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MATHEMATICAL MODEL OF CRIMINALS AND GUARDS

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ABSTRACT

This outline is concerned with a mathematical model describing the interaction of two sociological species, termed as Criminals and Guards. Using 15 major crime data covering the period 1995 to 2011 from the Delhi crime statistics, a simplest model is developed. We propose a system of two nonlinear first order ordinary differential equations with some parameters. Using the data and parameters the graph of the population fluctuation of the year of the criminals and Guards are drawn. The results suggest that if the Guards population rises, the criminals' population falls rapidly.

Keywords: *nonlinear, stability, pre-predator, equilibrium point, Jacobian, Eigenvalue.*

INTRODUCTION

Crime is a major social concern. All crimes have negative impact on societies. These negative impacts range from lost of lives, lost of properties, national security threats and so on. To prevent these criminal activities the Guards communities are to be engaged in the societies. The Guards include the central and state police service, the Armed Forces, BSF, CBI, CID etc. Police launched many strategies to combat crime. Although criminal activities still are in the increasing level according to the "Crime in India" 2013 published by the National Crime Records Bureau. So there is the need to come out a model which will help the societies to analyze and reduce the criminal activities.

ASSUMPTIONS FOR THE MODEL

Let, the criminal prone society allows Guards to interact with criminals. Then we consider the following assumptions-

1. If there are no Guards (predator), criminals (Prey), will grow exponentially at a rate proportional to their numbers.
2. If there are no criminals, then the Guards will decline at a rate proportional to the Guards population. Because, in criminal free society owner will not bear the cost of Guards population.
3. Criminals- Guards interaction is modeled by mass action terms proportional to the product of the two populations.

DERIVATION OF MODEL

Let $X(t)$ be the number of criminals at time t and $Y(t)$ be the number of Guards at time t and N be the population at time t in the society.

From assumption (1), if $Y(t)=0$, then

$$\frac{d}{dt}(X(t)) = AX(t),\ A>0 \tag{1}$$

Assumption (3) implies that an increase in Guards in a criminal's prone society will reduce the criminals at rates proportional to their products i.e.

$$\frac{d}{dt}(X(t)) = -BX(t)Y(t)\,,\ B>0 \tag{2}$$

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From assumption (2) if $X(t)=0$, then

$$\frac{dY(t)}{dt}=-CY(t),\ C>0 \tag{3}$$

Again from the assumption (3), as the Guards population grows in number to prevent the rapid growth of criminals, the encounter rate between the Guards and criminals are also increases at the rate proportional to the product of both populations.

$$\frac{dY(t)}{dt}=DX(t)Y(t),\ D>0 \tag{4}$$

Equations (1) & (2)

$$\Rightarrow \frac{d}{dt}(X(t))=AX(t)-BX(t)Y(t) \tag{5}$$

Equations (3) & (4)

$$\Rightarrow \frac{d}{dt}(Y(t))=-CY(t)+DX(t)Y(t) \tag{6}$$

Table-3.1: parameters, their meanings and values (Using Crimes and Guards data)

parameter	Parameter definition	values
A	Growth rate coefficient of criminals	0.12
B	Constant of proportionality that measures the probability that a Guards-criminals encounter removes one of the criminals.	0.002
C	the decline rate of Guards populations	0.05
D	growth rate coefficient that measure the efficiency of Guards populations to increase the encounter for reducing criminals	0.0007
X_0	Initial number of criminals	60
Y_0	Initial number of Guards	40
N_0	Initial population	100

If we put $B=\frac{A}{L}$,

$$(5)\Rightarrow \frac{dX}{dt}=AX\left(1-\frac{Y}{L}\right)\text{, constant } L>0 \tag{7a}$$

Putting $D=\frac{C}{V}$,

$$(6)\Rightarrow \frac{dY}{dt}=-CY\left(1-\frac{X}{V}\right)\text{, constant } V>0 \tag{7b}$$

The equations (7) represents a system of non-linear ordinary differential equations, known as Lotka-Volterra system.

