THE OSSERMAN SURFACES

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ABSTRACT.

A *hyper surface $M \subset R^{n+1}$ is pointwise Osserman surface if the eigenvalues of the Jacobi operator $J(X) = R(u,X,X)$, where R is the curvature tensor of M, are pointwise constants, for any tangent vector X in the tangent space M_p, at any point $p \in M$. In this short note we prove that M is Osserman surface if and only if M is locally Euclidean hyper surface or hyper surface of constant sectional curvature.*

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Let (N, g) be a Riemannian manifold with metric tensor g, and curvature tensor R', defined by the formula:

$$R'(X,Y) = \nabla'_X Y - \nabla'_Y X - \nabla'_{[X,Y]},$$

where ∇' is the Levi-Civita connection on *N*, and *X,Y* are arbitrary tangent vector fields in the tangent bundle $\aleph(N)$, on the manifold *N*.

Let (M, g) be an *n*-dimensional Riemannian manifold, which is isometric embedding in (N, g), and let *dim N=n+m*. If we consider locally in a neighborhood U_p, at a point $p \in M$, then we always can choose a smooth intersections $\xi_1, \xi_2, ..., \xi_m$ in the normal bundle $\aleph(M)^\perp$, which are independent vector fields, and which form an orthonormal basis at any point $p \in U_p$. If *X, Y* are smooth vector fields in the tangent bundle $\aleph(M)$, then

$$\left(\nabla'_X Y\right)_p = \left(\nabla_X Y\right)_p + \alpha(X,Y)_p,$$

where $\nabla_X Y$ is the covariant derivation, defined for the Riemannian connection ∇ on the submanifold (M, g), respectively $\alpha(X, Y)$ is the second fundamental form of (M, g). Also

$$\alpha(X,Y) = \sum_{i=1}^{m} h^i(X,Y)\xi_i$$

is the decomposition of $\alpha(X, Y)$, with respect to the orthonormal basis $\xi_1, \xi_2, ..., \xi_m \in M_p$, at a point $p \in M$.

According to the definition, for any smooth vector field $\xi \in \aleph(M)^\perp$, holds

$$(\nabla'_X \xi)_p = -\left(A_\xi(X)\right)_p + D_\xi(X)_p,$$

where A_ξ is a linear symmetric Weingarten operator in M_p, at a point $p \in M$, such that

$$g(A_\xi(X),Y) = g(\alpha(X,Y),\xi).$$

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The following formulae and equations are well known) [3]:

I. The Gauss formula:

$$\nabla'_X Y = \nabla_X Y + \alpha(X,Y) ;$$

II. The Weingarten formula:

$$(\nabla'_X \xi) = -A_\xi(X) + D_\xi(X) ;$$

III. The Gauss equation:

$$g(R'(X,Y,Z)U) = g(R(X,Y,Z)U) + g(\alpha(X,Z)\alpha(Y,U)) - g(\alpha(Y,Z)\alpha(X,U)),$$

where R is the curvature tensor of the Riemannian manifold (M,g), and where X,Y,Z,U are the tangent vector fields in the tangent bundle $\aleph(M)$.

Further we consider the case, when the Riemannian manifold N coincide with Euclidean vector space R^{n+1}, and the Riemannian submanifold M is a hyper surface in R^{n+1}. Then the Gauss equation has the form [3]:

$$R(X,Y,Z) = g(AY,Z)AX - g(AX,Z)AY \quad .$$

In the next we will use the Jacobi operator

$$J(X) = R(u,X,X) \quad ,$$

which is a linear symmetric operator, defined for any unit tangent vector $X \in M_p$, at a point $p \in M$ [1], [2], [4]. Following the terminology in [2] we introduce

Definition 1: *A hyper surface $M \subset R^{n+1}$ is pointwise Osserman surface if the eigenvalues of the Jacobi operator $J(X)$ are pointwise constants, for any tangent vector $X \in M_p$, at any point $p \in M$.*

Let M be a pointwise Osserman hyper surface in the Euclidean vector space R^{n+1}.

Let A be the Weingarten operator in M_p, and let $X_1, X_2,...,X_n$ be the eigenvector basis of A.

