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SOME RESULTS OF FIXED POINT THEOREM IN NON-NEWTONIAN METRIC SPACES

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ABSTRACT

The purpose of this paper is to study of fixed point theorems in non-Newtonian -metric spaces and obtains new results in it.

Keywords: *non-Newtonian metric spaces, fixed point, Fixed point theorem, Continuous Mapping, Complete metric space.*

1. INTRODUCTION

Banach [1992] Proved a fixed point theorem for contraction mapping in complete Metric space. It is well known as a Banach Fixed point theorem. Every contraction mapping of a complete metric space X into itself has a unique fixed point (Bonsall 1962). Aage and Salunke [3] proved the result on fixed point in Dislocated and Dislocated Quasi-Metric space. Dass and Gupta [1] generalized Banach's contraction principle in Metric Space. Rohades [2] introduced a partial ordering for various definitions contractive mappings. The study of non newtonian calculi have been started in 1972 by Grossman and Katz [6]. These provide an alternative to the classical calculus and they include the geometric, anageometric and bigeometric calculi, etc. In 2002 Cakmac and Basar [4], have introduced the concept of non Newtonian metric space. Also they have given the triangle and Minkowski's inequalities in the sense of non-Newtonian calculus. Recently, Binbasioglu, *et al.* [5] discussed some topological properties of the non newtonian metric space and also introduced the concept of fixed point theory for the non newtonian Metric Space. The non-Newtonian calculi are alternatives to the classical calculus of Newton and Leibnitz. They provide a wide variety of mathematical tools for use in science, engineering and mathematics.

2. PRELIMINARIES

Proposition 2.1 [4]: The triangle inequality with respect to non-Newtonian distance $|\cdot|_N$, for any $x, y \in \mathbb{R}(N)$ is given by $|x \dot{+} y|_N \leq |x|_N \dot{+} |y|_N$.

The non-Newtonian metric spaces provide an alternative to the metric spaces introduced in [4].

Definition 2.2 [4]: Let $X \neq \emptyset$ be a set. If a function $d_N: X \times X \to \mathbb{R}^+(N)$ satisfies the following axioms for all $x, y, z \in X$:
(NM1) $d_N(x,y) = \beta(0) = \dot{0}$ if and only if $x = y$,
(NM2) $d_N(x,y) = d_N(y,x)$,
(NM3) $d_N(x,y) \dot{\leq} d_N(x,z) \dot{+} d_N(z,y)$,
then it is called a non-Newtonian metric on X and the pair (X, d_N) is called a non-Newtonian metric space.

Proposition 2.3 [4]: Suppose that the non-Newtonian metric d_N on $\mathbb{R}(N)$ is such that $d_N(x,y) = |x \dot{-} y|_N$ for all $x, y \in \mathbb{R}(N)$, then $(\mathbb{R}(N), d_N)$ is a non-Newtonian metric space.

Proposition 2.4 [5]: Let (X, d_N) be a non-Newtonian metric space. Then we have the following inequality:
$|d_N(x,z) \dot{-} d_N(y,z)|_N \dot{\leq} d_N(x,y)$ for all $x, y, z \in X$.

Definition 2.5 [4]: Let (X, d_N^N) and (Y, d_N^Y) be two non-Newtonian metric spaces and let $f : X \to Y$ be a function. If f satisfies the requirement that, for every $\varepsilon \dot{>} \dot{0}$, there exists $\delta \dot{>} \dot{0}$ such that $f(B_\delta^N(x)) \subset B_\varepsilon^N(f(x))$, then f is said to be non-Newtonian continuous at $x \in X$.

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Example 2.6: Given a non-Newtonian metric space(X, d_N), define a non-Newtonian metric on $X \times X$ by $p((x_1, x_2), (y_1, y_2)) = d_N(x_1, y_1) \dot{+} d_N(x_2, y_2)$. Then the non-Newtonian metric $d_N : X \times X \to (\mathbb{R}^+(N), |\cdot|_N)$ is non-Newtonian continuous on $X \times X$. To show this, let $(y_1, y_2), (x_1, x_2) \in X \times X$.