DERIVATION OF FIXED POINTS

Thus for the fixed point (X^*,Y^*) for systems (7), we have

$$\frac{dX}{dt}=0=AX^*\left(1-\frac{Y^*}{L}\right) \text{ and } \frac{dY}{dt}=0=-CY^*\left(1-\frac{X^*}{V}\right)$$

Solving these, we have, $(X^*,Y^*)=(0,0)$ and $(X^*,Y^*)=(V,L)$ are two fixed points.
Hence $(0,0)$ and (V,L) are two fixed points of the systems (7).

Stability Analysis: The stability is determined by linearization of (7) with the help of Jacobian corresponding to the fixed points $(0,0)$ and (V,L)

The system (7) can be written as

$$\frac{dX}{dt} = AX(1-\frac{Y}{L}) = f(X,Y), say$$
$$\frac{dY}{dt} = -CY(1-\frac{X}{V}) = g(X,Y), \text{say}$$

The Jacobian matrix of the above systems at the fixed point (X^*,Y^*) is given by

$$\mathrm{J}(X^*,Y^*) = \begin{bmatrix} \frac{\partial f}{\partial X} & \frac{\partial f}{\partial Y} \\ \frac{\partial g}{\partial X} & \frac{\partial g}{\partial Y} \end{bmatrix}_{(X^*,Y^*)}$$

The Jacobian corresponding to the fixed point $(0,0)$ is

$$\mathrm{J}_* = \mathrm{J}(0,0) = \begin{bmatrix} A(1-\frac{Y}{L}) & -\frac{AX}{L} \\ \frac{CY}{V} & -C(1-\frac{X}{V}) \end{bmatrix}_{(0,0)}$$
$$\Rightarrow J_* = \mathrm{J}(0,0) = \begin{bmatrix} A & 0 \\ 0 & -C \end{bmatrix}$$

Now, $J_* - \lambda I = \begin{bmatrix} A-\lambda & 0 \\ 0 & -C-\lambda \end{bmatrix}$, where λ is an eigen value of J_*

The characteristic equation is, $\left|J_* - \lambda I\right| = 0 \Rightarrow \det\begin{bmatrix} A-\lambda & 0 \\ 0 & -C-\lambda \end{bmatrix} = 0$

Hence the eigenvalues are $\lambda_1 = A = 0.12$ and $\lambda_2 = -C = -0.05$

Since two eigenvalues are of opposite signs, so the fixed point $(0,0)$ is unstable.

Again, the Jacobian matrix corresponding to the fixed point (V,L) is

$$J_* = J(V,L) = \begin{bmatrix} A\left(1-\frac{Y}{L}\right) & -\frac{AX}{L} \\ \frac{CY}{V} & -C\left(1-\frac{X}{V}\right) \end{bmatrix}_{(V,L)}$$
$$= \begin{bmatrix} 0 & -\frac{AV}{L} \\ \frac{CL}{V} & 0 \end{bmatrix} \tag{8}$$

$$\det(J_* - \lambda I) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 0-\lambda & -\dfrac{AV}{L} \\ \dfrac{CL}{V} & 0-\lambda \end{bmatrix}\right)=0$$

$$\Rightarrow \lambda^2 + AC = 0 \Rightarrow \lambda = \pm\sqrt{-AC} \Rightarrow \lambda = \pm i\sqrt{AC}$$

This gives $\lambda_{1,2} = \pm i\sqrt{0.0060} \Rightarrow \lambda_{1,2} = \pm 0.08i$ (9)

This shows that the real parts of eigenvalues of the Jacobian matrix are zero for the fixed point (V, L).So the fixed point (V, L) is a neutrally stable centre. This analysis suggest that the criminals depends on the parameter connected with the Guards $Y = L$ Similarly Guards depends on the parameter associated with criminals $X = V$.

