



A COMMON FIXED POINT THEOREM FOR SIX MAPPINGS

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ABSTRACT

Using notion of compatibility, weak compatibility and commutatively we have generalized fixed point theorem for six mappings satisfying rational inequality.

1. INTRODUCTION AND PRELIMINARIES

The concept of common fixed point theorem for commuting mapping was given by Jungck [4]. The notion of weak commutativity was introduced by Sessa [6]. Imdad and Khan [5] has proved a common fixed point theorem for six mappings which was extension of Fisher [1] and Jeong- Rhoades [3].

Definition: 1.1 [6]: A pair of self-mapping (A, B) on a metric space (X, d) is said to be weakly commuting if $d(ABx, BAx) \leq d(Bx, Ax)$ for all x in X . Obviously, commuting mappings are weakly commuting but the converse is not necessarily true.

Definition 1.2[4]: A pair of self mappings (A, B) of a metric space (X, d) is said to be compatible if

$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = t \in X$. Obviously, weakly commuting mappings are compatible but the converse is not necessarily true.

The following theorem is given by Fisher [1]

Theorem 1.1: Let S and T be two self mappings of a complete metric space (X, d) such that for all x, y in X either

$$(a) \quad d(Sx, Ty) \leq \frac{b[d(x, Ty)]^2 + c[d(y, Sx)]^2}{d(x, Ty) + d(y, Sx)}$$

If $d(x, Ty) + d(y, Sx) \neq 0, 0 \leq b, c, b + c < 1$ or

$$(b) \quad d(Sx, Ty) = 0 \text{ if } d(x, Ty) + d(y, Sx) = 0$$

If one of S or T is continuous then S and T have a unique common fixed point.

Motivated by Fisher [2] and Imdad and Khan [5], in the present paper, an extension of theorem 1.1 is generalized for power n by improving the contraction condition and choosing suitable weak commutativity conditions.

2 MAIN RESULT

We prove the following

Theorem 2.1: Let A, B, S, T, I and J be self mappings of a complete metric space (X, d) satisfying $AB(X) \subset J(X)$, $ST(X) \subset I(X)$ and for each $x, y \in X$ either

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$$d(ABx, STy) \leq \alpha_1 \left[\frac{[d(ABx, Jy)]^n + [d(STy, Ix)]^n}{[d(ABx, Jy)]^{n-1} + [d(STy, Ix)]^{n-1}} \right] + \alpha_2 [d(ABx, Ix) + d(STy, Jy)] + \alpha_3 d(Ix, Jy) \quad (2.1)$$

$$\text{if } d(ABx, Jy)^{n-1} + d(STy, Ix)^{n-1} \neq 0, \alpha_i \geq 0 \quad (i = 1, 2, 3) \text{ with atleast one } \alpha_i \text{ non zero and} \\ 2\alpha_1 + 2\alpha_2 + \alpha_3 < 1 \text{ or } d(ABx, STy) = 0 \text{ if } [d(ABx, Jy)]^{n-1} + [d(STy, Ix)]^{n-1} = 0 \quad (2.2)$$

If either

(i) {AB, I} are compatible, I or AB is continuous and (ST, J) are weakly compatible,

or

(ii) {ST, J} are compatible, J or ST is continuous and (AB, I) are weakly compatible then AB, ST, I and J have a unique common fixed point. Furthermore if the pairs (A, B), (A, I), (B, I), (S, T), (S, J), (T, J) are commuting mappings then A, B, S, T, I and J have a unique common fixed point."

Proof: Let x_0 be an arbitrary point in X. Since $AB(X) \subset J(X)$ we can find a point x_1 in X such that $ABx_0 = Jx_1$. Also since $ST(X)$ subset of $I(X)$ we can choose a point x_2 with $STx_1 = Ix_2$. Using this argument repeatedly one can construct a sequence $\{z_n\}$ such that $z_{2n} = ABx_{2n} = Jx_{2n+1}$, $z_{2n+1} = STx_{2n+1} = Ix_{2n+2}$ for $n = 0, 1, 2, \dots$

For brevity let us put

$u_{2n} = d(ABx_{2n}, STx_{2n+1})$ and $u_{2n+1} = d(STx_{2n+1}, ABx_{2n+2})$ for $n = 0, 1, 2, \dots$. Now we distinguish two cases:

