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COMMON FIXED POINT THEOREM FOR COMPATIBLE MAPS OF TYPE (β) AND TYPE (α) USING INTEGRAL TYPE MAPPING FOR IMPLICIT RELATIONS IN FUZZY TWO METRIC SPACES

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ABSTRACT

In this paper, common fixed point theorem of integral type of compatible mappings of type(β) and type (α) satisfying integral mapping of Two Fuzzy metric space. Establish some new fixed point theorems in complete metric spaces.

1. INTRODUCTION

Fixed point theory became one of the most interesting area of research in the last forty years for instance research about control theory, differential equations, integral equations, economics, and etc. The fixed point theorem, generally known as the Banach contraction mapping principle, appeared by Banach in 1922. Later, the Banach contraction principle has been widely generalized and extended a common fixed point theorem by removing the assumption of continuity, relaxing compatibility to compatible maps of type (α) or (β). weak compatibility and replacing the completeness of the space with a set of alternative conditions for functions satisfying an implicit relation in FM-space.

In our chapter the following implicit relation: Let $I = [0,1]$,$*$ be a continuous t-norm and F be the set of all real continuous functions $F: I^6 \to R$ satisfying the following conditions

1. F is no increasing in the fifth and sixth variables,
2. if, for some constant $k \in (0,1)$ we have
   a. $F\left(u(kt), v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right)\right) \geq 1$, or
   b. $F\left(u(kt), v(t), u(t), v(t), u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right), 1\right) \geq 1$

   for any fixed $t > 0$ and any nondecreasing functions $u, v: (0,\infty) \to I$ with $0 \leq u(t), v(t) \leq 1$ then there exists $h \in (0,1)$ with $u(ht) \geq v(t) * u(t)$,
3. if, for some constant $k \in (0,1)$ we have
   $F(u(kt), u(t), 1, 1, u(t), u(t)) \geq 1$
   for any fixed $t > 0$ and any nondecreasing function $u: (0,\infty) \to I$ then $u(kt) \geq u(t)$.
   Beside this the concepts of Fuzzy 2-metric spaces are as follows,

2. PRELIMINARIES

Definition 2.1: A triplet $(X, M, \star)$ is said to be a Fuzzy 2- metric space if X is an arbitrary set, $\star$ is a continuous $t-$norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following condition for all $x, y, z, s, t > 0$,

2.1 (FM − 1) $M(x, y, \theta, t) > 0$
2.2 (FM − 2) $M(x, y, \theta, t) = 1$ if and only if $x = y = \theta$.
2.3(FM − 3) $M(x, y, \theta, t) = M(y, \theta, x, t) = M(\theta, x, y, t)$
2.4 (FM − 4) $M(x, y, \theta, t) \star M(y, z, \theta, s) \star M(z, x, \theta, q) \leq M(x, y, z, t + s + q)$
2.5(FM − 5) $M(x, y, \theta, \bullet): (0,\infty) \to (0,1]$ is continuous.

Then M is called a Fuzzy 2- metric on X. The function $M(x, y, \theta, t)$ denote the degree of nearness between x, y and θ with respect to t.

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Example 2.2: Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and

$$M(x, y, \theta, t) = \frac{t}{t + d(x, y, \theta)}$$

For all $x, y \in X$ and all $t > 0$. Then $(X, M, \star)$ is a Fuzzy 2- metric space.

It is called the Fuzzy 2- metric space induced by d.

We note that, $M(x, y, \theta, t)$ can be realized as the measure of nearness between x and y with respect to t. It is known that $M(x, y, \cdot)$ is non decreasing for all $x, y \in X$. Let $M(x, y, \star)$ be a Fuzzy 2- metric space for $t > 0$, the open ball

$$B(x, r, \theta, t) = \{y \in X: M(x, y, \theta, t) > 1 - r\}.$$

Now, the collection $\{B(x, r, \theta, t): x \in X, 0 < r < 1, t > 0\}$ is a neighborhood system for a topology τ on X induced by the Fuzzy 2- metric M. This topology is Housdroff and first countable.

