SOME NEW CONCEPTS OF CONTINUITY IN TOPOLOGICAL SPACES

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ABSTRACT

The purpose of this paper is to introduce and investigate several continuous functions namely $g_{s_{\alpha}}^{**}$ -continuous functions and contra $g_{s_{\alpha}}^{**}$ -continuous functions along with their several characterizations. Further we introduce new types of graphs called $g_{s_{\alpha}}^{**}$-closed graphs, contra $g_{s_{\alpha}}^{**}$ -closed graphs and investigated several characterizations of such notions.

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Keywords: $g_{s_{\alpha}}^{**}$ -continuous functions, contra $g_{s_{\alpha}}^{**}$ -continuous functions, $g_{s_{\alpha}}^{**}$-closed graph, contra $g_{s_{\alpha}}^{**}$-closed graph, locally $g_{s_{\alpha}}^{**}$-indiscrete space.

1. INTRODUCTION

In recent literature, we find many topologists have focused their research in the direction of investigating types of generalized continuity. The notion of contra-continuity was first investigated by Dontchev[7]. A good number of researchers have initiated different types of contra-continuous functions which are found in the papers [4],[5],[6]. In 1970, Levine [10] discussed the notion of generalized closed sets in topological spaces. Extensive research on generalizing closedness was done in recent years. In 1963, Levine [11] introduced the concepts of semi-open sets in topological spaces. W. Dunham [9] introduced the concept of generalized closure and defined a new topology $\tau^*$ and investigated some of their properties. Quite recently the authors Robert.A and Pious Missier.S introduced and studied semi−open [15] sets and semi-$\alpha$−open [15] sets using the generalized closure operator. Recently Santhini et.al [16] introduced $g_{s_{\alpha}}^{**}$ -closed sets in topological spaces. In 1969, Long [12] introduced closed graphs in topological spaces. In this paper, by means of $g_{s_{\alpha}}^{**}$-closed sets, we introduce namely, $g_{s_{\alpha}}^{**}$-continuous functions and contra $g_{s_{\alpha}}^{**}$-continuous functions along with their several properties, characterizations and mutual relationships. Further we introduce new types of graphs, called $g_{s_{\alpha}}^{**}$-closed graphs, contra $g_{s_{\alpha}}^{**}$-closed graphs via $g_{s_{\alpha}}^{**}$-open sets. Several characterizations and properties of such notions are investigated.

2. PRELIMINARIES

In this section, we recall some basic definitions and properties used in our paper.

Definition 2.1: A subset A of a space $(X, \tau)$ is said to be

(i) semi-open [11] if $A \subseteq \text{cl}(\text{int}A)$.

(ii) semi-open if [15] $A \subseteq \text{cl}_{(\text{int}A)}$.

(iii) semi-$\alpha$-open [15] if $A \subseteq \text{cl}_{(\text{int}A)}$.

(iv) a g-closed set [2] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

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(v) a $\alpha$-closed set \([17]\) if $\text{cl}(A) \subseteq U$ whenever $A$ and $U$ is semi-open in $X$.
(vi) a generalized-semi closed set(briefly gs-closed) \([5]\) if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.
(vii) a $g^*$ -closed set \([14]\) if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is g-open in $X$.
(viii) a generalized semi pre-closed set(briefly gsp-closed)\([8]\) if $\text{spcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

**Definition 2.2:** A subset $A$ of a space $(X, \tau)$ is called generalized gs*-closed set (briefly gs*-closed) \([16]\) if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-$\alpha$-open in $(X, \tau)$.

The class of all gs*-open subsets of $X$ is denoted by $\text{gs}_a^**O(X, \tau)$ and the class of all gs*-open subsets of $X$ containing $x$ is denoted by $\text{gs}_a^**O(X, x)$.

**Definition 2.3:** A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called a

1. semi-continuous \([11]\) if $f^{-1}(V)$ is semi-closed set in $(X, \tau)$ for every closed set $V$ in $(Y, \sigma)$.
2. semi*$\alpha$-continuous \([13]\) if $f^{-1}(V)$ is semi*-closed set in $(X, \tau)$ for every closed set $V$ in $(Y, \sigma)$.
3. semi*$\alpha$-continuous \([15]\) if $f^{-1}(V)$ is semi*$\alpha$-closed set in $(X, \tau)$ for every closed set $V$ in $(Y, \sigma)$.
4. g-continuous \([2]\) if $f^{-1}(V)$ is g-closed set in $(X, \tau)$ for every closed set $V$ in $(Y, \sigma)$.
5. generalized semi-continuous(briefly gs-continuous) \([5]\) if $f^{-1}(V)$ is gs-closed set in $(X, \tau)$ for every closed set $V$ in $(Y, \sigma)$.
6. generalized semi-precontinuous (briefly gsp-continuous) \([8]\) if $f^{-1}(V)$ is gsp-closed set in $(X, \tau)$ for every closed set $V$ in $(Y, \sigma)$.
7. $\omega$-continuous \([17]\) if $f^{-1}(V)$ is $\omega$-closed set in $(X, \tau)$ for every closed set $V$ in $(Y, \sigma)$.
8. $g^*$-continuous \([14]\) if $f^{-1}(V)$ is $g^*$-closed set in $(X, \tau)$ for every closed set $V$ in $(Y, \sigma)$.

