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pg**- Connected space

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ABSTRACT

*In this paper we introduce pg**- connected space, pg**-component, pg**- connected modulo I space andestablish results about the relation between them.*

Key words: *pg**- connected space, pg**-component, pg**- connected modulo I space.*

1. INTRODUCTION

Levine [3] introduced the class of g-closed sets in 1970. Veerakumar[7] introduced g*-closed sets. P M Helen [5] introduced g**-closed sets. A.S.Mashhour, M.E Abd El. Monsef and S.N.EI.Deeb [5] introduced a new class of pre-open sets in 1982. Ideal topological spaces have been first introduced by K.Kuratowski [2] in 1930. The purpose of this paper is to introduce pg**- connected space, pg**-component and pg**- connected modulo I space and investigate their properties.

2. PRELIMINARIES

Definition 2.1: A subset A of a topological space (X,τ) is called a pre-open set [4] if $A \subseteq int(cl(A)$ and a pre-closed set if $cl(int(A)) \subseteq A$.

Definition 2.2: A subset A of topological space (X,τ) is called
1. generalized closed set (g-closed) [3] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ).
2. g*-closed set [7] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in (X,τ).
3. g**-closed set [5] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g*-open in(X,τ).
4. pg**- closed set[6] if $pcl(A) \subseteq U$whenever $A \subseteq U$ and U is g*-open in(X,τ).

Definition 2.3: A function $f: (X,\tau) \rightarrow (Y,\sigma)$ is called
1. pg**-irresolute[6] if $f^{-1}(V)$ is a pg**-closed set of (X,τ) for every pg**-closed set V of (Y,σ).
2. pg**-continuous[6] if $f^{-1}(V)$ is a pg**-closed set of $(X,\tau)$for every closed set V of (Y,σ).
3. pg**-resolute[6] if $f(U)$ is pg**- open in Y whenever U is pg**- open in X.

Definition 2.4: An ideal [2] I on a nonempty set X is a collection of subsets of X which satisfies the following properties.(i) $A \in I, B \in I \Rightarrow A \cup B \in I$ $(ii) A \in I, B \subset A \Rightarrow B \in I$. A topological space (X,τ) with an ideal Ion X is called an ideal topological space and is denoted by (X,τ,I).

3. pg**- Connected space

Definition 3.1: Let X be a topological space. A pg**-separation of X is a pair A and B of disjoint nonempty pg**- open subsets of X whose union is X. The space X is said to be pg**- Connected if there does not exist a pg**-separation of X. If there exist a pg**-separation then X is said to be pg**-disconnected.

Note: If $X = A \cup B$ is a pg**-separation then $A^c = B$ and $B^c = A$ and hence A and B are pg**- closed.

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Remark 3.2: A space X is pg**- connected if and only if the only subsets of X that are both pg**- open and pg**- closed in X are the empty set and X itself.

Proof is obvious.

Example 3.3: An infinite set with finite complement topology is pg**- connected since it is impossible to find two disjoint pg**- open sets.

Example 3.4: Any indiscrete topological space $(X, \tau)$with more than one point is pg**- disconnected since every subset is pg**- open.

Theorem 3.5: Every pg**- connected space is connected but not conversely.

Proof: Obvious, since every open set is pg**- open.

Theorem 3.6: Every pg**- connected space is g**- connected but not conversely.

Proof: Obvious, since every g**- open set is pg**- open.

Example 3.7: The space in example (3.4) is connected but not pg**- connected.

Example 3.8: The space $X = \{a, b, c\}$ with topology $\tau = \{\varphi, X, \{a, c\}\}$ is g**-connected but not pg**- connected.

Example 3.9: $\mathbb{R}$ withusualtopology is connected and g**- connected but not pg**- connected.

Since $\mathbb{Q}$ and $\mathbb{Q}^c$ are pg**- open but not open and g**- open.