This effect is due to the particular coupling of variables. The presence of $Y \neq 0$, means that available criminals has to be just enough to make growth rate due to encounter, $\dfrac{CX}{V}$ equal Guards rate C for a steady criminals to persists. Similarly, when $X \neq 0$, Guards can only keep them under control when criminals growth rate A and encounter rate $\dfrac{A}{L}$ are equal.

COMPONENT GRAPH

Using the data and parameters we can draw the graph of the population fluctuation of the year of criminals and Guards as shown in the figure below.

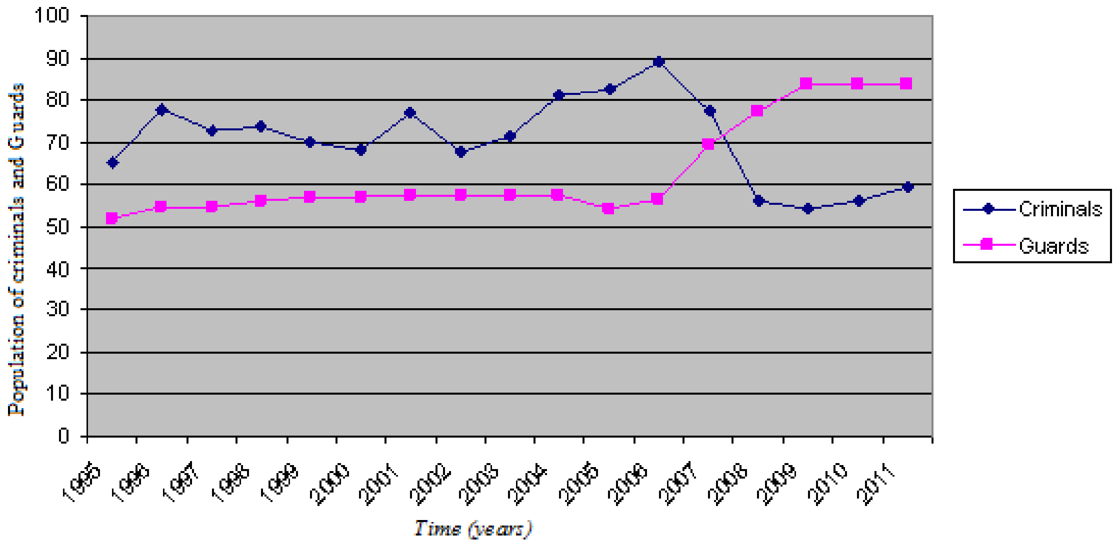

Figure-3.1: Criminal-Guards component graph

NUMERICAL SOLUTIONS OF THE MODEL

The numbers of criminals $X(t)$ and Guards $Y(t)$ to be determined from the ordinary differential equation (ODE) (7*a*) and (7*b*) by Runge-Kutta's numerical solution methods. A function $X(t)$ and $Y(t)$ for all t in an interval is called a solution of ODE (7a) and (7b) respectively. The value X_0 of $X(t)$ and Y_0 of $Y(t)$ at some t_0 can be estimated, and must surely be a critical factor in predicting later values of $X(t)$ and $Y(t)$. The condition $X(t_0) = X_0$ and $Y(t_0) = Y_0$ are respectively called an initial condition of (7a) and (7*b*).

Measuring time forward from the time t_0, we have created a problem whose solution $X(t)$ and $Y(t)$ are predicted number of criminals and Guards at future times-

In (7a), for the given constants A and L, and the value t_0 and X_0, we find a function $X(t)$ for which

$$X' = AX\left(1-\frac{Y}{L}\right),\ X(t_0) = X_0 \tag{10}$$

on some t-interval containing t_0. The ODE and the initial condition in (10) form an initial value problem (IVP) for $X(t)$.

