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# A NOTE ON EQUIVALENT DEFINITIONS OF A GROUP

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***ABSTRACT***

*The equivalent definition actually specifies a constant to be called the identity element (neutral element), and a unary operation that plays the role of the inverse map. To show the equivalence, we really need to show that the identity element and inverse map of a group are already uniquely determined by the binary operation.*

## 1. INTRODUCTION

In mathematics, a **group** is an algebraic structure consisting of a set of elements equipped with an operation that combines any two elements to form a third element. The operation satisfies four conditions called the group axioms, namely closure, associativity, identity and invertibility.

The main difference is that the above definition (textbook definition) only postulates *existence* of an identity element (neutral element) and inverses, but does not include them as part of the group structure.

**1.1 Definition:** Let * be a binary operation defined on G. an element $e \in G$ is called a left identity if $e * a = a$ for all $a \in G$. Then e is called a right identity if $a * e = a$ for all $a \in G$.

**1.2 Example**

1. In C we define $z \circ z = |z\ |z$. Here all elements z such that $|z| = 1$ are left identities.
2. In R we define $a * b = ab^2$. Here 1 and -1 are right identities.
3. In N we define $a * b = a$. Here every element is a right identity.

**1.3 Definition:** Let * be a binary operation defined on G. Let $e \in G$ be the identity element. Let $a \in G$. An element $a' \in G$ is called a left inverse of a if $a' * a = e$. a' is called a right inverse of a if $a * a' = e$.

**1.4 Note:** The identity element e of a group G is both a left identity and a right identity. The inverse of any element $a \in G$ is both a left inverse and a right inverse.

**1.5 Theorem:** Let G be a non empty set with an associative binary operation defined on it such that there exists a left identity e in G and each element $a \in G$ has a left inverse a' with respect to e. Then G is a group.

**Proof:**
a' is a left inverse of a so that $a'a = e$.

let a'' be a left inverse of a' so that $a''a' = e$

then $aa' = e(aa')$ [since e is left identity]
$= (a''a')(aa')$
$= a''(a'a)a'$ [associative]
$= a''(ea')$
$= a''a'$ [since e is left identity]
$= e.$

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Hence, a’ is also a right inverse of a.

Also, a = ea = (aa’)a = a(a’a) = ae.

Hence, e is also a right identity.

Thus ea = a = ae and a’a = aa’ = e and for all $a \in G$. Hence G is a group.

**1.6 Theorem:** Let G be a non empty set with an associative binary operation defined on it such that there exists a right identity e in G and each element $a \in G$ has a right inverse a’ with respect to e. Then G is a group.

The proof is Similar to previous theorem.

**1.7 Note:** If G is a non empty set with an associative binary operation * defined on it such that there exists a left identity and a right inverse for each element, then (G, *) need not be a group.

For example, consider (R, *) where a * b = |a|b.

Clearly * is a binary operation on $R^{*}$.

Now, a * (b*c) = (a * b) * c = |a||b|c and hence * is associative.
(-1) * a = |-1|a = a

Hence -1 is a left identity.

Now, when a < 0;
a * (1/a) = |a|(1/a) = (-a)(1/a) = -1 and

when a > 0;
a * (-1/a) = |a|(-1/a) = (a) (-1/a) = -1.

Hence if a < 0, (1/a) is the right inverse of a and if a>0, (-1/a) is the right inverse of a. However ($R^*$, *) is not a group since the equation y * a = a has two solutions namely 1 and -1.

**1.8 Theorem:** Let G be a non empty set with an associative binary operation defined on it such that the equation ax = b and ya = b have unique solutions for x and y in G. Then G is a group.

**Proof:**
Let $a \in G$. Then there exists a unique $e \in G$ such that ea = a.

Now, let b be any other element in G. then there exists a unique x in G such that ax = b

Now, eb = e(ax) = (ea)x = ax = b
eb = b for all $b \in G$ so that e is a left identity.

Let $a \in G$. Then ya = a has a unique solution a’.
a'a = e so that a’ is the left inverse of a.

Hence by theorem 4.1.5, G is a group.

**1.9 Theorem:** Let G be a finite set with an associative binary operation defined on G in which both cancellation laws hold good. Then G is group.

**Proof:**
Let G = { $a_1, a_2$, ……, $a_n$ }

Now let $a, b \in G$

Consider the elements $aa_1$, $aa_2$, ……..$aa_n$.

All these elements are distinct, for if $aa_r = aa_s$ then $a_r = a_s$ (by cancellation law).

 

Hence $aa_1, aa_2, \ldots\ldots aa_n$ are just the elements $a_{1,} a_2, \ldots\ldots a_n$ of G in some order and hence $aa_i = b$ for some i.

Thus the equation ax = b has a unique solution for x in G. Similarly taking the elements $aa_1, aa_2, \ldots\ldots aa_n$ we can prove that the equation ya = b has a unique solution for y in G.

Hence by previous theorem, G is a group.

**1.10 Note:** The above theorem is not t rue if G is infinite. For example, consider (N, +). Clearly + is an associative binary operation defined on N and both cancellation laws holf good in N.

But (N, +) is not a group.

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