Theorem 3.10: Let(X, τ) be a topological space. The following conditions are equivalent:

(i) X is pg**- connected.

(ii) If A and B are disjoint pg**- open subsets of X with $X = A \cup B$, then either $A = \varphi$ (hence $B = X$) or $B = \varphi$ (hence $A = X$).

(iii) If C and D are disjoint pg**- closed subsets of X with $X = C \cup D$, then either $C = \varphi$ (hence $D = X$) or $D = \varphi$ (hence $C = X$).

Proof:
$(\boldsymbol{i}) \Longrightarrow (\boldsymbol{ii})$: Let X be pg**- connected and let A and B be pg**- open subsets of X with $X = A \cup B$ and $A \cap B = \varphi$. Since $A = X \setminus B$, A is also pg**- closed, so either $A = \varphi$ or $A = X$, (ii) follows.

$(\boldsymbol{ii}) \Longrightarrow (\boldsymbol{i})$: Assume (ii) and let G be a subset of X which is both pg**- open and pg**- closed and hence $X \setminus$ G is also both pg**- open and pg**- closed. Since $X = G \cup X \setminus$ G, (ii) gives that either $G = \varphi$ or $G = X$.

$(\boldsymbol{ii}) \Leftrightarrow (\boldsymbol{iii})$: This follows from the fact that if A and B are disjoint pg**- open sets with $X = A \cup B$, then A and B are also pg**- closed. Similarly if A and B are disjoint pg**- closed sets with $X = A \cup B$, then A and B are also pg**- open.

Definition 3.11: Let Y be a subset of a topological space X. A pg**- separation of Y is a pair of disjoint nonempty pg**- open subsets$A$ and B of X whose union is Y. The space Y is said to be pg**- connected if there does not exist a pg**- separation of.Y is said to be pg**- disconnected if there exist a pg**- separation of Y.

Theorem 3.12: If the sets A and B form a pg**-separation of X, and if Y is a pg**- open and pg**- connected subset of X, then Y lies entirely within either A or B.

Proof: $X = A \cup B$ is a pg**- separation of X. Suppose Y intersects both A and B then
$Y = (A \cap Y) \cup (B \cap Y)$ is a pg**- separation of Y which is a contradiction.

Theorem 3.13: Let C be a pg**-connected subset of a topological space X and let D be a subset such that $C \subset D \subset pg^{**}cl(C)$, then D is pg**-connected.

Proof: Suppose D is pg**-disconnected, then $D = A \cup B$ is a pg**-separation of D. Since C is pg**-connected and $C \subset D = A \cup B$, then either $C \subset A$ or $C \subset B$. To be specific, that C is disjoint from B. This implies $pg^{**}cl(C) \cap B = \varphi$, and $D \subset pg^{**}cl(C)$. Therefore $D \cap B = \varphi$, this is not true. Hence D is pg**-connected.

Theorem 3.14: Let C be a pg**-connected subset of a topological space X. Then $pg ** cl(c)$ is also pg**-connected.

Proof follows from taking $D = pg ** cl(C)$ in theorem (3.13).

Theorem 3.15: If C is a pg**-dense subset of a topological space (X, τ) and if C is also pg**-connected, then X is pg**-connected.

Proof: Follows from $pg^{**}cl(C) = X$.

Theorem 3.16: Let (X, τ) and (Y, σ) be two topological spaces and $f : (X, \tau) \to (Y, \sigma)$ be a function. Then,

1. f is onto, pg**- continuous and X is pg**- connected $\Longrightarrow$ Y is connected.
2. f is onto, continuous and X is pg**- connected $\Longrightarrow$ Y is connected.
3. f is strongly pg**- continuous and X is connected $\Longrightarrow$ Y is pg**- connected.
4. f is onto and pg**- resolute then Y is pg**- connected $\Longrightarrow$ X is connected.
5. f is a bijection and open then Y is pg**- connected $\Longrightarrow$X is connected.
6. f is onto, pg**- irresolute and X is pg**- connected $\Longrightarrow$ Y is pg**- connected.
7. f is a bijection and pg**- resolute then Y is pg**- connected $\Longrightarrow$ X is pg**- connected.