(i) Suppose that $u_{2n} + u_{2n+1} \neq 0$ for $n = 0, 1, 2, \dots$

Then using the inequality (2.1.1), we have

$$u_{2n+1} = d(z_{2n+1}, z_{2n+2}) = d(STx_{2n+1}, ABx_{2n+2}) \\ \leq \alpha_1 \left[\frac{[d(ABx_{2n+1}, Jx_{2n+1})]^n + [d(STx_{2n+1}, Ix_{2n+1})]^n}{[d(ABx_{2n+2}, Jx_{2n+1})]^{n-1} + [d(STx_{2n+1}, Ix_{2n+2})]^{n-1}} \right] \\ + \alpha_2 [d(ABx_{2n+1}, Ix_{2n+2}) + d(STx_{2n+1}, Jy_{2n+2})] + \alpha_3 d(Ix_{2n+2}, Jy_{2n+1}) \\ d(z_{2n+1}, z_{2n+2}) \leq \frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_1 - \alpha_2} d(z_{2n}, z_{2n+1})$$

or

Similarly we can show that

$$d(z_{2n}, z_{2n+1}) \leq \frac{\alpha_1 + \alpha_2 + \alpha_3}{1 - \alpha_1 - \alpha_2} d(z_{2n-1}, z_{2n})$$

$$\text{thus for every } n \text{ we have } d(z_n, z_{n+1}) \leq kd(z_{n-1}, z_n) \quad (2.3)$$

which shows that $\{z_n\}$ is a cauchy sequence in the complete metric space (X, d) and so has a limit point z in X. hence the sequence $ABx_{2n} = Jx_{2n+1}$ and $STx_{2n+1} = Ix_{2n+2}$ which are subsequences also converge to the point z .

let us now assume that I is continuous so that the sequence $\{I^2x_{2n}\}$ and $\{IABx_{2n}\}$ converges to Iz . Also in view of compatibility of $\{I, AB\}$, $\{ABIx_{2n}\}$ converges to Iz .

$$d(ABIx_{2n}, STx_{2n+1}) \leq \alpha_1 \left[\frac{[d(ABIx_{2n}, Jx_{2n+1})]^n + [d(STx_{2n+1}, I^2x_{2n})]^n}{[d(ABIx_{2n}, Jx_{2n+1})]^{n-1} + [d(STx_{2n+1}, I^2x_{2n})]^{n-1}} \right] \\ + \alpha_2 [d(ABz, Iz) + d(STx_{2n+1}, Jx_{2n+1})] + \alpha_3 [d(Iz, Jx_{n+1})]$$

Which on letting $n \rightarrow \infty$ reduces to $(1 - \alpha_1 - \alpha_3)d(Iz, z) \leq 0$

yielding thereby $Iz = z$

Now,

$$d(ABz, STx_{2n+1}) \leq \alpha_1 \left[\frac{[d(ABz, Jx_{2n+1})]^n + [d(STx_{2n+1}, Iz)]^n}{[d(ABz, Jx_{2n+1})]^{n-1} + [d(STx_{2n+1}, Iz)]^{n-1}} \right] \\ + \alpha_2 [d(ABz, Iz) + d(STx_{2n+1}, Jx_{2n+1})] + \alpha_3 [d(Iz, Jx_{n+1})]$$

On letting and using $Iz = z$ we get

$$d(ABz, z) \leq (\alpha_1 + \alpha_2)d(ABz, z)$$

This implies $ABz = z$

Since $AB(x) \subset J(x)$ then there always exists a point z' such that

$$Jz' = z \text{ so that } STz = ST(Jz')$$

Now

$$d(z, STz) = d(ABz, STz) \\ \leq \alpha_1 \left[\frac{[d(ABz, Jz')]^n + [d(STz', Iz)]^n}{\alpha(ABz, Jz')^{n-1} + d(STz', Iz)^{n-1}} \right] + \alpha_2 [d(ABz, Iz) + d(STz', Jz')] + \alpha_3 d(Iz, Jz') \\ \leq (\alpha_1 + \alpha_2)[d(STz', z)]$$

Hence, $STz' = z = Jz'$ which shows that z' is a common point of AB, I, ST and J . Now using the weak compatibility of (ST, J) , we have $STz = ST(Jz') = J(STz') = Jz$ which shows that z is also a coincidence point of the pair (ST, J) . Now

$$d(z, STz) = d(ABz, STz) \\ \leq \alpha_1 \left[\frac{[d(ABz, Jz)]^n + [d(STz, Iz)]^n}{\alpha(ABz, Jz)^{n-1} + d(STz, Iz)^{n-1}} \right] + \alpha_2 [d(ABz, Iz) + d(STz, Jz)] + \alpha_3 d(Iz, Jz) \\ \leq (\alpha_1 + \alpha_3)d(z, STz)$$

Hence $z = STz = Jz$ which shows that z is a common fixed point of AB, I, ST and J .