**Definition 2.4:** A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be

1. contra-continuous \([7]\) if $f^{-1}(V)$ is closed in $(X, \tau)$ for every open set $V$ in $(Y, \sigma)$.
2. contra semi-continuous \([6]\) if $f^{-1}(V)$ is semi-closed in $(X, \tau)$ for every open set $V$ in $(Y, \sigma)$.
3. contra semi*$\alpha$-continuous \([13]\) if $f^{-1}(V)$ is semi*$\alpha$-closed in $(X, \tau)$ for every open set $V$ in $(Y, \sigma)$.
4. contra semi*$\alpha$-continuous \([15]\) if $f^{-1}(V)$ is semi*$\alpha$-closed in $(X, \tau)$ for every open set $V$ in $(Y, \sigma)$.
5. contra gs-continuous \([3]\) if $f^{-1}(V)$ is gs-closed in $(X, \tau)$ for every open set $V$ in $(Y, \sigma)$.
6. contra gsp-continuous \([1]\) if $f^{-1}(V)$ is gsp-closed in $(X, \tau)$ for every open set $V$ in $(Y, \sigma)$.
7. contra g-continuous \([4]\) if $f^{-1}(V)$ is g-closed in $(X, \tau)$ for every open set $V$ in $(Y, \sigma)$.
8. contra $g^*$-continuous \([14]\) if $f^{-1}(V)$ is $g^*$-closed in $(X, \tau)$ for every open set $V$ in $(Y, \sigma)$.

**Definition 2.5:** A space $X$ is locally indiscrete \([18]\) if every open set in $X$ is closed.

**Definition 2.6:**
(i) A space $(X, \tau)$ is called a $T_{\alpha}^{**}$-space \([16]\) if every gs*-closed set in it is closed.
(ii) A space $(X, \tau)$ is called a $T_{\alpha}^{**}$-space \([16]\) if every gs-closed set in it is gs*-closed.

3. $g_{a}^{**}$-Continuous and $g_{a}^{**}$-Irresolute functions

In this section, the concepts of gs*-continuity and gs*-irresoluteness are introduced and studied.

**Definition 3.1:** A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called gs*-continuous if $f^{-1}(V)$ is gs*-closed set in $(X, \tau)$ for every closed set $V$ in $(Y, \sigma)$.

**Example 3.2:** Let $X = Y = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{\varnothing, Y, \{a, b\}, \{a\}\}$. Then $f: (X, \tau) \rightarrow (Y, \sigma)$ defined by $f(a) = c$, $f(b) = a$, $f(c) = b$ is gs*-continuous.

**Theorem 3.3:**
(1) Every continuous function is gs*-continuous.
(2) Every $\omega$-continuous function is gs*-continuous.
(3) Every $g^*$-continuous function is gs*-continuous.
(4) Every semi-continuous function is gs*-continuous.
(5) Every semi*$\alpha$-continuous function is gs*-continuous.
(6) Every gs*-continuous function is gs-continuous.
(7) Every gs*-continuous function is gsp-continuous.

**Proof:**
(1) Let $V$ be a closed set in $Y$. Since, $f$ is continuous, $f^{-1}(V)$ is closed in $X$. By theorem 3.2 \([16]\), $f^{-1}(V)$ is gs*-closed in $X$ and so $f$ is gs*-continuous.
(2) Similar to the proof of (1).
Remark 3.4: The converses of the above theorems are not be true as seen from the following examples.

**Example 3.5:** Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{c, d\}, \{a, c, d\}\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{b\}\}$. Then $f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = b$, $f(b) = c$, $f(c) = d$, $f(d) = a$ is $g$s*-continuous but not continuous.

**Example 3.6:** Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}, \{b\}\}$ and $\sigma = \{\emptyset, Y, \{a, b\}\}$. Then $f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = f(b) = a$, $f(c) = b$ is $g$s*-continuous but not $\alpha$-continuous.

**Example 3.7:** Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a, b, c\}, \{a\}\}$ and $\sigma = \{\emptyset, Y, \{a, b\}\}$. Then $f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = b$, $f(b) = c$, $f(c) = a$ is $g$s*-continuous but not $g^*$s continuous.

**Example 3.8:** Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b, c\}, \{a\}\}$ and $\sigma = \{\emptyset, Y, \{a, b, c\}\}$. Then $f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = c$, $f(b) = a$, $f(c) = b$ is $g$s*-continuous but not $g$s*-continuous.

**Example 3.9:** Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a, b, c, d\}, \{a, d\}\}$ and $\sigma = \{\emptyset, Y, \{a, b, c\}\}$. Then $f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = f(b) = a$, $f(c) = b$, $f(d) = c$ is $g$s*-continuous but not semi-continuous.

**Example 3.10:** Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}, \{a\}\}$ and $\sigma = \{\emptyset, Y, \{a, b\}\}$. Then $f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = c$, $f(b) = a$, $f(c) = b$ is $g$s*-continuous but not $g$s*-continuous.

**Example 3.11:** Let $X = Y = \{a, b, c\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{b\}\}$. Then $f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = c$, $f(b) = a$, $f(c) = b$ is $g$s*-continuous but not $g$s*-continuous.