Since we have $|d_N(y_1, y_2) \dot{-} d_N(x_1, x_2)|_N \dot{\leq} d_N(x_1, y_2) \dot{+} d_N(x_2, y_2)$, it is clear that d_N is non-Newtonian continuous on $X \times X$. Now, we emphasize some properties of convergent sequences in a non-Newtonian metric space.

Definition 2.7 [4]: A sequence (x_n) in a metric space $X = (X, d_N)$ is said to be convergent if for every given $\varepsilon \dot{>} \dot{0}$ there exist an $n_0 = n_0(\varepsilon) \in N$ *and* $x \in X$ such that $d_N(x_n, x) \dot{<} \varepsilon$ for all $n > n_0$, and it is denoted by $x_n \overset{N}{\to} x$, as $n \to \infty$.

Definition 2.8 [5]: A sequence (x_n) in a non-Newtonian metric space $X = (X, d_N)$ is said to be non-Newtonian Cauchy if for every $\varepsilon \dot{>} \dot{0}$ there exists an $n_0 = n_0(\varepsilon) \in N$ such that $d_N(x_n, x_m) \dot{<} \varepsilon$ for all $m, n > n_0$. Similarly, if for every non-Newtonian open ball $B_\varepsilon^N(x)$, there exists a natural number n_0 such that $n > n_0$, $x_n \in B_\varepsilon^N(x)$, then the sequence (x_n) is said to be non-Newtonian convergent to x.

The space X is said to be non-Newtonian complete if every non-Newtonian Cauchy sequence in X converges [4].

Lemma 2.9 [5]: Let (X, d_N) be a non-Newtonian metric space, (x_n) a sequence in X and $x \in X$. Then
$x_n \overset{N}{\to} x \ (n \to \infty)$ if and only if $d_N(x_n, x) \overset{N}{\to} \dot{0} \ (n \to \infty)$.

Theorem 2.10 [5]: Let (X, d_N^X) and (Y, d_N^Y) be two non-Newtonian metric spaces, $f : X \to Y$ a mapping and (x_n) any sequence in X. Then f is non-Newtonian continuous at the point $x \in X$ if and only if $f(x_n) \overset{N}{\to} f(x)$ for every sequence (x_n) with $x_n \overset{N}{\to} x \ (n \to \infty)$.

Theorem 2.11 [5]: Let (X, d_N) be a non-Newtonian metric space and $S \subset X$. Then

(i) a point $x \in X$ belongs to $\bar{S}$ if and only if there exists a sequence (x_n) in S such that $x_n \overset{N}{\to} x \ (n \to \infty)$,

(ii) the set S is non-Newtonian closed if and only if every non-Newtonian convergent sequence in S has a non-Newtonian limit point that belongs to S.

We now define the fixed point theorem on non-Newtonian metric spaces and give some examples.

Definition 2.12 [5]: Let X be a set and T a map from X to X. A fixed point of T is a point $x \in X$ such that $Tx = x$. In other words, a fixed point of T is a solution of the functional equation $Tx = x, x \in X$.

Definition 2.13 [5]: Suppose that (X, d_N) is a non-Newtonian complete metric space and $T : X \to X$ is any mapping. The mapping T is said to satisfy a non-Newtonian Lipchitz condition with $k \in \mathbb{R}(N)$ if

$d_N(T(x), T(y)) \dot{\leq} k \dot{\times} d_N(x, y)$ holds for all $x, y \in X$.

If $k \dot{<} \dot{1}$, then T is called a non-Newtonian contraction mapping.

Theorem 2.14 [5]: Let T be a non-Newtonian contraction mapping on a non-Newtonian complete metric space X. Then T has a unique fixed point.

Theorem 2.15 [5]: Let T be a mapping on a non-Newtonian complete metric space X into itself. Let T be a non-Newtonian contraction on a closed ball $\bar{B}_{\dot{r}}^N(x_0) = \{x \in X : d_N(x, x_0) \dot{\leq} \dot{r}\}$.