Proof: (1) Suppose $Y = A \cup B$ is a separation of Y then $X = f^{-1}(Y) = f^{-1}(A) \cup f^{-1}(B)$ is a pg**- separation of X which is a contradiction. Therefore Y is connected.

Proofs for (2) to (7) are similar to the above proof.

Remark 3.17: The property of being "pg**- connected" is a pg**- topological property. This follows from (6) and (7) of theorem (3.16).

Theorem 3.18: A topological space (X, τ) is pg**- disconnected if and only if there exists a pg**- continuous map of X onto discrete two point space $Y = \{0, 1\}$.

Proof: (X, τ) is pg**- disconnected and $Y = \{0, 1\}$ is a space with discrete topology. Let $X = A \cup B$ be a pg**- separation of X. Define $f : X \to Y$ such that $f(A) = 0$ and $f(B) = 1$. Obviously f is onto, pg**- continuous map.

Conversely, let $f : X \to Y$ be pg**- continuous, onto map. Then $X = f^{-1}(0) \cup f^{-1}(1)$ is a pg**- separation of X.

Theorem 3.19: The union of a collection $\{A_\alpha\}$ of pg**- connected subsets of X that have a point p in common is pg**- connected.

Proof: Let $\cup A_\alpha = B \cup C$ be pg**- separation of $\cup A_\alpha$. Then B and C are disjoint non empty pg**- open sets in X. $p \in \cap A_\alpha \Longrightarrow p \in B$ or $p \in C$. Assume that $p \in B$. Then by theorem (3.12), A_α lies entirely within B for all α (since $p \in B$). Therefore C is empty which is a contradiction.

Corollary 3.20: Let $\{A_n\}$ be a sequence of pg**- open pg**- connected subsets of X such that $A_n \cap A_{n+1} \neq \varphi$, for all n. Then $\cup A_n$ is pg**- connected.

Proof: This can be proved by induction on n. By theorem (3.14), the result is true for $n = 2$. Assume that the result to be true when n = k. Now to prove the result when $n = k + 1$. By the hypothesis $\bigcup_{i=1}^{k} A_i$ is pg**- connected. Now $(\bigcup_{i=1}^{k} A_i) \cap A_{k+1} \neq \varphi$. Therefore $\bigcup_{i=1}^{k+1} A_i$ is pg**- connected. By induction hypothesis the result is true for all n.

Corollary 3.21: Let $\{A_\alpha\}_{\alpha \in A}$ be an arbitrary collection of pg**-open pg**-connected subsets of X. Let A be a pg**- open pg**- connected subset of X. If $A \cap A_\alpha \neq \varphi$, forall α then $A \cup (\cup A_\alpha)$ is pg**- connected.

Proof: Suppose that $A \cup (\cup A_\alpha) = B \cup C$ be a pg**- separation of the subset $A \cup (\cup A_\alpha)$. Since $A \subseteq B \cup C$, by theorem (3.10) $A \subseteq B$ or $A \subseteq C$. Without loss of generality assume that $A \subseteq B$. Let $\alpha \in A$ be arbitrary. $A_\alpha \subseteq B \cup C \Longrightarrow A_\alpha \subseteq B$ or $A_\alpha \subseteq C$. But $A \cap A_\alpha \neq \varphi \Longrightarrow A_\alpha \subseteq B$. Since α is arbitrary, $A_\alpha \subseteq B, \forall$. Hence $A \cup (\cup A_\alpha) \subseteq B$, contradicting the fact that C is nonempty. Therefore $A \cup (\cup A_\alpha)$ is pg**- connected.

Definition 3.22: A space (X, τ) is said to be *totally pg**- disconnected* if its only pg**- connected subsets are one point sets.