**Remark 3.12:** $g$s*-continuous and g-continuous functions are independent of each other.

**Example 3.13:** Let $X = \{a, b, c, d\}$, $Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}, \{a, b\}\}$ and $\sigma = \{\emptyset, Y, \{a, b\}\}$. Then $f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = b$, $f(b) = f(c) = f(d) = a$ is $g$s*-continuous but not $g$s*-continuous.

**Example 3.14:** Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}\}$ and $\sigma = \{\emptyset, Y, \{b\}, \{a, b\}\}$. Then $f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = b$, $f(b) = c$, $f(c) = d$, $f(d) = a$ is $g$s*-continuous but not g-continuous.

**Remark 3.15:** $g$s*-continuous and semi*$\alpha$-continuous functions are independent of each other.

**Example 3.16:** Let $X = \{a, b, c, d\}$, $Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b, c\}\}$ and $\sigma = \{\emptyset, Y, \{b\}, \{a, b\}\}$. Then $f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = f(b) = a$, $f(c) = c$, $f(d) = b$ is $g$s*-continuous but not $g$s*-continuous.

4. Characteristics of $g$s*-continuous functions

**Theorem 4.1:** The following are equivalent for a function $f: (X, \tau) \to (Y, \sigma)$. Assume that $g$s*-O(X, $\tau$) is closed under any union.

(i) $f$ is $g$s*-continuous.

(ii) For each $x \in X$ and each open set $F$ in $Y$ containing $f(x)$, there exists a $g$s*-open set $U$ in $X$ containing $x$ such that $f(U) \subseteq F$.

**Proof:**

(i) $\Rightarrow$ (ii): Let $x \in X$ and $F$ be an open set in $Y$ containing $f(x)$. Since $f$ is $g$s*-continuous, $f^{-1}(F)$ is $g$s*-open in $X$ containing $x$. Take $U = f^{-1}(F)$ then $U$ is a $g$s*-open set in $X$ containing $x$ such that $f(U) \subseteq F$.

(ii) $\Rightarrow$ (i): Let $F$ be an open set in $Y$ such that $x \in f^{-1}(F)$. Then $F$ is an open set containing $f(x)$. By (i), there exists a $g$s*-open set $U_x$ in $X$ containing $x$ such that $f(U_x) \subseteq F$ which implies $U \subseteq f^{-1}(F)$. Therefore $f^{-1}(F) = U \cup \{U_x : x \in f^{-1}(F)\}$. Since $U_x$ is $g$s*-open and $g$s*-O(X, $\tau$) is closed under any union. Hence $f^{-1}(F)$ is open and so $f$ is $g$s*-continuous.

**Theorem 4.2:** A function $f: (X, \tau) \to (Y, \sigma)$ is $g$s*-continuous if and only if $f^{-1}(V)$ is $g$s*-open in $X$ for every open set $V$ in $Y$.

**Proof:** Since $f^{-1}(V) = (f^{-1}(V))^\tau$, proof follows.
Remark 4.3: The composition of two $gs_{**}$-continuous functions is not $gs_{**}$-continuous.

Example 4.4: Let $X = Y = Z = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a\}, \{a, b\}\}$, $\sigma = \{\varnothing, Y, \{a\}, \{a, b\}\}$ and $\mu = \{\varnothing, Z, \{a\}\}$. Then $f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = a, f(b) = c, f(c) = b$ and $g: (Y, \sigma) \to (Z, \mu)$ defined by $g(a) = b, g(b) = a, g(c) = c$. Then $f$ and $g$ are $gs_{**}$-continuous but $g \circ f: (X, \tau) \to (Z, \mu)$ is not $gs_{**}$-continuous.

Theorem 4.5: Let $f: (X, \tau) \to (Y, \sigma)$ and $g: (Y, \sigma) \to (Z, \mu)$ be any functions. Then

(i) $g \circ f: (X, \tau) \to (Z, \mu)$ is $gs_{**}$-continuous if $g$ is continuous and $f$ is $gs_{**}$-continuous.

(ii) $g \circ f: (X, \tau) \to (Z, \mu)$ is $gs_{**}$-continuous if $g$ is continuous and $f$ is $gs_{**}$-continuous.

Proof:

(i) Let $V$ be any closed set in $Z$. Since $g$ is continuous, $g^{-1}(V)$ is closed in $Y$. By Theorem 4.2, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $gs_{**}$-closed in $X$. Hence $g \circ f$ is $gs_{**}$-continuous.

(ii) Similar to the proof of (i).

Theorem 4.6: Let $X$ and $Y$ be any topological spaces and $\sigma$ be a $T_{**}$-space then the following holds.

(i) $g \circ f: (X, \tau) \to (Z, \mu)$ is $gs_{**}$-continuous if $g$ is $gs_{**}$-continuous and $f$ is $gs_{**}$-continuous.

(ii) $g \circ f: (X, \tau) \to (Z, \mu)$ is semi-continuous if $g$ is $gs_{**}$-continuous and $f$ is semi-continuous.

(iii) $g \circ f: (X, \tau) \to (Z, \mu)$ is $gs_{**}$-continuous if $g$ is $gs_{**}$-continuous and $f$ is $gs_{**}$-continuous.

Proof: (i) Let $U$ be any closed set in $Z$. Since $g$ is $gs_{**}$-continuous, $g^{-1}(U)$ is $gs_{**}$-closed in $Y$. By Theorem 3.2, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is $gs_{**}$-closed in $X$ and hence $g \circ f$ is $gs_{**}$-continuous.