Suppose that $d_N(x_0, Tx_0) \dot{<} (\dot{1} \dot{-} k)\dot{r}$. Then the iterative sequence defined by $x_n = T^n x_0 = Tx_{n-1}$ converges to an $x \in \bar{B}_{\dot{r}}^N(x_0)$ and this x is the unique fixed point of T.

3. MAIN RESULTS

Theorem 3.1: Let (X, d_N) be a complete non-Newtonian metric space and suppose there exist non negative constants $\alpha_1, \alpha_2, \alpha_3$ *with* $\alpha_1 \dot{+} \alpha_2 \dot{+} \alpha_3 < \dot{1}$. Let f: X → X be a continuous mapping satisfying

$$d_N(fx, fy) \dot{\leq} \alpha_1 \dot{x} d_N(x, y) \dot{+} \alpha_2 \dot{x} d_N(x, fx) \dot{+} \alpha_3 \dot{x} d_N(y, fy) \qquad (3.1)$$

For all x, y ∈ X. *Then f has a unique fixed point.*

Proof: Let $\{x_n\}$ be a sequence in X, defined as follows. Let $x_0 \in X, f(x_0) = x_1, f(x_1) = x_2, \ldots., f(x_n) = x_{n+1}, \ldots.$

Consider

$$\begin{aligned} d_N(x_n, x_{n+1}) &= d_N(fx_{n-1}, fx_n) \\ &\dot{\leq} \alpha_1 \dot{x} d_N(x_{n-1}, x_n) \dot{+} \alpha_2 \dot{x} d_N(x_{n-1}, fx_{n-1}) \dot{+} \alpha_3 \dot{x} d_N(x_n, fx_n) \\ &= \alpha_1 \dot{x} d_N(x_{n-1}, x_n) \dot{+} \alpha_2 \dot{x} d_N(x_{n-1}, x_n) \dot{+} \alpha_3 \dot{x} d_N(x_n, x_{n+1}) \end{aligned}$$

Therefore,

$$\begin{aligned} d_N(x_n, x_{n+1}) &\leq \frac{\alpha_1 \dot{+} \alpha_2}{1 \dot{-} \alpha_3} d_N(x_{n-1}, x_n) \\ &= \lambda d_N(x_{n-1}, x_n), \end{aligned}$$

Where $\lambda = \frac{\alpha_1 \dot{+} \alpha_2}{1 \dot{-} \alpha_3}$.

Similarly, we have $d_N(x_{n-1}, x_n) \dot{\leq} \lambda\, d_N(x_{n-2}, x_{n-1})$. In this way, we get

$$d_N(x_n, x_{n+1}) \dot{\leq} \lambda^n d_N(x_0, x_1).$$

Since $\dot{0} \dot{\leq} \lambda < \dot{1}$, so for n $\to \infty$, $\lambda^n \to \infty$ we have $d_N(x_n, x_{n+1}) \to \dot{0}$.

Hence $\{x_n\}$ is a Cauchy sequence in the complete non nutonian metric space X, so there is a point $t_0 \in X$, such that $x_n \to t_0$. Since f is continuous

$$f(t_0) = \lim f(x_n) = \lim x_{n+1} = t_0$$

Thus $f(t_0) = t_0$, so f has a fixed point.

Uniqueness: If x ∈ X is a fixed point of f**,** then x = f(x), by (3.1)

$$d_N(x,x) = d_N(fx, fx) \dot{\leq} (\alpha_1 \dot{+} \alpha_2 \dot{+} \alpha_3) d_N(x,x)$$

which is true only if $d_N(x,x) = \dot{0}$, *since* $\dot{0} \dot{\leq} \alpha_1 \dot{+} \alpha_2 \dot{+} \alpha_3 < \dot{1}$ *and* $d_N(x,x) \dot{\geq} \dot{0}$. *Thus* $d_N(x,x) = \dot{0}$ *for a fixed point x of f*

Let x, y be fixed points f**.** Then by (3.1)

$$\begin{aligned} d_N(x,y) &= d_N(fx, fy) \\ &\dot{\leq} \alpha_1 d_N(x,y) \dot{+} \alpha_2 d_N(x,x) \dot{+} \alpha_3 d_N(y,y) \end{aligned}$$

i.e. $d_N(x,y) \dot{\leq} \alpha_1 d_N(x,y)$ *and from this it follows that* $d_N(x,y) = \dot{0}$, *since* $d_N(x,y) \dot{\geq} \dot{0}, \dot{0} \dot{\leq} \alpha_1 < \dot{1}$,.