Example 3.23: Let (X, τ) be an indiscrete topological space with more than one point. Here all subsets are pg**- open. If $A = \{x_1, x_2\}$ then $A = \{x_1\} \cup \{x_2\}$ is a pg**-separation of A. Therefore any subset with more than one point is pg**- disconnected. Hence (X, τ) is totally pg**- disconnected.

Example 3.24: An infinite set with finite complement topology is not totally pg**- disconnected.

Remark 3.25: Totally pg**- disconnectedness implies pg**- disconnectedness.

Definition 3.26: A point $x \in X$ is said to be in pg**-boundary of A $(pg{**}Bd(A))$ if every pg**- open set containing x intersects both A and $X - A$.

Example 3.27: Any infinite subset A of $\mathbb{R}$ whose complement is also infinite has every real number as its pg**- boundary point.

Theorem 3.28: Let (X, τ) be a topological space and let A be a subset of X. If C is pg**- open pg**- connected subset of X that intersects both A and $X - A$ then C intersects $pg{**}Bd(A)$.

Proof: Given that $C \cap A \neq \varphi$ and $C \cap A^c \neq \varphi$. Now $C = (C \cap A) \cup (C \cap A^c)$ is a nonempty disjoint union. Suppose both are pg**- open then it is a contradiction to the fact that C is pg**- connected. Hence either $C \cap A$ or $C \cap A^c$ is not pg**- open. Suppose that $C \cap A$ is not pg**- open. Then there exist $x \in C \cap A$ which is not pg**-interior point of $C \cap A$. Let U be a pg**- open set containing x. Then $U \cap C$ is a pg**- open set containing x and hence $(U \cap C) \cap (C \cap A)^c \neq \varphi$. This implies U intersects both A and A^c and therefore $x \in pg{**}Bd(A)$. Hence $C \cap pg{**}Bd(A) \neq \varphi$.

Next we extend the intermediate value theorem for pg**- connected space.

Theorem 3.29: (Generalisation of Intermediate value theorem) Let $f : X \to \mathbb{R}$ be a pg**- continuous map, where X is pg**- connected space and $\mathbb{R}$ with usual topology. If x, y are two points of X and $a = f(x)$ and $b = f(y)$ then for every real number r between a and b, there exists a point c of X such that $f(c) =$ r.

Proof: Assume the hypothesis of the theorem. Suppose there is no point c of X such that $f(c) =$ r, then $A = (-\infty, r)$ and $B = (r, \infty)$ are disjoint open sets in $\mathbb{R}$ and $X = f^{-1}(A) \cup f^{-1}(B)$ which is a pg**-separation of X, contradicting the fact that X is pg**- connected. Therefore there exists $c \in X$ such that $f(c) =$ r.

Remark 3.30: The above theorem holds even if,

- f is continuous and X is pg**- connected.
- f is pg**- irresolute and X is pg**- connected.
- f is strongly pg**- continuous and X is pg**- connected.
- f is strongly pg**- continuous and X is connected.

4. pg**-components

Definition 4.1: Let(X, τ) be a topological space. Define an equivalence relation on X by setting $x \sim y$ if and only if there exists a pg**- connected subset of X containing both x and y. The equivalence classes are called *pg**- components* of X. pg**- component containing x is denoted by $C_x = \{y \in X / y \sim x\}$.

(i) $x \sim x$, since $\{x\}$ is pg**- connected. Hence $\sim$ is reflexive.

(ii) If $x \sim y$, then there exists a pg**- connected subset of X containing both x and y and hence $y \sim x$. Therefore $\sim$ is symmetric.

(iii) Let $x \sim y$ and $y \sim z$. Then there exists a pg**- connected subset $A$of X containing both x and y and a pg**- connected subset B of X containing both $y$and z. Since A and B are pg**- connected have a point y in common $A \cup B$ is a pg**- connected subset of X containing x, y and z. Therefore $\sim$ is transitive.