Similarly we have the IVP for $Y(t)$ of (7*b*) as

$$Y' = -CY\left(1-\frac{X}{V}\right),\ Y(t_0) = Y_0 \tag{11}$$

We can write the IVP (10) and (11) as below-

$$X' = f\left(t, X, Y\right), \qquad X(t_0) = X_0 \tag{12a}$$

$$Y' = g\left(t, X, Y\right), \qquad Y(t_0) = Y_0 \tag{12b}$$

where $f(t, X, Y) = AX\left(1-\frac{Y}{L}\right)$

and $g\left(t, X, Y\right) = -CY\left(1-\frac{X}{V}\right)$

The IVP of the system (12) has a unique solution on an interval containing t_0 if the rate functions $f(t, X, Y)$ and $g(t, X, Y)$ are well enough behaved.

We discuss basic numerical procedures called Runge-Kutta method for finding approximate values for the solutions $X(t)$ and $Y(t)$ of the IVP (12) at a discrete set of times near t_0.

RUNGE-KUTTA (RK4) METHOD

Let us consider to approximate the value of the solution of IVP (12) at some future time T. Let $T > t > t_0$ and take increasing sequence $t_1, t_2, ..., t_N$ with $t_N = T$ and $t_1 > t_0$ and define the step function $h_n = t_n - t_{n-1}$ at step n for $n = 1, 2, ..., N$.

From a given X_0 and a given function $f(t, X, Y, h)$, a one-step method computes an approximation X_n to $X(t_n)$ using the discretization scheme

$$X_n = X_{n-1} + h_n f\left(t_{n-1}, X_{n-1}, Y_{n-1}, h_n\right),\ n = 1, 2, ..., N.$$

Similarly, (for Y_n),

$$Y_n = Y_{n-1} + h_n g\left(t_{n-1}, X_{n-1}, Y_{n-1}, h_n\right), n = 1, 2, ..., N$$

To compute X_n, Y_n, only the value of X_{n-1} and Y_{n-1} is required, so it is called one-step method. This method uses the given value X_0 to generate X_1, X_1 to generate X_2, and so on, until the process terminates with the calculation of X_n, which is an approximation of *X*(*T*). In this method, if the averages of the above slope function $f\left(t_{n-1}, X_{n-1}, Y_{n-1}, h_n\right)$ at two or more points over the interval $\left[t_{n-1}, t_n\right]$ are used to calculate X_n then this method is said to be Runge-Kutta Fourth Order Method (RK4). The RK4 method involves a weighted average of slopes at the midpoint $t_{n-1} + \frac{h}{2}$ and at the end points t_{n-1} and t_n. For the IVP of the system (12) the RK4 is the one-step method is given by

$$t_n = t_{n-1} + h$$

$$X_n = X_{n-1} + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$Y_n = Y_{n-1} + \frac{h}{6}\left(p_1 + 2p_2 + 2p_3 + p_4\right)$$

Where h is the fixed step size and

$$k_1 = f\left(t_{n-1}, X_{n-1}, Y_{n-1}\right)$$
$$p_1 = g\left(t_{n-1}, X_{n-1}, Y_{n-1}\right)$$
$$k_2 = f\left(t_{n-1} + \frac{h}{2}, X_{n-1} + \frac{h}{2}k_1, Y_{n-1} + \frac{h}{2}p_1\right)$$
$$p_2 = g\left(t_{n-1} + \frac{h}{2}, X_{n-1} + \frac{h}{2}k_1, Y_{n-1} + \frac{h}{2}p_1\right)$$
$$k_3 = f\left(t_{n-1} + \frac{h}{2}, X_{n-1} + \frac{h}{2}k_2, Y_{n-1} + \frac{h}{2}p_2\right)$$
$$p_3 = g\left(t_{n-1} + \frac{h}{2}, X_{n-1} + \frac{h}{2}k_2, Y_{n-1} + \frac{h}{2}p_2\right)$$
$$k_4 = f\left(t_{n-1} + h, X_{n-1} + hk_3, Y_{n-1} + hp_3\right),$$
$$p_4 = g\left(t_{n-1} + h, X_{n-1} + hk_3, Y_{n-1} + hp_3\right)$$