Similarly $d_N(x,y) = \dot{0}$.

Hence $x = y$, *i.e. Uniqueness of the fixed point follows*

Theorem 3.2: Let (X, d_N) be a complete non-Newtonian metric space and let f: X→ X be a continuous mapping satisfying

$$d_N(fx, fy) \dot{\leq} \alpha \max\{d_N(x,y), d_N(x, fx), d_N(y, fy)\} \tag{3.2}$$

For all x, y ϵ X. If $\dot{0} \dot{\leq} \alpha < \dot{1}$, then f has a unique fixed point.

Proof: Let $\{x_n\}$ be a sequence in X, defined as follows.

Let $x_0 \in X$, $f(x_0) = x_1$, $f(x_1) = x_2$, $f(x_n) = x_{n+1}, \ldots$.

Consider

$$\begin{aligned} d_N(x_n, x_{n+1}) &= d_N(fx_{n-1}, fx_n) \\ &\dot{\leq} \alpha \max\{d_N(x_{n-1}, x_n), d_N(x_{n-1}, fx_{n-1}), d_N(x_n, fx_n)\} \\ &= \alpha \max\{d_N(x_{n-1}, x_n), d_N(x_{n-1}, x_n), d_N(x_n, x_{n+1})\} \end{aligned}$$

Hence

$$d_N(x_n, x_{n+1}) \dot{\leq} \alpha \{d_N(x_{n-1}, x_n)\}$$

Similarly we will show that

$$d_N(x_{n-1}, x_n) \dot{\leq} \alpha\, d_N(x_{n-2}, x_{n-1})$$

And

$$d_N(x_n, x_{n+1}) \dot{\leq} \alpha^2 d_N(x_{n-2}, x_{n-1})$$

Thus

$$d_N(x_n, x_{n+1}) \dot{\leq} \alpha^n d_N(x_0, x_1)$$

Since $\dot{0} \dot{\leq} \alpha < \dot{1}$, as n $\to \infty$, $\alpha^n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence in X. Thus $\{x_n\}$ converges to some t_0. Since f is continuous, we have

$$f(t_0) = \lim f(x_n) = \lim x_{n+1} = t_0$$

Uniqueness: Let x be a fixed point of f, then by (3.2)

$$d_N(x, x) = d_N(f x, f x) \dot{\leq} \alpha \max\{d_N(x, x)\}$$

i.e. $d_N(x, x) \dot{\leq} \alpha d_N(x, x)$, which gives $d_N(x, x) = \dot{0}$, Since $\dot{0} \dot{\leq} \alpha < \dot{1}$ and $d_N(x, x) \dot{\geq} \dot{0}$. Thus $d_N(x, x) = \dot{0}$ if x is a fixed point of f**.**

Let, x, y ϵ X be fixed points of f**.** That is, fx = x, fy = y. Then by (3.2),

$$\begin{aligned} d_N(x, y) &= d_N(fx, fy) \\ &\dot{\leq} \max\{d_N(x, y), d_N(x, x), d_N(y, y)\} \\ &= \alpha d_N(x, y) \end{aligned}$$

which is true only if $d_N(x, y) = \dot{0}$ (*since* $d_N(x, x) = \dot{0} = d_N(y, y), \dot{0} \dot{\leq} \alpha < \dot{1}$.

Similarly $d_N(y, x) = \dot{0}$ and hence $x = y$. Thus x is a fixed point of f is unique.

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