Definition 2.3: A sequence $\{x_n\}$ in a Fuzzy 2- metric space $(X, M, \star)$ is said to be a converges to x iff for each $\varepsilon > 0$ and each $t > 0$, $n_0 \in N$ such that $M(x_n, x, \theta, t) > 1 - \varepsilon$ for all $n \geq n_0$.

Definition 2.4: A sequence $\{x_n\}$ in a Fuzzy 2- metric space $(X, M, \star)$ is said to be a G- Cauchy sequence converges to x iff for each $\varepsilon > 0$ and each $t > 0$, $n_0 \in N$ such that $M(x_m, x_n, \theta, t) > 1 - \varepsilon$ for all $m, n \geq n_0$.

A Fuzzy 2- metric space $(X, M, \star)$ is said to be complete if every G- Cauchy sequence in it converges to a point in it.

3. MAIN THEOREM

Common Fixed Point Theorem for Compatible Maps of Type (β) and Type (α) Using integral type mapping

Integral type contraction principle is one of the most popular contraction principle in fixed point theory. The first known result in this direction was given by Branciari [13] in general setting of lebgesgue integrable function and proved following fixed point theorems in metric spaces.

Theorem 3.1: Let $(X, M, \star)$ be a complete Fuzzy 2- metric space and let A, B, S, T, P and Q be mappings from X into itself such that the following conditions are satisfied:

3.1 (a) $P(X) \subset ST(X)$ and $Q(X) \subset AB(X)$,

3.1 (b) (P, AB) is compatible of type (β) and (Q, ST) is weak compatible,

3.1 (c) there exists $k \in (0,1)$ such that for every $x, y \in X$ and $t > 0$

$$\int_0^{F\left(\substack{M^2(Px,Qy,\theta,kt),M^2(Px,ABx,\theta,t),\\ M^2(Qy,STy,\theta,t)\,,,M^2(ABx,Qy,\theta,t)}\right)} \xi(v)\ dv \geq 1$$

Where $\xi : [0, +\infty] \to [0, +\infty]$ is a lebgesgue integrable mapping which is summable on each compact subset of $[0, +\infty]$ non negative and such that $\forall\ \varepsilon > 0$, $\int_0^{\varepsilon} \xi(v)\ dv > 0$. Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof: Let $x_0 \in X$, then from 3.1 (a) we have $x_1, x_2 \in X$ such that

$$Px_0 = STx_1 \text{ and } Qx_1 = ABx_2$$

Inductively, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that for $n \in N$

$$Px_{2n-2} = STx_{2n-1} = y_{2n-1} \text{ and } Qx_{2n-1} = ABx_{2n} = y_{2n}$$

Put $x = x_{2n}$ and $y = x_{2n+1}$ in 3.1.(c)then we have

$$\int_0^{F\left(\substack{M^2(Px_{2n},Qx_{2n+1},\theta,kt),M^2(Px_{2n},ABx_{2n},\theta,t)\\ ,M^2(Qx_{2n+1},STx_{2n+1},\theta,t)\,,,M^2(ABx_{2n},Q_{2n+1},\theta,t)}\right)} \xi(v)\ dv > 1$$

$$\int_0^{F\left(\substack{M^2(y_{2n+1},y_{2n+2},\theta,kt),M^2(y_{2n+1},y_{2n},\theta,t),\\ M^2(y_{2n+2},y_{2n+1},\theta,t),M^2(y_{2n},y_{2n+2},\theta,t)}\right)} \xi(v)\ dv > 1$$

$$\int_0^{F\left(\substack{M^2(y_{2n+1},y_{2n+2},\theta,kt),\\ M^2(y_{2n+2},y_{2n+1},\theta,t),M^2(y_{2n+1},y_{2n+1},\theta,t),\\ \star M^2\left(y_{2n+1},y_{2n+2},\theta,\frac{t}{2}\right)}\right)} \xi(v)\ dv > 1$$