Example 4.2: Let (X, τ) be an indiscrete topological space with more than one point. Then each pg**- component of X consists of a single point.

Theorem 4.3: Any two pg**- components are either identical or disjoint.

Proof: Follows from the definition of pg**- component.

Theorem 4.4: The pg**-components of X are pg**-connected subsets of X whose union is X, such that each nonempty pg**-connected subset of X intersects only one of the pg**-components.

Proof: Each pg**- connected subset A of X intersects only one of the pg**- components. For, if A intersects the pg**- components C_1 and C_2 of X, say in points x_1 and x_2 then $x_1 \sim x_2$, this implies $C_1 = C_2$. To prove the pg**- component C is pg**- connected, choose a point $x_0 \in C$.

Now for every $x \in C$, $x_0 \sim x$. Therefore there exists a pg**- connected subset A_x containing x and x_0, implies $A_x \subset C$. Therefore $\underset{x \in C}{\overset{U}{}} A_x = C$. Since A_x are pg**- connected subsets having the point x_0 in common, C is pg**- connected.

Corollary 4.5: C_x is the union of all pg**- connected sets containing x.

Theorem: C_x is the largest pg**- connected set containing x. If there is another pg**- connected subset A of X such that $x \in A$, then $A \subset C_x$.

Proof: Let $t \in A \Rightarrow x, t \in A$, where A is pg**- connected, this implies $t \sim x$. Therefore $t \in C_x$ and hence $\subset C_x$. Hence C_x is the largest pg**- connected set containing x.

Theorem 4.6: Let(X, τ) be a topological space, then the following are true.

(i) Each point in X is contained in exactly one pg**-component of X.
(ii) Each pg**- connected subset of X is contained in a pg**-component of X.
(iii) A pg**- connected subset of X which is pg**-clopen is a pg**-component of X.
(iv) If C is pg**- component of X then $C = pg^{**}cl(C)$. If (X, τ) is a pg**-multiplicative space then every pg**-component is pg**-closed.

Proof:

(i) Let $x \in X$ and consider the collection $\{C_i\}$ of all pg**- connected subsets of X containing x, this collection is non empty since $\{x\}$ itself is pg**- connected. $C = \cup C_i$ is a maximal pg**- connected subset of X which contains x and therefore a pg**-component of X. Suppose C^* is another pg**-component of X containing x, it clearly among the C_i's and is therefore contained in C, since C^* is also pg**-component we must have $C = C^*$.
(ii) A pg**- connected subset of X is contained in the pg**-component which contains any one of its points.
(iii) Let A be a pg**- connected subset of X which is pg**-clopen, then (by (ii)) A is contained in some pg**-component C. If A is a proper subset of C, then $C \cap A$ and $C \cap A^c$ forms a pg**-separation of C which is a contradiction to the fact that C, being a pg**-component, is pg**- connected. Therefore $A = C$.
(iv) If the pg**-component $C \neq pg^{**}cl(C)$ then its pg**-closure $(pg^{**}cl(C))$ is a pg**- connected (3.14) subset of X which properly contains C, this is the contradiction to the maximality of C as pg**- connected subset of X. Hence $C = pg^{**}cl(C)$. If X is a pg**-multiplicative space, then C is pg**-closed.

Theorem 4.7: Let X be a totally pg**- disconnected space. Then $C_x = \{x\}$, where C_x is a pg**-component of x.

Proof: Let X be a totally pg**- disconnected space, and then its only pg**- connected subsets are one point sets. Suppose $y \epsilon C_x$ such that $x \neq y$ then C_x is not pg**- connected which is contradiction to the fact that the pg**-components of X are pg**- connected subsets of X (4.4). Therefore in a totally pg**- disconnected space the pg**-component of $x$is$\{x\}$.

