \alpha_3 \left[d\left(Iz, Jx_{n+1}\right)\right]\right] \right]$$

Which on letting $n \to \infty$ reduces to $\left(1-\alpha_1-\alpha_3\right) d\left(Iz, z\right) \le 0$

yielding thereby $Iz = z$

Now,

$$d\left(ABz, STx_{2n+1}\right) \le \alpha_1 \left[ \frac{\left[d\left(ABz, Jx_{2n+1}\right)\right]^n + \left[d\left(STx_{2n+1}, Iz\right)\right]^n}{\left[d\left(ABz, Jx_{2n+1}\right)\right]^{n-1} + \left[d\left(STx_{2n+1}, Iz\right)\right]^{n-1}} \right] \\ + \alpha_2 \left[d\left(ABz, Iz\right) + d\left(STx_{2n+1}, Jx_{2n+1}\right)\right] + \alpha_3 \left[d\left(Iz, Jx_{n+1}\right)\right]$$

On letting and using $Iz = z$ we get

$$d\left(ABz, z\right) \le \left(\alpha_1 + \alpha_2\right) d\left(ABz, z\right)$$

This implies $AB\,z = z$

Since $AB\left(x\right) \subset J\left(x\right)$ then there always exists a point $z'$ such that $Jz' = z$ so that $STz = ST\left(Jz'\right)$

Now

$$\begin{aligned} d(z, STz) &= d(ABz, STz) \\ &\le \alpha_1 \left[ \frac{\left[d\left(ABz, Jz'\right)\right]^n + \left[d\left(STz', Iz\right)\right]^n}{\alpha\left(ABz, Jz'\right)^{n-1} + d\left(STz', Iz\right)^{n-1}} \right] + \alpha_2 \left[d\left(ABz, Iz\right) + d\left(STz', Jz'\right)\right] + \alpha_3\, d\left(Iz, Jz'\right) \\ &\le \left(\alpha_1 + \alpha_2\right)\left[d\left(STz', z\right)\right] \end{aligned}$$

Hence, $STz' = z = Jz'$ which shows that $z'$ is a common point of *AB, I, ST* and *J*. Now using the weak compatibility of $\left(ST, J\right)$, we have $STz = ST(Jz') = J(STz') = Jz$ which shows that $z$ is also a coincidence point of the pair (ST, J). Now

$$\begin{aligned} d\left(z, STz\right) &= d\left(ABz, STz\right) \\ &\le \alpha_1 \left[ \frac{\left[d\left(ABz, Jz\right)\right]^n + \left[d\left(STz, Iz\right)\right]^n}{\alpha\left(ABz, Jz\right)^{n-1} + d\left(STz, Iz\right)^{n-1}} \right] + \alpha_2 \left[d\left(ABz, Iz\right) + d\left(STz, Jz\right)\right] + \alpha_3 d\left(Iz, Jz\right) \\ &\le \left(\alpha_1 + \alpha_3\right) d\left(z, STz\right) \end{aligned}$$

Hence $z = STz = Jz$ which shows that $z$ is a common fined point of $AB, I, ST$ and $J$.

Now suppose that $AB$ is continuous so that the sequences $\left\{AB^2 x_{2n}\right\}$ and $\left\{ABIx_{2n}\right\}$ converge to $ABz$, since $\left(AB, I\right)$ are compatible it follows that $\left\{IABx_{2n}\right\}$ also converge to $ABz$, thus

$$d\left(AB^2x_{2n}, STx_{2n+1}\right) \le \alpha_1 \left[ \frac{\left[d\left(AB^2x_{2n}, Jx_{2n+1}\right)\right]^n + \left[d\left(STx_{2n+1}, IABx_{2n}\right)\right]^n}{\left[d\left(AB^2x_{2n}, Jx_{2n+1}\right)\right]^{n-1} + \left[d\left(STx_{2n+1}, IABx_{2n}\right)\right]^{n-1}} \right] + \alpha_2 \left[ d\left(AB^2x_{2n}, IABx_{2n}\right) + d\left(STx_{2n+1}, Jx_{2n+1}\right)\right] + \alpha_3\, d\left(IABx_{2n}, Jx_{2n+1}\right)$$