(ii)-(iii) similar to the proof of (i).

Theorem 4.7: If a function $f: X \to Y$ is $gs_{**}$-continuous where $X$ is a $T_{**}$-space then $f$ is continuous.(resp. semi-continuous)

Proof: Let $V$ be a closed set in $Y$. Since $f$ is $gs_{**}$-continuous, $f^{-1}(V)$ is $gs_{**}$-closed in $X$. Since $X$ is a $T_{**}$-space, $f^{-1}(V)$ is closed in $X$ and so $f$ is continuous.

Theorem 4.8: If a function $f: X \to Y$ is $gs_{**}$-continuous where $X$ is a $T_{**}$-space then $f$ is $gs$-continuous.

Proof: Let $V$ be a closed set in $Y$. Since $f$ is $gs_{**}$-continuous, $f^{-1}(V)$ is $gs_{**}$-closed in $X$. Since $X$ is a $T_{**}$-space, $f^{-1}(V)$ is closed in $X$ By theorem 3.2[16], $f^{-1}(V)$ is $gs$-closed in $X$ and so $f$ is $gs$-continuous.

Definition 4.9: A function $f: (X, \tau) \to (Y, \sigma)$ is called a $gs_{**}$-irresolute if $f^{-1}(V)$ is $gs_{**}$-closed set in $(X, \tau)$ for every $gs_{**}$-closed set $V$ in $(Y, \sigma)$.

Example 4.10: Let $X = Y = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a\}, \{b, c\} \}$ and $\sigma = \{\varnothing, Y, \{a, b\}\}$. Then $f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = b, f(b) = a, f(c) = c$ is $gs_{**}$-irresolute.

Theorem 4.11:

(1) Every $gs_{**}$-irresolute function is $gs_{**}$-continuous.

(2) Every $gs_{**}$-irresolute function is $gs$-continuous.

(3) Every $gs_{**}$-irresolute function is $gs$-continuous.

Proof:

(1) Let $V$ be a closed set in $Y$. By theorem 3.2[16], $V$ is $gs_{**}$-closed in $Y$. Since $f$ is $gs_{**}$-irresolute, $f^{-1}(V)$ is $gs_{**}$-closed set in $X$ and so $f$ is $gs_{**}$-continuous.

(2)-(3) similar to the proof of (1).

Remark 4.12: The converses of the above theorems are not true as seen from the following example.

Example 4.13: Let $X = Y = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a\}\}$ and $\sigma = \{\varnothing, Y, \{a\}\}$. Then $f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = b, f(b) = a, f(c) = c$, is $gs_{**}$-continuous but not $gs_{**}$-irresolute.

Example 4.14: Let $X = Y = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\sigma = \{\varnothing, Y, \{a\}, \{b, c\}\}$. Then $f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = a, f(b) = b, f(c) = b$, is $gs$-continuous but not $gs_{**}$-irresolute.

Example 4.15: Let $X = Y = \{a, b, c\}$, $\tau = \{\varnothing, X, \{a\}\}$ and $\sigma = \{\varnothing, Y, \{a\}\}$. Then $f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = a, f(b) = c, f(c) = b$, is $gs$-continuous but not $gs_{**}$-irresolute.
Theorem 4.16: Let f: (X, τ) → (Y, σ) and g: (Y, σ) → (Z, μ) be any functions. Then the following holds.
(i) g ◦ f: (X, τ) → (Z, μ) is gs_α**-irresolute if g is gs_α**-irresolute and f is gs_α***-irresolute.
(ii) g ◦ f: (X, τ) → (Z, μ) is gs_α**-continuous if g is gs_α**-continuous and f is gs_α**-irresolute.

Proof:
(i) Let V be gs_α**-irresolute in Z. Then g_1(V) is gs_α**-closed in Y. Also f is gs_α**-irresolute, f_1(g_1(V)) = (g ◦ f)_1(V) is gs_α**-closed set in X. Hence g ◦ f is gs_α**-irresolute.
(ii) Similar to the proof of (i).

Theorem 4.17: A function f: (X, τ) → (Y, σ) is gs_α**-irresolute if and only if f_1(V) is gs_α**-open in X for every gs_α**-open set V in Y.

Proof: Since f_1(V^c) = (f_1(V))^c, the proof follows.

Theorem 4.18: If a function f: X → Y is gs_α**-continuous where X is a _c _T_x-**-space then f is gs_α**-irresolute.

Proof: Let U be a gs_α**-closed set in Y. Since Y is a _c _T_x-**-space, then U is closed in Y. By theorem 3.2 [16], U is gs_α**-closed set in Y. Since f is gs_α**-irresolute, f_1(U) is gs_α**-closed in X and so f is gs_α**-irresolute.

Theorem 4.19: Let X and Z be any topological spaces and Y be a _c _T_x-**-space then g ◦ f: (X, τ) → (Z, μ) is gs_α**-continuous if g is gs_α**-irresolute and f is gs_α**-continuous.

Proof: Let U be any closed set in Z. Since g is gs_α**-irresolute, g_1(U) is gs_α**-closed in Y. But X is a _c _T_x-**-space which implies g_1(U) is closed in Y. Since f is gs_α**-continuous, f_1(g_1(U)) = (g ◦ f)_1(U) is gs_α**-closed in X and hence g ◦ f is gs_α**-continuous.