Now suppose that AB is continuous so that the sequences $\{AB^2x_{2n}\}$ and $\{ABIx_{2n}\}$ converge to ABz , since (AB, I) are compatible it follows that $\{IABx_{2n}\}$ also converge to ABz , thus

$$d\left(AB^2x_{2n}, STx_{2n+1}\right) \leq \alpha_1 \left[\frac{\left[d\left(AB^2x_{2n}, Jx_{2n+1}\right)\right]^n + \left[d\left(STx_{2n+1}, IABx_{2n}\right)\right]^n}{\left[d\left(AB^2x_{2n}, Jx_{2n+1}\right)\right]^{n-1} + \left[d\left(STx_{2n+1}, IABx_{2n}\right)\right]^{n-1}} \right] \\ + \alpha_2 \left[d\left(AB^2x_{2n}, IABx_{2n}\right) + d\left(STx_{2n+1}, Jx_{2n+1}\right) \right] + \alpha_3 d\left(IABx_{2n}, Jx_{2n+1}\right)$$

which on letting $n \rightarrow \infty$ reduces to

$$d\left(ABz, z\right) \leq (\alpha_1 + \alpha_3) d\left(ABz, z\right)$$

which implies $ABz = z$ as earlier, there exists z' in X such that

$$ABz = z = Jz' \text{ then}$$

$$d\left(AB^2x_{2n}, STz'\right) \leq \alpha_1 \left[\frac{\left[d\left(AB^2x_{2n}, Jz'\right)\right]^n + \left[d\left(STz', IABx_{2n}\right)\right]^n}{\left[d\left(AB^2x_{2n}, Jz'\right)\right]^{n-1} + \left[d\left(STz', IABx_{2n}\right)\right]^{n-1}} \right] \\ + \alpha_2 \left[d\left(AB^2x_{2n}, IABx_{2n}\right) + d\left(STz', Jz'\right) \right] + \alpha_3 d\left(IABx_{2n}, Jz'\right)$$

This on letting $n \rightarrow \infty$ reduces to

$$d\left(z, STz'\right) \leq (\alpha_1 + \alpha_2) d\left(z, STz'\right)$$

This gives $STz' = z = Jz'$ thus z' is a coincidence point of (ST, J) . since, the pair (ST, J) is weakly compatible hence $STz = ST(Jz') = J(STz') = Jz$ which shows that $STz = Jz$ further,

$$d\left(ABx_{2n}, STz\right) \leq \alpha_1 \left[\frac{\left[d\left(ABx_{2n}, Jz\right)\right]^n + \left[d\left(STz, Ix_{2n}\right)\right]^n}{\left[d\left(ABx_{2n}, Jz\right)\right]^{n-1} + \left[d\left(STz, Ix_{2n}\right)\right]^{n-1}} \right] \\ + \alpha_2 \left[d\left(ABx_{2n}, Ix_{2n}\right) + d\left(STz, Jz\right) \right] + \alpha_3 d\left(Ix_{2n}, Jz\right)$$

which on letting $n \rightarrow \infty$ reduces to

$$d\left(z, STz\right) \leq (\alpha_1 + \alpha_3) d\left(z, STz\right) \\ d\left(STz, z\right) = 0$$

$$\text{it follows that } STz = z = Jz$$

The point z therefore is in the range of ST and since $ST(X) \subset I(X)$ there exist a point z'' in X such that $Iz'' = z$ thus

$$d\left(ABz'', z\right) = d\left(ABz'', STz\right) \leq \alpha_1 \frac{\left[d\left(ABz'', Jz\right)\right]^n + \left[d\left(STz, Iz''\right)\right]^n}{\left[d\left(ABz'', Jz\right)\right]^{n-1} + \left[d\left(STz, Iz''\right)\right]^{n-1}} \\ + \alpha_2 \left[d\left(ABz'', Iz''\right) + d\left(STz, Jz\right) \right] + \alpha_3 d\left(Iz'', Jz\right)$$

Letting $n \rightarrow \infty$

$$d\left(ABz'', z\right) \leq (\alpha_1 + \alpha_2) d\left(ABz'', z\right)$$

which shows that $ABz'' = z$

Also since (AB, I) are compatible and hence using weakly commuting we obtain

$$\begin{aligned} d(ABz, Iz) &= d(AB(Iz''), I(ABz'')) \\ &\leq d(Iz'', ABz'') = d(z, z) = 0 \end{aligned}$$

Therefore $ABz = Iz = z$

Thus we have proved that z is common fixed point of AB, ST, I and J .