which on letting $n \to \infty$ reduces to

$$d\left(ABz, z\right) \le \left(\alpha_1 + \alpha_3\right) d\left(ABz, z\right)$$

which implies $ABz = z$ as earlier, there exists $z'$ is $X$ such that

$ABz = z = Jz'$ then

$$d\left(AB^2x_{2n}, STz'\right) \le \alpha_1 \left[ \frac{\left[d\left(AB^2x_{2n}, Jz'\right)\right]^n + \left[d\left(STz', IABx_{2n}\right)\right]^n}{\left[d\left(AB^2x_{2n}, Jz'\right)\right]^{n-1} + \left[d\left(STz', IABx_{2n}\right)\right]^{n-1}} \right] + \alpha_2 \left[ d\left(AB^2x_{2n}, IABx_{2n}\right) + d\left(STz', Jz'\right)\right] + \alpha_3\, d\left(IABx_{2n}, Jz'\right)$$

This on letting $n \to \infty$ reduces to

$$d\left(z, STz'\right) \le \left(\alpha_1 + \alpha_2\right) d\left(z, STz'\right)$$

This gives $STz' = z = Jz'$ thus $z'$ is a coincidence point of ($ST$, $J$). since, the pair ($ST$, $J$) is weakly compatible hence $STz = ST(Jz') = J(STz') = Jz$ which shows that $STz = Jz$ further,

$$d\left(ABx_{2n}, STz\right) \le \alpha_1 \left[ \frac{\left[d\left(ABx_{2n}, Jz\right)\right]^n + \left[d\left(STz, Ix_{2n}\right)\right]^n}{\left[d\left(ABx_{2n}, Jz\right)\right]^{n-1} + \left[d\left(STz, Ix_{2n}\right)\right]^{n-1}} \right] + \alpha_2 \left[ d\left(ABx_{2n}, Ix_{2n}\right) + d\left(STz, Jz\right)\right] + \alpha_3\, d\left(Ix_{2n}, Jz\right)$$

which on letting $n \to \infty$ reduces to

$$d\left(z, STz\right) \le \left(\alpha_1 + \alpha_3\right) d\left(z, STz\right)$$

$$d\left(STz, z\right) = 0$$

it follows that $STz = z = Jz$

The point $z$ therefore is in the range of $ST$ and since $ST\left(X\right) \subset I\left(X\right)$ there exist a point $z'$ in $X$ such that $Iz'' = z$ thus

$$d\left(ABz'', z\right) = d\left(ABz'', STz\right) \le \alpha_1 \frac{\left[d\left(ABz'', Jz\right)\right]^n + \left[d\left(STz, Iz''\right)\right]^n}{\left[d\left(ABz'', Jz\right)\right]^{n-1} + \left[d\left(STz, Iz''\right)\right]^{n-1}} + \alpha_2 \left[ d\left(ABz'', Iz''\right) + d\left(STz, Jz\right)\right] + \alpha_3\, d\left(Iz'', Jz\right)$$

Letting $n \to \infty$

$$d\left(ABz'' z\right) \le \left(\alpha_1 + \alpha_2\right) d\left(ABz'', z\right)$$

 

which shows that $ABz'' = z$

Also since $(AB, I)$ are compatible and hence using weakly commuting we obtain

$$\begin{aligned} d(ABz, Iz) &= d\left(AB(Iz''), I(ABz'')\right) \\ &\le d(Iz'', ABz'') = d(z, z) = 0 \end{aligned}$$

Therefore $ABz = Iz = z$

Thus we have proved that $z$ is common fixed point of $AB, ST, I$ and $J$.

If the mappings $ST$ or $J$ is continuous instead of $AB$ or $I$ then proof of $z$ is a common fixed point of $AB, ST, I$ and $J$ is similar.