Theorem 4.20: Let X and Z be any topological spaces and Y be a _c _T_x-**-space then g ◦ f: (X, τ) → (Z, μ) is gs_α**-continuous if g is gs-continuous and f is gs_α**-irresolute.

Proof: Let U be any closed set in Z. Since g is gs-continuous, g_1(U) is gs-closed in Y. But Y is a _c _T_x-**-space implies g_1(U) is closed in Y. Since f is gs_α**- -irresolute, f_1(g_1(U)) = (g ◦ f)_1(U) is gs_α**-closed in X. Consequently g ◦ f is gs_α**-continuous.

5. Contra gs_α**-continuous functions

In this section, we define contra gs_α**-continuous functions and derives some of their properties.

Definition 5.1: A function f: X → Y is said to be contra gs_α**-continuous if f_1(V) is gs_α**-closed in X for every open set V in Y.

Example 5.2: Let X = Y = {a, b, c}, τ = {∅, X, {a}, {b, c}} and σ = {{∅, Y, {a}}. Then f: (X, τ) → (Y, σ) defined by f(a) = c, f(b) = a, f(c) = b is a contra gs_α**-continuous.

Theorem 5.3: The following are equivalent for a function f: (X, τ) → (Y, σ).
Assume that gs_α**O(X, τ) is closed under any union.
(1) f is contra gs_α**-continuous.
(2) For every closed set F of Y, f_1(F) is gs_α**-open in X.
(3) For each x ∈ X and each closed set F of Y containing f(x), there exists gs_α**-open set U containing x in X such that f(U) ⊆ F.

Proof:
(1) ⇒(2): Let F be a closed set in Y. Then Y−F is an open set in Y. By (1), f_1(Y−F) = X − f_1(F) is gs_α**-closed in X, which implies f_1(F) is gs_α**-open in X.

(2) ⇒(1): Similar to the proof of (1).

(2) ⇒(3): Let F be a closed set in Y containing f(x). Then x ∈ f_1(F). By (2), f_1(F) is gs_α**-open in X containing x.

Let U = f_1(F). Then U is gs_α**-open in X containing x and f(U) = f(f_1(F)) ⊆ F.

(3) ⇒(2): Let F be a closed set in Y containing f(x) which implies x ∈ f_1(F). From (3), there exists gs_α**-open set U_x in X containing x such that f(U_x) ⊆ F which implies U_x ⊆ f_1(F). Therefore f_1(F) = U_x ∪ {U_x : x ∈ f_1(F)} and since U_x is gs_α**-open and gs_α**O(X, τ) is closed under any union, f_1(F) is gs_α**-open in X.
Remark 5.4: Composition of two contra gs**-continuous functions is not contra gs**-continuous.

Example 5.5: $X = \{a, b, c, d\}, Y = Z = \{a, b, c\}, \tau = \{\emptyset, X, \{a, b\}, \{a, b, c\}, \{a, b, d\}\},$ and $\sigma = \{\emptyset, Y, \{a\}, \{a, b\}\}$ and $\mu = \{\emptyset, Z, \{a\}\}.$ Then $f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = f(b) = c, f(c) = b,$ and $g: (Y, \sigma) \to (Z, \mu)$ defined by $g(a) = b, g(b) = c,$ and $g(c) = a$ are gs**-continuous but $g \circ f: (X, \tau) \to (Z, \mu)$ is not gs**-continuous.

Theorem 5.6:
(i) Every contra-continuous function is contra gs**-continuous.
(ii) Every contra semi-continuous function is contra gs**-continuous.
(iii) Every contra semi*-continuous function is contra gs**-continuous.
(iv) Every contra gs**-continuous function is contra gs-continuous.
(v) Every contra gs**-continuous function is contra gsp-continuous.

Proof:
(i) Let $V$ be any open set in $Y.$ Since $f$ is contra-continuous, $f^{-1}(V)$ is closed in $X.$ By theorem 3.2[16], $f^{-1}(V)$ is gs**-closed in $X.$ Hence $f$ is contra gs**-irresolute.
(ii) - (v). Similar to the proof of (i).

Remark 5.7: The converses of the above theorems are not true as seen from the following examples.

Example 5.8: Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{a, b\}\}. Then f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = c, f(b) = a, f(c) = b$ is contra gs**-continuous but not contra-continuous.

Example 5.9: Let $X = \{a, b, c, d\}, Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{a, b\}\}. Then f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = f(d) = b, f(c) = d, f(d) = c$ is contra gs**-continuous but not contra semi*-continuous.

Example 5.10: Let $X = \{a, b, c, d\}, Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a, b\}, \{a, c, d\}\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{a, b\}\}. Then f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = b, f(b) = a, f(c) = d, f(d) = c$ is contra gs**-continuous but not contra semi*-continuous.

Example 5.11: Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{a, b\}\}. Then f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = b, f(b) = a, f(c) = c$ is contra gs-continuous but not contra gs**-continuous.

Example 5.12: Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{b\}\}$ and $\sigma = \{\emptyset, Y, \{a, b\}\}. Then f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = c, f(b) = a, f(c) = b$ is contra gsp-continuous but not contra gs**-continuous.