5. pg**- connected modulo *I*

Definition 5.1: Let (X, τ, I) be an ideal topological space then $X = A \cup B$ is said to be pg**-separation modulo I if A and B arenon empty pg**- open sub sets of X such that $A \cap B \in I$.(X, τ, I) is said to be pg**- connected modulo I if there is no pg**-separation modulo I for X.

Definition 5.2: Let Y be a subset of X. $Y = A \cup B$ is said to be pg**-separation modulo I of Y if A and B are non empty pg**- open sub sets of X and $A \cap B \in I$.

If there is no pg**-separation modulo I for Y then we say Y is pg**- connected modulo I subset.

Theorem 5.3: $X = A \cup B$ is pg**-separation of X implies $X = A \cup B$ is a pg**-separation modulo I of X for any ideal I.

Proof: It follows since $\varphi \in I$.

Theorem 5.4: (X, τ, I) is pg**- connected modulo I for some ideal I implies (X, τ) is pg**- connected. Equivalently If (X, τ) is pg**- disconnected then (X, τ, I) is pg**- disconnected modulo I for some ideal I.

Proof follows from theorem (5.3).

Remark 5.5: The converse is false as seen in the following example.

Example 5.6: Let(X, τ) be infinite cofinite topological space and $I = \wp(X)$. Then X is pg**- connected. On the other hand $X - \{x\}, X - \{y\}$ are pg**-open and non empty, and $(X - \{x\}) \cup (X - \{y\})$ is a pg**-separation modulo I of X. Therefore (X, τ, I) is not pg**- connected modulo I.

Theorem 5.7: Let (X, τ, I) be an ideal topological space, $X = A \cup B$ is a pg**-separation modulo I of X and Y is pg**-open pg**- connected subset of X modulo I then Y lies entirely within either A or B.

Proof: $X = A \cup B$ is a pg**-separation of X modulo I .Therefore A and B are non emptypg**- open sets and $A \cap B \in I$.

Now $Y = (Y \cap A) \cup (Y \cap B)$, $(Y \cap A)$ and $(Y \cap B)$ are pg**-open sets and $(Y \cap A) \cap (Y \cap B) = Y \cap (A \cap B) \in I$. If $(Y \cap A)$ and $(Y \cap B)$ are both non empty then $Y = (Y \cap A) \cup (Y \cap B)$ is a pg**-separation of Y modulo I which is a contradiction. Therefore $(Y \cap A) = \varphi$ or $(Y \cap B) = \varphi$ and hence Y lies entirely within either A or B.

Theorem 5.8: Let (X, τ, I) and (Y, σ, J) be two ideal topological spaces and $f : (X, \tau, I) \to (Y, \sigma, J)$ be a bijection where $J = f(I)$, then

1. f is pg**- continuous and X is pg**- connected modulo $I \Longrightarrow Y$ is connected modulo J .
2. f is continuous and X is pg**- connected modulo $I \Longrightarrow Y$ is connected modulo J.
3. f is strongly pg**- continuous and X is connected $\Longrightarrow Y$ is pg**- connected modulo J .
4. f is pg**-resolute then Y is pg**- connected modulo $J \Longrightarrow X$ is connected modulo I.
5. f is a bijection and open then Y is pg**- connected modulo $J \Longrightarrow X$ is connected modulo I .
6. f is pg**- irresolute and X is pg**- connected modulo $I \Longrightarrow Y$ is pg**- connected modulo J.
7. f is pg**-resolute then Y is pg**- connected modulo $J \Longrightarrow X$ is pg**- connected modulo I .

Proof: (1) Assume that Y is not connected modulo J. Let $Y = A \cup B$ be a pg**- separation modulo J. Therefore A and B are nonempty pg**- open subsets of Y such that $A \cap B \in J$. Then $X = f^{-1}(A) \cup f^{-1}(B)$ is a pg**- separation modulo I since $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) \in I$ which is a contradiction. Therefore Y is connected modulo J.

Proofs for (2) to (7) are similar to the above proof.

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