Now, we have

$$X' = f(t, X, Y), \quad X(t_0) = X_0$$

and $$Y' = g(t, X, Y), \quad Y(t_0) = Y_0$$

$$t = t_0 = 0, t > 0$$

where $$f(t, X, Y) = AX(t)\left(1 - \frac{Y(t)}{L}\right)$$

and $$g(t, X, Y) = -CY(t)\left(1 - \frac{X(t)}{V}\right)$$

Let, initial population be $X(0) = X_0 = 60$, $Y(0) = Y_0 = 40$, $N_0 = 100$, and $A = 0.12$, $C = 0.05$, $L = 56.2, V = 74.1$

$$k_1 = f\left(t_{n-1}, X_{n-1}, Y_{n-1}\right) = AX_{n-1}\left(t_{n-1}\right)\left(1 - \frac{Y_{n-1}\left(t_{n-1}\right)}{L}\right)$$

$$p_1 = -CY_{n-1}\left(t_{n-1}\right)\left(1 - \frac{X_{n-1}\left(t_{n-1}\right)}{V}\right)$$

$$k_2 = f\left(t_{n-1} + \frac{h}{2}, X_{n-1} + \frac{h}{2}k_1, Y_{n-1} + \frac{h}{2}p_1\right),$$

$$= A\left[X_{n-1}\left(t_{n-1} + \frac{h}{2}\right) + \frac{h}{2}k_1\right]\left[1 - \frac{1}{L}\left\{Y_{n-1}\left(t_{n-1} + \frac{h}{2}\right) + \frac{h}{2}p_1\right\}\right]$$

$$p_2 = -C\left\{Y_{n-1}\left(t_{n-1} + \frac{h}{2}\right) + \frac{h}{2}p_1\right\}\left[1 - \frac{1}{V}\left\{X_{n-1}\left(t_{n-1} + \frac{h}{2}\right) + \frac{h}{2}k_1\right\}\right]$$

$$k_3 = A\left\{X_{n-1}\left(t_{n-1} + \frac{h}{2}\right) + \frac{h}{2}k_2\right\}\left[1 - \frac{1}{L}\left\{Y_{n-1}\left(t_{n-1} + \frac{h}{2}\right) + \frac{h}{2}p_2\right\}\right]$$

$$p_3 = -C\left\{Y_{n-1}\left(t_{n-1} + \frac{h}{2}\right) + \frac{h}{2}p_2\right\}\left[1 - \frac{1}{V}\left\{X_{n-1}\left(t_{n-1} + \frac{h}{2}\right) + \frac{h}{2}k_2\right\}\right]$$

$$k_4 = A\left\{X_{n-1}\left(t_{n-1}+h\right)+hk_3\right\}\left[1-\frac{1}{L}\left\{Y_{n-1}\left(t_{n-1}+h\right)+hp_3\right\}\right]$$

$$p_4 = -C\left\{Y_{n-1}\left(t_{n-1}+h\right)+hp_3\right\}\left[1-\frac{1}{V}\left\{X_{n-1}\left(t_{n-1}+h\right)+hk_3\right\}\right]$$

Thus the general RK4 method for the criminals is given by

$$t_n = t_{n-1}+h$$

$$X_n = X_{n-1}+\frac{h}{6}\left(k_1+2k_2+2k_3+k_4\right), n=1,2,\ldots,N$$

and the RK4 method for the Guards is given by

$$t_n = t_{n-1}+h$$

$$Y_n = Y_{n-1}+\frac{h}{6}\left(p_1+2p_2+2p_3+p_4\right),\ n=1,2,\ldots,N$$

CONCLUSIONS

From the Fig: 3.1, It is seen that as time progresses in years, Guards population and criminals population clearly fluctuate at a cyclic time interval. When the criminal population peaks, Guards population begins to rise rapidly. However, as the Guards population rises, the criminal population falls rapidly.

The general RK4 method of Criminals and Guards equation are derived, which yields the numerical solutions of criminals and Guards.

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