Remark 5.13: From the above results we have the following diagram.

In the above diagram $A \to B$ denotes $A$ implies $B$ but not conversely.

Remark 5.14: Contra g-continuous function and contra gs**-continuous functions are independent of each other.

Example 5.15: Let $X = \{a, b, c, d\}, Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{d\}, \{a, d\}, \{c, d\}, \{a, c, d\}\}$ and $\sigma = \{\emptyset, Y, \{a, b\}\}. Then f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = f(b) = a, f(c) = d, f(d) = b$ is contra g-continuous but not contra gs**-continuous.

Example 5.16: Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\sigma = \{\emptyset, Y, \{a, b\}\}. Then f: (X, \tau) \to (Y, \sigma)$ defined by $f(a) = c, f(b) = a, f(c) = b$ is contra gs**-continuous but not contra g-continuous.
Remark 5.17: Contra gs*-continuous function and contra semi*-α-continuous functions are independent of each other.

Example 5.18: Let \( X = \{a, b, c, d\}, Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \) and \( \sigma = \{\emptyset, Y, \{a\}, \{b\}\} \). Then \( f: (X, \tau) \rightarrow (Y, \sigma) \) defined by \( f(a) = f(d) = b, f(b) = a, f(c) = c \) is contra gs*-continuous but not contra semi*-α-continuous.

Example 5.19: Let \( X = \{a, b, c, d\}, Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \) and \( \sigma = \{\emptyset, Y, \{a, b\}\} \). Then \( f: (X, \tau) \rightarrow (Y, \sigma) \) defined by \( f(a) = f(c) = a, f(a) = c, f(d) = b \) is contra semi*-α-continuous but not contra gs*-continuous.

Theorem 5.20:
(i) If \( f: X \rightarrow Y \) is gs*-continuous and \( h: Y \rightarrow Z \) is contra-continuous then \( h \circ f: X \rightarrow Z \) is contra gs*-continuous.
(ii) If \( f: X \rightarrow Y \) is contra gs*-continuous and \( h: Y \rightarrow Z \) is continuous then \( h \circ f: X \rightarrow Z \) is contra gs*-continuous.
(iii) If \( f: X \rightarrow Y \) is contra gs*-continuous and \( h: Y \rightarrow Z \) is contra-continuous then \( h \circ f: X \rightarrow Z \) is contra gs*-continuous.

Proof:
(i) Let \( V \) be an open set in \( Z \). Since \( h \) is contra-continuous, \( h^{-1}(V) \) is closed in \( Y \). If \( f \) is gs*-continuous, \( f^{-1}(h^{-1}(V)) = (h \circ f)^{-1}(V) \) is gs*-closed in \( X \) and hence \( h \circ f \) is gs*-continuous.
(ii) - (iii) Similar to the proof of (i).

Remark 5.21: The concept of gs*-continuity and contra gs*-continuity are independent.

Example 5.22: Let \( X = Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{a, b\}\} \) and \( \sigma = \{\emptyset, Y, \{a\}, \{b\}\} \). Then \( f: (X, \tau) \rightarrow (Y, \sigma) \) defined by \( f(a) = b, f(b) = a, f(c) = c \) is contra gs*-continuous but not gs*-continuous.

Example 5.23: Let \( X = Y = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{b\}\} \) and \( \sigma = \{\emptyset, Y, \{a\}, \{b\}\} \). Then \( f: (X, \tau) \rightarrow (Y, \sigma) \) defined by \( f(a) = b, f(b) = a, f(c) = c \) is gs*-continuous but not contra gs*-continuous.

Theorem 5.24: If \( f: (X, \tau) \rightarrow (Y, \sigma) \) is gs*-irresolute and \( g: (Y, \sigma) \rightarrow (Z, \mu) \) is a contra gs*-continuous function then \( g \circ f: X \rightarrow Y \) is contra gs*-continuous.

Proof: Let \( V \) be an open set in \( Z \). Since \( g \) is contra gs*-continuous, \( g^{-1}(V) \) is gs*-closed in \( Y \). Since \( f \) is gs*-irresolute, \( f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V) \) is gs*-closed in \( X \) and hence \( g \circ f \) is contra gs*-continuous.

Theorem 5.25: If a function \( f: X \rightarrow Y \) is contra gs*-continuous and \( Y \) is regular, then \( f \) is gs*-continuous.

Proof: Let \( x \in X \) and \( V \) be an open set in \( Y \) containing \( f(x) \). Since \( Y \) is regular there exists an open set \( W \) in \( Y \) containing \( f(x) \) such that \( \text{cl}(W) \subseteq V \). Since \( f \) is contra gs*-continuous. By theorem 4.1, there exists gs*-open set \( V \) in \( X \) containing \( x \) such that \( f(U) \subseteq \text{cl}(W) \). Then \( f(U) \subseteq \text{cl}(W) \subseteq V \). Therefore \( f \) is gs*-continuous.

Theorem 5.26: Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a function and \( X \) is a \( s_{T_{s^*}} \)-space. Then the following are equivalent.
(i) \( f \) is conra semi-continuous.
(ii) \( f \) is contra gs*-continuous.