From condition 3.2 (a) we have

$$\int_0^{M^2(y_{2n+1},y_{2n+2},\theta,kt)} \xi(v)\ dv \geq \int_0^{M^2\left(y_{2n},y_{2n+1},\theta,\frac{t}{2}\right)\star M^2\left(y_{2n+2},y_{2n+1},\theta,\frac{t}{2}\right)} \xi(v)\ dv$$

We have

$$\int_0^{M^2(y_{2n+1},y_{2n+2},\theta,kt)} \xi(v)\ dv \geq \int_0^{M^2\left(y_{2n},y_{2n+1},\theta,\frac{t}{2}\right)} \xi(v)\ dv$$

Since $\xi(v)$ is a lebesgue integrable function so we have

$$M(y_{2n+1}, y_{2n+2}, \theta, kt) \geq M\left(y_{2n}, y_{2n+1}, \theta, \frac{t}{2}\right)$$

Similarly we have

$$M(y_{2n+2}, y_{2n+3}, \theta, kt) \geq M\left(y_{2n+1}, y_{2n+2}, \theta, \frac{t}{2}\right)$$

Thus we have

$$M(y_{n+1}, y_{n+2}, \theta, kt) \geq M\left(y_n, y_{n+1}, \theta, \frac{t}{2}\right)$$
$$M(y_{n+1}, y_{n+2}, \theta, t) \geq M\left(y_n, y_{n+1}, \theta, \frac{t}{2^k}\right)$$
$$M(y_n, y_{n+1}, \theta, t) \geq M\left(y_0, y_1, \theta, \frac{t}{2^{nk}}\right) \to 1 \text{ as } n \to \infty,$$

and hence $M(y_n, y_{n+1}, \theta, t) \to 1$ as $n \to \infty$ for all $t > 0$.

For each $\epsilon > 0$ and $t > 0$, we can choose $n_0 \in N$ such that
$$M(y_n, y_{n+1}, \theta, t) > 1 - \epsilon \text{ for all } n > n_0.$$

For any $m, n \in N$ we suppose that $m \geq n$. Then we have
$$M(y_n, y_m, \theta, t) \geq M\left(y_n, y_{n+1}, \theta, \frac{t}{m-n}\right) \star M\left(y_{n+1}, y_{n+2}, \theta, \frac{t}{m-n}\right) \star \ldots . \star M\left(y_{m-1}, y_m, \theta, \frac{t}{m-n}\right)$$
$$M(y_n, y_m, \theta, t) \geq (1-\epsilon) \star (1-\epsilon) \star \ldots \ldots \star (1-\epsilon)(m-n)\text{times}$$
$$M(y_n, y_m, \theta, t) \geq (1-\epsilon)$$
And hence $\{y_n\}$ is a Cauchy sequence in X.

Since $(X, M, \star)$ is complete, $\{y_n\}$ converges to some point $z \in X$. Also its subsequences converges to the same point $z \in X$. That is

$$\{Px_{2n+2}\} \to z \text{ and } \{STx_{2n+1}\} \to z \qquad 3.2\ (i)$$

$$\{Qx_{2n+1}\} \to z \text{ and } \{ABx_{2n}\} \to z \qquad 3.2\ (ii)$$

As (P, AB) is compatible pair of type (β), we have
$$M(PPx_{2n}, (AB)(AB)x_{2n}, \theta, t) = 1, \qquad \text{for all } t > 0$$

Or $\quad M(PPx_{2n}, ABz, \theta, t) = 1$

Therefore, $\quad PPx_{2n} \to ABz.$

Put $x = (AB)x_{2n}$ and $y = x_{2n+1}$ in 5.2.1(c) we have

$$\int_0^{F\left(\begin{array}{c} M^2(P(AB)x_{2n},Qy,\theta,kt), \\ M^2(P(AB)x_{2n},AB(AB)x_{2n},\theta,t), M^2(Qx_{2n+1},STx_{2n+1},\theta,t), \\ M^2(AB(AB)x_{2n},Qx_{2n+1},\theta,t) \end{array}\right)} \xi(v)\ dv > 1$$