If the mappings ST or J is continuous instead of AB or I then proof of z is a common fixed point of AB, ST, I and J is similar.

Let v be another fixed point of I, J, AB and ST

then

$$\begin{aligned} d(z, v) &= d(ABz, STv) \\ &\leq \alpha_1 \left[\frac{[d(ABz, Jv)]^n + [d(STv, Iz)]^n}{[d(ABz, Jv)]^{n-1} + [d(STv, Iz)]^{n-1}} \right] \\ &\quad + \alpha_2 [d(ABz, Iz) + d(STv, Jv)] \\ &\quad + \alpha_3 d(Iz, Jv) \end{aligned}$$

$$d(z, v) \leq (\alpha_1 + \alpha_3) d(z, v) \text{ yielding thereby } z = v$$

Finally we need to show that z is also a common fixed point of A, B, S, T, I and J . For this let z be the unique common fixed point of both the pairs (AB, I) and (ST, J) . Then

$$Az = A(ABz) = A(BAz) = AB(Az), \quad Az = A(Iz) = I(Az)$$

$$Bz = B(ABz) = B(A(Bz)) = BA(Bz) = AB(Bz), \quad Bz = B(Iz) = I(Bz)$$

which shows that Az and Bz is a common fixed point of (AB, I) yielding thereby $Az = z = Bz = Iz = ABz$ in the view of uniqueness of the common fixed point of the pair (AB, I) .

Similarly using the commutativity of $(S, T), (SJ)$ and (T, J) it can be shown that $Sz = z = Tz = Jz = STz$.

Now we need to show that $Az = Sz (Bz = Tz)$ also remains a common fixed point of both the pairs (AB, I) and (ST, J) . For this

$$\begin{aligned} d(Az, Sz) &= d(A(BAz), S(TSz)) \\ &= d(AB(Az), ST(Sz)) \\ &\leq \alpha_1 \left[\frac{[d(AB(Az), J(Sz))]^n + [d(ST(Sz), I(Az))]^n}{[d(AB(Az), J(Sz))]^{n-1} + [d(ST(Sz), I(Az))]^{n-1}} \right] \end{aligned}$$

$$+\alpha_2 \left[d(AB(Az), I(Az)) + d(ST(Sz), J(Sz)) \right] + \alpha_3 d(I(Az), J(Sz))$$

implies that $d(Az, Sz) = 0$ (as $d(AB(Az), J(Sz)) + d(ST(Sz), I(Az)) = 0$) yielding thereby $Az = Sz$.

Similarly it can be shown that $Bz = Tz$. Thus z is the unique common fixed of A, B, S, T, I and J .

(ii) Suppose that $d(ABx, Jy) + d(STy, lx) = 0$ implies $d(ABx, STy) = 0$. Then we argue as follows.

Suppose that there exists an n such that $z_n = z_{n+1}$. Then, also $z_{n+1} = z_{n+2}$. Suppose not. Then from (2.3) we have $0 < d(z_{n+1}, z_{n+2}) \leq kd(z_{n+1}, z_n)$ yielding thereby $z_{n+1} = z_{n+2}$. Thus $z_n = z_{n+k}$ for $k = 1, 2, \dots$ it then follows that there exist two point w_1 and w_2 such that $v_1 = ABw_1 = Jw_2$ and $v_2 = STw_2 = lw_1$. Since $d(ABw_1, Jw_2) + d(STw_2, lw_1) = 0$ from (2.2) $d(ABw_1, STw_2) = 0$ i.e. $v_1 = ABw_1 = STw_2 = v_2$. Also note that $Iv_1 = I(ABw_1) = AB(Iw_1) = ABv_1$. Similarly $STv_2 = Jv_2$. Define $y_1 = ABv_1, y_2 = STv_2$. Since $d(ABv_1, Jv_2) + d(STv_2, Iv_1) = 0$, it follow from (2.2) that $d(ABv_1, STv_2) = 0$, i.e. $y_1 = y_2$. Thus $ABv_1 = Iv_1 = STv_2 = Jv_2$. But $v_1 = v_2$. follows AB, I, ST and J have a common coincidence point. Define $w = ABv_1$, it then follows that w is also a common coincidence point of AB, I, ST and J . If $ABw \neq ABv_1 = STv_1$, then $d(ABw, STv_1) > 0$. But, since $d(ABw, Jv_1) + d(STv_1, lw) = 0$, it follows from (2.2) that $d(ABw, STv_1) = 0$, i.e. $ABw = STv_1$, a contradiction. Therefore $ABw = ABv_1 = w$ and w is a common fixed point of AB, ST, I and J .