Proof:
(i) \( \Rightarrow \) (ii): By theorem 5.6, proof follows.
(ii) \( \Rightarrow \) (i): Let \( V \) be any open set in \( Y \). Since \( f \) is contra gs*-continuous, \( f^{-1}(V) \) is gs*-closed in \( X \). Since \( X \) is a \( s_{T_{s^*}} \)-space, \( f^{-1}(V) \) is closed in \( X \) and hence \( f^{-1}(V) \) is semi-closed in \( X \) if \( f \) is contra semi-continuous.

Theorem 5.27: Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be a function and \( X \) is a \( s_{T_{s^*}} \)-space. Then the following are equivalent.
(i) \( f \) is contra gs*-continuous.
(ii) \( f \) is contra gs-continuous.

Proof: Similar to the proof of theorem 5.26.

Theorem 5.28: If \( f \) is gs*-continuous and if \( Y \) is locally indiscrete then \( f \) is contra gs*-continuous.

Proof: Let \( V \) be an open set in \( Y \). Since \( Y \) is locally indiscrete, \( V \) is closed in \( X \). Since \( f \) is gs*-continuous, \( f^{-1}(V) \) is gs*-closed in \( X \) hence \( f \) is contra gs*-continuous.
Theorem 5.29: If a function $f: (X, \tau) \to (Y, \sigma)$ is continuous and $X$ is locally indiscrete then $f$ is contra $g_{s\alpha}$ continuous.

Proof: Let $V$ be an open set in $(Y, \sigma)$. Since $f$ is continuous, $f^{-1}(V)$ is open in $X$. Since $X$ is locally indiscrete, $f^{-1}(V)$ is closed set in $X$. By theorem 3.2, $f^{-1}(V)$ is $g_{s\alpha}$-closed in $X$ and hence $f$ is contra $g_{s\alpha}$ continuous.

Theorem 5.30: If a function $f: (X, \tau) \to (Y, \sigma)$ is contra $g_{s\alpha}$ continuous and $X$ is a $\sigma_f$-space then $f: (X, \tau) \to (Y, \sigma)$ is contra $g_{s\alpha}$ continuous.

Proof: Let $V$ be an open set in $Y$. Since $f$ is contra $g_{s\alpha}$-continuous, $f^{-1}(V)$ is $g_{s\alpha}$-closed in $X$. Since $X$ is $\sigma_f$-space, $f^{-1}(V)$ is closed and $g_{s\alpha}$-closed in $X$ and hence $f$ is contra $g_{s\alpha}$-continuous.

Definition 5.31: A space $X$ is called locally $g_{s\alpha}$-indiscrete if every $g_{s\alpha}$-open set is closed in $X$.

Theorem 5.32: If a function $f: (X, \tau) \to (Y, \sigma)$ is $g_{s\alpha}$-continuous and the space $X$ is locally $g_{s\alpha}$-indiscrete then $f$ is contra continuous.

Proof: Let $V$ be an open set in $Y$. Since $f$ is $g_{s\alpha}$-continuous, $f^{-1}(V)$ is $g_{s\alpha}$-open in $X$. Since $X$ is locally $g_{s\alpha}$-indiscrete, $f^{-1}(V)$ is closed in $X$ and by theorem 3.2, $f^{-1}(V)$ is $g_{s\alpha}$-closed in $X$. Consequently $f$ is contra $g_{s\alpha}$-continuous.

Theorem 5.33: If a function $f: (X, \tau) \to (Y, \sigma)$ is $g_{s\alpha}$-irresolute where $Y$ is a locally $g_{s\alpha}$-indiscrete space and $g: (Y, \sigma) \to (Z, \mu)$ is contra $g_{s\alpha}$-continuous function then $g \circ f$ is $g_{s\alpha}$-continuous.

Proof: Let $V$ be any closed set in $Z$. Since $g$ is contra $g_{s\alpha}$-continuous, $g^{-1}(V)$ is $g_{s\alpha}$-open in $Y$. But $Y$ is locally $g_{s\alpha}$-indiscrete implies $g^{-1}(V)$ is closed in $Y$. By theorem 3.2, $g^{-1}(V)$ is $g_{s\alpha}$-closed in $Y$. Since $f$ is $g_{s\alpha}$-irresolute, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $g_{s\alpha}$-closed in $X$ and hence $g \circ f$ is $g_{s\alpha}$-continuous.

Theorem 5.34: If a function $f: (X, \tau) \to (Y, \sigma)$ is $g_{s\alpha}$-continuous and the space $(X, \tau)$ is locally $g_{s\alpha}$-indiscrete space then $f$ is contra $g_{s\alpha}$-continuous.

Proof: Let $V$ be any open set in $(Y, \sigma)$. Since $f$ is $g_{s\alpha}$-continuous, $f^{-1}(V)$ is $g_{s\alpha}$-open in $X$. Since $X$ is locally $g_{s\alpha}$-indiscrete, $f^{-1}(V)$ is closed in $X$. By theorem 3.2, $f^{-1}(V)$ is $g_{s\alpha}$-closed set in $X$ and hence $f$ is contra $g_{s\alpha}$-continuous.

6. Contra $g_{s\alpha}$-closed graph

Definition 6.1: The graph $G(f)$ of a function $f: X \to Y$ is said to be $g_{s\alpha}$-closed (resp.contra $g_{s\alpha}$-closed) if for each $(x, y) \in (X \times Y) - G(f)$, there exist an $U \in g_{s\alpha}$-O(X,x) and an open (resp. closed) set $V$ in $Y$ such that $(U \times V) \cap G(f) = \emptyset$.