Taking $n \to \infty$ and 3.1(a) we get

$$\int_0^{M^2((AB)z,z,\theta,kt)} \xi(v)\ dv \geq \int_0^{M^2((AB)z,z,\theta,t)} \xi(v)\ dv$$

Since $\xi(v)$ is a lebesgue integrable function which implies
$$M((AB)z, z, \theta, kt) \geq M((AB)z, z, \theta, t)$$

We have

$$ABz = z. \qquad 3.2(iii)$$

Put $x = z$ and $y = x_{2n+1}$ in 3.1(c) we have

$$\int_0^{F\begin{pmatrix} M^2(Pz,Q\,x_{2n+1},\theta,kt),*M^2(Pz,ABz,\theta,t) \\ M^2(Q\,x_{2n+1},ST\,x_{2n+1},\theta,t),, \\ M^2(ABz,Q\,x_{2n+1},\theta,t) \end{pmatrix}} \xi(v)\ dv > 1$$

Taking $n \to \infty$ 3.1 (a) and using equation 3.1 (i) we have

That is $$\int_0^{M^2(Pz,z,\theta,kt)} \xi(v)\ dv \geq \int_0^{M^2(Pz,z,\theta,t)} \xi(v)\ dv$$

Since $\xi(v)$ is a lebesgue integrable function so we have

$$M(Pz, z, \theta, kt) \geq M(Pz, z, \theta, t)$$

we get $Pz = z$

So we have $ABz = Pz = z.$

Putting $x = Bz$ and $y = x_{2n+1}$ in 3.1(d), we get

$$\int_0^{F\begin{pmatrix} M^2(PBz,Qx_{2n+1},\theta,kt),M^2(PBz,ABBz,\theta,t) \\ M^2(Qx_{2n+1},STx_{2n+1},\theta,t),,M^2(ABBz,Qx_{2n+1},\theta,t) \end{pmatrix}} \xi(v)\ dv > 1$$

Taking $n \to \infty$, 3.1(a) and using 3.1.(i) we get

$$\int_0^{M^2(Bz,z,,\theta kt)} \xi(v)\ dv \geq \int_0^{M^2(Bz,z,\theta,t)} \xi(v)\ dv$$

Since $\xi(v)$ is a lebesgue integrable function which follows

$$M(Bz, z, \theta, kt) \geq M(Bz, z, \theta, t)$$

We have $Bz = z$

And also we have $ABz = z$ implies $Az = z$

Therefore $Az = Bz = Pz = z.$ 5.2.1 (iv)

As $P(X) \subset ST(X)$ there exists $u \in X$ such that

$$z = Pz = STu$$

Putting $x = x_{2n}$ and $y = u$ in 3.1(c) we get

$$\int_0^{F\begin{pmatrix} M^2(Px_{2n},Qu,\theta,kt),M^2(Px_{2n},ABx_{2n},\theta,t) \\ M^2(Qu,STu,\theta,t),,M^2(ABx_{2n},Qu,\theta,t) \end{pmatrix}} \xi(v)\ dv > 1$$

Taking $n \to \infty$ and using 3.1. (i) and 3.1(ii) we get

$$\int_0^{F\begin{pmatrix} M^2(z,Qu,\theta,kt),,M^2(z,z,\theta,t) \\ M^2(Qu,STu,\theta,t),M^2(z,Qu,\theta,t) \end{pmatrix}} \xi(v)\ dv > 1$$

$$\int_0^{M^2(z,Qu,\theta,kt)} \xi(v)\ dv \geq \int_0^{M^2(z,Qu,\theta,t)} \xi(v)\ dv$$

Since $\xi(v)$ is a lebesgue integrable function which implies

$$M(z, Qu, \theta, kt) \geq M(z, Qu, \theta, t)$$

we have $Qu = z$

Hence $STu = z = Qu.$

Hence (Q, ST) is weak compatible, therefore, we have

$$QSTu = STQu$$

Thus $Qz = STz.$

Putting $x = x_{2n}$ and $y = z$ in 3.1(c) we get

$$\int_0^{F\begin{pmatrix} M^2(Px_{2n},Qz,\theta,kt),M^2(ABx_{2n},STz,\theta,t),M^2(Px_{2n},ABx_{2n},\theta,t) \\ M^2(Qz,STz,\theta,t),M^2(Px_{2n},STz,\theta,t),M^2(ABx_{2n},Qz,\theta,t) \end{pmatrix}} \xi(v)\ dv > 1$$