Let $v$ be another fixed point of $I, J, AB$ and $ST$

then

$$\begin{aligned} d(z, v) &= d(ABz, STv) \\ &\le \alpha_1 \left[ \frac{\left[d(ABz, Jv)\right]^n + \left[d(STv, Iz)\right]^n}{\left[d(ABz, Jv)\right]^{n-1} + \left[d(STv, Iz)\right]^{n-1}} \right] \\ &+ \alpha_2 \left[ d(ABz, Iz) + d(STv, Jv) \right] \\ &+ \alpha_3\, d(Iz, Jv) \end{aligned}$$

$d(z, v) \le (\alpha_1 + \alpha_3)\, d(z, v)$ yielding thereby $z = v$

Finally we need to show that $z$ is also a common fixed point of $A, B, S, T, I$ and $J$. For this let $z$ be the unique common fixed point of both the pairs $(AB, I)$ and $(ST, J)$. Then

$$Az = A(ABz) = A(BAz) = AB(Az),\ Az = A(lz) = I(Az)$$

$$Bz = B(ABz) = B(A(Bz)) = BA(Bz) = AB(Bz). \quad Bz = B(lz) = I(Bz)$$

which shows that $Az$ and $Bz$ is a common fixed point of $(AB, I)$ yielding thereby $Az = z = Bz = Iz = ABz$ in the view of uniqueness of the common fixed point of the pair $(AB, I)$.

Similarly using the commutatively of $(S, T), (S\ J)$ and $(T, J)$ it can be shown that $Sz = z = Tz = Jz = STz$.

Now we need to show that $Az = Sz\,(Bz = Tz)$ also remains a common fixed point of both the pairs $(AB, I)$ and $(ST, J)$. For this

$$\begin{aligned} d(Az, Sz) &= d\left(A(BAz), S(TSz)\right) \\ &= d\left(AB(Az), ST(Sz)\right) \\ &\le \alpha_1 \left[ \frac{\left[d\left(AB(Az), J(Sz)\right)\right]^n + \left[d\left(ST(Sz), I(Az)\right)\right]^n}{\left[d\left(AB(Az), J(Sz)\right)\right]^{n-1} + \left[d\left(ST(Sz), I(Az)\right)\right]^{n-1}} \right] \end{aligned}$$

$$+\alpha_2\left[d\left(AB(Az),I(Az)\right)+d\left(ST(Sz),J(Sz)\right)\right]+\alpha_3 d\left(I(Az),J(Sz)\right)$$

implies that $d(Az,Sz)=0$ $\left(\text{as } d\left(AB(Az),J(Sz)+d\left(ST(Sz),I(Az)\right)=0\right)\right)$ yielding thereby $Az=Sz$.

Similarly it can be shown that $Bz=Tz$. Thus $z$ is the unique common fixed of $A,B,S,T,l$ and $J$.

(ii) Suppose that $d(ABx,Jy)+d(STy,lx)=0$ implies $d(ABx,STy)=0$. Then we argue as follows.

Suppose that there exists an $n$ such that $z_n=z_{n+1}$. Then, also $z_{n+1}=z_{n+2}$. Suppose not. Then from (2.3) we have $0<d(z_{n+1},z_{n+2})\le kd(z_{n+1},z_n)$ yielding thereby $z_{n+1}=z_{n+2}$. Thus $z_n=z_{n+k}$ for $k=1,2,....$ it then follows that there exist two point $w_1$ and $w_2$ such that $v_1=ABw_1=Jw_2$ and $v_2=STw_2=lw_1$. Since $d(ABw_1,Jw_2)+d(STw_2,Iw_1)=0$ from (2.2) $d(ABw_1,STw_2)=0$ .i.e. $v_1=ABw_1=STw_2=v_2$. Also note that $Iv_1=I(ABw_1)=AB(Iw_1)=ABv_1$. Similarly $STv_2=Jv_2$. Define $y_1=ABv_1,y_2=STv_2$. Since $d(ABv_1,Jv_2)+d(STv_2,Iv_1)=0$, it follow from (2.2) that $d(ABv_1,STv_2)=0$, i.e. $y_1=y_2$. Thus $ABv_1=Iv_1=STv_2=Jv_2$. But $v_1=v_2$. follows $AB,I,ST$ and $J$ have a common coincidence point. Define $w=ABv_1$, it then follows that $w$ is also a common coincidence point of $AB,I,ST$ and $J$. If $ABw\ne ABv_1=STv_1$, then $d(ABw,STv_1)>0$. But, since $d(ABw,Jv_1)+d(STv_1,lw)=0$, it follows from (2.2) that $d(ABw,STv_1)=0$, i.e. $ABw=STv_1$, a contradiction. Therefore $ABw=ABv_1=w$ and $w$ is a common fixed point of $AB,ST,I$ and $J$.