The rest of the proof is identical to the case (1), hence it is omitted. This completes the proof.

Corollary 2.2: Theorem 2.1 remains true if contraction conditions (2.1.1) and (2.1.2) are replaced by any of the following conditions:

$$(i) \text{ Either } d(ABz, STy) \leq \alpha_1 \left[\frac{[d(ABx, Jy)]^n + [d(STy, lx)]^n}{[d(ABx, Jy)]^{n-1} + [d(STy, lx)]^{n-1}} \right] + \alpha_2 [d(ABx, lx) + d(STy, Jy)]$$

$$\text{if } d(ABx, Jy) + d(STy, lx) \neq 0, \quad \alpha_1, \alpha_2 > 0, 2\alpha_1 + 2\alpha_2 < 1 \text{ or} \quad (A)$$

$$d(ABx, STy) = 0 \text{ if } d(ABx, Jy) + d(STy, lx) = 0$$

$$(ii) \text{ Either } d(ABx, STy) \leq \alpha_1 \left[\frac{[d(ABx, Jy)]^n + [d(STy, lx)]^n}{[d(ABx, Jy)]^{n-1} + [d(STy, lx)]^{n-1}} \right] + \alpha_3 (lx, Jy)$$

$$\text{if } d(ABx, Jy) + d(STy, lx) \neq 0, \quad \alpha_1, \alpha_3 > 0, 2\alpha_1 + \alpha_3 < 1 \text{ or} \quad (B)$$

$$d(ABx, STy) = 0 \text{ if } d(ABx, Jy) + d(STy, lx) = 0$$

$$(iii) \text{ Either } d(ABx, STy) \leq \alpha_1 \left[\frac{[d(ABx, Jy)]^n + [d(STy, lx)]^n}{[d(ABx, Jy)]^{n-1} + [d(STy, lx)]^{n-1}} \right]$$

$$\text{if } d(ABx, Jy) + d(STy, lx) \neq 0, \quad \alpha_1 > 0, \alpha_1 < 1/2 \text{ or} \quad (C)$$

$$d(ABx, STy) = 0 \text{ if } d(ABx, Jy) + d(STy, lx) = 0$$

$$(iv) d(ABx, STy) \leq \alpha_1 [d(ABx, Jy) + d(STy, Jy)]$$

$$+\alpha_2 [d(ABx, lx) + d(STy, Jy)] + \alpha_3 d(lx, Jy) \text{ if } 2\alpha_1 + 2\alpha_2 + \alpha_3 < 1 \quad (D)$$

$$(v) \ d(ABx, STy) \leq \alpha_1 [d(ABx, Jy) + d(STy, lx)] \text{ if } \alpha_1 < \frac{1}{2} \quad (E)$$

$$(vi) \ d(ABx, STy) \leq \alpha_2 [d(ABx, lx) + d(STy, Jy)] \text{ if } \alpha_2 < \frac{1}{2} \quad (F)$$

$$(vii) \ d(ABx, STy) \leq \alpha_3 d(lx, Jy) \text{ if } \alpha_3 < 1 \quad (G)$$

Proof: Corollaries corresponding to the contraction conditions (A), (B) and (C) can be deduced directly from Theorem 2.1 by choosing $\alpha_3 = 0$, $\alpha_2 = 0$ and $\alpha_2 = \alpha_3 = 0$, respectively. The corollary corresponding the contraction condition (D) also follows from Theorem 2.1 by noting that

$$\frac{[d(ABx, Jy)]^n + [d(STy, lx)]^n}{[d(ABx, Jy)]^{n-1} + [d(STy, lx)]^{n-1}} \leq \frac{[d(ABx, Jy) + d(STy, lx)]^n}{[d(ABx, Jy) + d(STy, lx)]^{n-1}} \\ \leq [d(ABx, Jy) + d(STy, lx)]$$

Finally one may note that the contraction conditions (E), (F) and (G) are special cases of the contraction condition (D).

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