Taking $n \to \infty$ and using 3.1(ii) we get

$$\int_0^{F\begin{pmatrix} M^2(z,Qz,\theta,kt),,M^2(z,z,\theta,t) \\ M^2(Qz,STz,\theta,t),,M^2(z,Qz,\theta,t) \end{pmatrix}} \xi(v)\ dv > 1$$

$$\int_0^{M^2(z,Qz,\theta,kt)} \xi(v)\ dv \geq \int_0^{M^2(z,Qz,\theta,t)} \xi(v)\ dv$$

Since $\xi(v)$ is a lebesgue integrable function and hence

$$M(z,Qz,\theta,kt) \geq M(z,Qz,\theta,t)$$

we get $\quad Qz = z.$

Putting $x = x_{2n}$ and $y = Tz$ in 3.1(c) we get

$$\int_0^{F\begin{pmatrix} M^2(Px_{2n},QTz,\theta\ ,kt),M^2(Px_{2n},ABx_{2n},\theta,t) \\ M^2(QTz,STTz,\theta,t),M^2(ABx_{2n},QTz,\theta,t) \end{pmatrix}} \xi(v)\ dv > 1$$

As $QT = TQ$ and $ST = TS$ we have

$$QTz = TQz = Tz$$

And $\quad ST(Tz) = T(STz) = TQz = Tz.$

Taking $n \to \infty$ we get

$$\int_0^{F\begin{pmatrix} M^2(z,Tz\ ,\theta,kt),,M^2(z,z,\theta,t) \\ M^2(Tz,Tz,\theta,t),M^2(z,Tz,\theta,t) \end{pmatrix}} \xi(v)\ dv > 1$$

$$\int_0^{M^2(z,Tz,\theta\ ,kt)} \xi(v)\ dv \geq \int_0^{M^2(z,Tz\ ,\theta,t)} \xi(v)\ dv$$

Since $\xi(v)$ is a lebesgue integrable function therefore

$$M(z,Tz\ ,\theta,kt) \geq M(z,Tz,\theta\ ,t)$$

We have $\quad Tz = z$

Now $\quad STz = Tz = z$ implies $Sz = z.$

Hence $\quad Sz = Tz = Qz = z \qquad$ 3.1(v)

Combining 3.1(iv) and 3.1(v) we have

$$Az = Bz = Pz = Sz = Tz = Qz = z$$

Hence z is the common fixed point of A, B, S, T, P and Q.

Uniqueness: Let u be another common fixed point of A, B, S, T, P and Q. Then

$$Au = Bu = Su = Tu = Pu = Qu = u$$

Putting $x = u$ and $y = z$ in 3.1(c) then we get

$$\int_0^{F\begin{pmatrix} M^2(Pu,Qz,\theta,kt),,M^2(Pu,ABu,\theta,t) \\ M^2(Qz,STz,\theta,t),M^2(ABu,Qz,\theta,t) \end{pmatrix}} \xi(v)\ dv > 1$$

Taking limit both side then we get

$$\int_0^{F\begin{pmatrix} M^2(u,z,\theta,kt),(u,u,\theta,t) \\ M^2(z,z,\theta,t),M^2(u,z,\theta,t) \end{pmatrix}} \xi(v)\ dv > 1$$

$$\int_0^{M^2(u,z,\theta,kt)} \xi(v)\ dv \geq \int_0^{M^2(u,z,\theta,t)} \xi(v)\ dv$$

Since $\xi(v)$ is a lebesgue integrable function so we have

$$M(u, z, \theta, kt) \geq M(u, z, \theta, t)$$

We get $z = u.$

That is z is a unique common fixed point of A, B, S, T, P and Q in X.

Remark 3.1: If we take $\xi(v) = 1$ then we get Theorem 3.1.

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