The rest of the proof is identical to the case (1), hence it is omitted. This completes the proof.

**Corollary 2.2:** Theorem 2.1 remains true if contraction conditions (2.1.1) and (2.1.2) are replaced by any of the following conditions:

(i) $Either\, d(ABz,STy)\le\alpha_1\left[\dfrac{\left[d(ABx,Jy)\right]^n+\left[d(STy,Ix)\right]^n}{\left[d(ABx,Jy)\right]^{n-1}+\left[d(STy,lx)\right]^{n-1}}\right]$

$$+\alpha_2\left[d(ABx,lx)+d(STy,Jy)\right]$$

$if\ d(ABx,Jy)+d(STy,lx)\ne 0,\qquad \alpha_1,\alpha_2>0,\ 2\alpha_1+2\alpha_2<1$ or (A)

$d(ABx,STy)=0\,if\ d(ABx,Jy)+d(STy,lx)=0$

(ii) $Either\, d(ABx,STy)\le\alpha_1\left[\dfrac{\left[d(ABx,Jy)\right]^n+\left[d(STy,Ix)\right]^n}{\left[d(ABx,Jy)\right]^{n-1}+\left[d(STy,lx)\right]^{n-1}}\right]+\alpha_3(lx,Jy)$

$if\ d(ABx,Jy)+d(STy,lx)\ne 0,\qquad \alpha_1,\alpha_3>0,\ 2\alpha_1+\alpha_3<1$ or (B)

$d(ABx,STy)=0\,if\ d(ABx,Jy)+d(STy,lx)=0$

(iii) $Either\, d(ABx,STy)\le\alpha_1\left[\dfrac{\left[d(ABx,Jy)\right]^n+\left[d(STy,Ix)\right]^n}{\left[d(ABx,Jy)\right]^{n-1}+\left[d(STy,lx)\right]^{n-1}}\right]$

$if\ d(ABx,Jy)+d(STy,lx)\ne 0,\qquad \alpha_1>0,\ \alpha_1<½$ or (C)

$d(ABx,STy)=0\,if\ d(ABx,Jy)+d(STy,lx)=0$

(iv) $d(ABx,STy)\le\alpha_1\left[d(ABx,Jy)+d(STy,Jy)\right]$

 

$$+\alpha_2\left[d\left(ABx,lx\right)+d\left(STy,Jy\right)\right]+\alpha_3\, d\left(lx,Jy\right) if\ \ 2\alpha_1+2\alpha_2+\alpha_3<1 \qquad \text{(D)}$$

(v) $d\left(ABx,STy\right)\le\alpha_1\left[d\left(ABx,Jy\right)+d\left(STy,lx\right)\right] if\ \alpha_1<\frac{1}{2}$ (E)

(vi) $d\left(ABx,STy\right)\le\alpha_2\left[d\left(ABx,lx\right)+d\left(STy,Jy\right)\right] if\ \alpha_2<½$ (F)

(vii) $d\left(ABx,STy\right)\le\alpha_3\, d\left(lx,Jy\right)\ if\ \ \alpha_3<1$ (G)

**Proof:**. Corollaries corresponding to the contraction conditions (A), (B) and (C) can be deduced directly from Theorem 2.1 by choosing $\alpha_3=0,\alpha_2=0$ and $\alpha_2=\alpha_3=0$, respectively. The corollary corresponding the contraction condition (D) also follows from Theorem 2.1 by noting that

$$\frac{\left[d\left(ABx,Jy\right)\right]^n+\left[d\left(STy,lx\right)\right]^n}{\left[d\left(ABx,Jy\right)\right]^{n-1}+\left[d\left(STy,lx\right)\right]^{n-1}}\le\frac{\left[d\left(ABx,Jy\right)+d\left(STy,lx\right)\right]^n}{\left[d\left(ABx,Jy\right)+d\left(STy,lx\right)\right]^{n-1}}$$
$$\le\left[d\left(ABx,Jy\right)+d\left(STy,lx\right)\right]$$

Finally one may note that the contraction conditions (E), (F) and (G) are special cases of the contraction condition (D).

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