Lemma 6.2: A function $f: X \to Y$ is $g_{s\alpha}$-closed (resp.contra $g_{s\alpha}$-closed) if for each $(x, y) \in (X \times Y)$, $G(f)$ there exists $U \in g_{s\alpha}$-O(X,x) and an open (resp. closed) set $V$ in $Y$ containing $y$ such that $f(U) \cap V = \emptyset$.

Proof: We shall prove that $f(U) \cap V = \emptyset$ if $(U \times V) \cap G(f) = \emptyset$. Let $(U \times V) \cap G(f) \neq \emptyset$. Then there exists $(x, y) \in (U \times V)$ and $(x, y) \in G(f)$ which implies $x \in U, y \in V$ and $y = f(x) \in V$. Therefore $f(U) \cap V \neq \emptyset$.

Theorem 6.3: If a function $f: X \to Y$ is $g_{s\alpha}$-continuous and $Y$ is a $T_1$-space then $G(f)$ is contra $g_{s\alpha}$-closed in $X \times Y$.

Proof: Let $(x, y) \in (X \times Y)-G(f)$. Then $y \neq f(x)$. Since $Y$ is $T_1$, there exists an open set $V$ of $Y$ such that $f(x) \in V, y \not\in V$. Since $f$ is $g_{s\alpha}$-continuous, by theorem 4.1 there exists a $g_{s\alpha}$-open set $U$ of $X$ containing $x$ such that $f(U) \subset V$. Therefore $f(U) \cap (Y - V) = \emptyset$ where $Y - V$ is closed in $Y$ containing $y$. By lemma 6.2, $G(f)$ is a $g_{s\alpha}$-closed graph in $X \times Y$.

Theorem 6.4: Let $f: X \to Y$ be a function and $g: X \times Y$ be the graph of $f$ defined by $g(x) = (x, f(x))$ for every $x \in X$. If $g$ is contra $g_{s\alpha}$-continuous, then $f$ is contra $g_{s\alpha}$-Continuous.

Proof: Let $U$ be an open set in $Y$, then $X \times U$ is an open set in $X \times Y$. Since $g$ is contra $g_{s\alpha}$-continuous, $f^{-1}(U) = g^{-1}(X \times U)$ is $g_{s\alpha}$-closed in $X$. Thus $f$ is contra $g_{s\alpha}$-continuous.
Definition 6.5:

(i) $\alpha$-T$_0$ if for every pair of distinct points $x$, $y$ in $X$ there exists a $\alpha$-open set $U$ containing one of the points but not the other.

(ii) $\alpha$-T$_1$ if for every pair of distinct points $x$, $y$ in $X$ there exists a $\alpha$-open set $U$ containing $x$ not $y$ and a $\alpha$-open set $V$ containing $y$ but not $x$.

(iii) $\alpha$-T$_2$ if for every pair of distinct points $x$, $y$ in $X$ there exists disjoint $\alpha$-open sets $U$ and $V$ containing $x$ and $y$ respectively.

Theorem 6.6: If $f : (X, \tau) \rightarrow (Y, \sigma)$ is an injective function with the $\alpha$-closed graph $G(f)$ then $X$ is $\alpha$-T$_1$.

Proof: Let $x$ and $y$ be two distinct points of $X$, then $f(x) \neq f(y)$. Thus $(x, f(y)) \in X \times Y - G(f)$. Since $G(f)$ is $\alpha$-closed, there exists a $\alpha$-open set $U$ containing $x$ and an open set $V$ containing $f(y)$ such that $f(U) \cap V = \emptyset$. By theorem 3.2 \cite{16}, $U$ and $V$ are $\alpha$-open sets containing $x$ and $y$ such that $f(U) \cap f(V) = \emptyset$. Hence $y \notin U$. Similarly there exist $\alpha$-open sets $M$ and $N$ containing $y$ and $f(x)$ such that $f(M) \cap N = \emptyset$. Hence $x \notin M$. It follows that $X$ is $\alpha$-T$_1$.

Theorem 6.7: If $f : (X, \tau) \rightarrow (Y, \sigma)$ is a surjective function with the $\alpha$-closed graph $G(f)$ then $Y$ is $\alpha$-T$_1$.

Proof: Let $y$ and $z$ be two distinct points of $Y$. Since $f$ is surjective there exist a point $x$ in $X$ such that $f(x) = z$. Therefore $(x, y) \notin G(f)$, by lemma 6.2, there exists a $\alpha$-open set $U$ containing $x$ and an open set $V$ containing $y$ such that $f(U) \cap V = \emptyset$. By theorem 3.2 \cite{16}, $U$ and $V$ are $\alpha$-open sets containing $x$ and $y$ such that $f(U) \cap V = \emptyset$. It follows that $z \notin V$. Similarly there exist $w \in X$ such that $f(w) = y$. Hence $f(w) \notin G(f)$. Similarly there exist $\alpha$-open sets $M$ and $N$ containing $w$ and $z$ respectively such that $f(M) \cap N = \emptyset$. Thus $y \notin N$. Hence the space $Y$ is $\alpha$-T$_1$.

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