Let $\lambda_1, \lambda_2, ... , \lambda_n$ be the eigenvalues, corresponding to the eigenvectors $X_1, X_2,...,X_n$. Then for any indexes $I < j$ $(i, j = 1, 2,...,n)$ holds:

$$R\left(X_i, X_j, X_k\right) = g\left(AX_j, X_k\right)AX_i - g\left(AX_i, X_k\right)AX_j =$$

$$= \begin{cases} 0 & , \quad k \neq i, j; \\ -\lambda_i \lambda_j & , \quad k = i; \\ \lambda_i \lambda_j & , \quad k = j. \end{cases} \tag{1}$$

It is easy to see that the matrix of the Jacobi operator $J(X_1)$, with respect to the orthonormal basis $X_1, X_2,...,X_n$, has the form:

$$\left(J\left(X_1\right)\right) = \begin{pmatrix} \lambda_1 \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 \lambda_3 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 \lambda_4 & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & \lambda_1 \lambda_n \end{pmatrix},$$

where $J\left(X_1\right)$ we consider as a linear symmetric operator in the tangent subspace

$$X_1^\perp = span\left\{\left(X_2, X_3,..., X_n\right) \subset M_p\right\}, p \in M.$$

Similarly we can check that the matrix of the Jacobi operator $J\left(X_2\right)$, with respect to the orthonormal basis $X_1, X_2,...,X_n$, has the form:

$$
\left(J\left(X_2 \right) \right) = \begin{pmatrix} \lambda_2\lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2\lambda_3 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2\lambda_4 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \lambda_2\lambda_n \end{pmatrix},
$$

where $J\left(X_2 \right)$ we consider as a linear symmetric operator in the tangent subspace

$$
X_2^{\perp} = span\left\{ \left(X_1, X_3, \ldots, X_n \right) \subset M_p \right\}, \ p \in M.
$$

From our condition M to be pointwise Osserman hyper surface in R^{n+1}, it follows that the eigenvalues $\lambda_1\lambda_2, \lambda_1\lambda_3, \ldots, \lambda_{1n}$ and $\lambda_2\lambda_1, \lambda_2\lambda_3, \ldots, \lambda_{2n}$ of the Jacobi operators $J\left(X_1 \right)$ and $J\left(X_2 \right)$ coincide, hence

$$
\lambda_1\lambda_s = \lambda_2\lambda_s, \quad s{=}3, 4, \ldots, n. \tag{2}
$$

From this equality it follows that at least one $\lambda_s {=} 0$, for any indexes $s{=}3, 4, \ldots, n$, and then all eigenvalues K_{sj} ($s{\neq}j{=}1, 2, \ldots, n$), of the Jacobi operator $J\left(X_s \right)$, are equal to 0. Since we assume M to be pointwise Osserman hyper surface in R^{n+1}, then all eigenvalues of any Jacobi operator $J\left(X_s \right)$ ($s{=}1, 2, \ldots, n$) are equal to 0. That means that the matrix of the curvature operator $\mathfrak{R}$ on the second exterior product $\wedge^2 M_p$, with respect to the orthonormal 2-vector basis $X_s \wedge X_t$ $\left(s < t, \ s, t = 1, 2, \ldots, n \right) \in \wedge^2 M_p$, is zero matrix, which means that $\mathfrak{R} \equiv 0$. From the last equality it follows that the curvature tensor R of hyper surface M is vanishing, which means that M is locally Euclidean hyper surface in R^{n+1}[3]. If all $\lambda_s \neq 0$, for any indexes $s = 3, 4, \ldots, n$ in the equalities (2), then $\lambda_1{=}\lambda_2$ and if these values are equal to 0, then all eigenvalues of the Jacobi operators $J\left(X_1 \right)$ and $J\left(X_2 \right)$ are equal to 0. Now from the assumption M to be pointwise Osserman hyper surface in R^{n+1}, it follows that all eigenvalues of any Jacobi operator $J\left(X_s \right)$ ($s{=}1,2,\ldots,n$), are equal to 0, and then M is locally Euclidean hyper surface in R^{n+1}, again. If $\lambda_1 = \lambda_2 \neq 0$, then using all Jacobi operators $J\left(X_s \right)$ ($s{=}3,4,\ldots,n$), we get $\lambda_1 = \lambda_2 = \ldots = \lambda_n \neq 0$, which means that M is hyper surface of constant sectional curvature[3]. Thus we prove

Theorem 1: *M is pointwise Osserman hyper surface in* R^{n+1} *if and only if one of the following cases is true*:
1) *M is a locally Euclidean hyper surface*;
2) *M is a hyper surface of constant sectional curvature .*

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