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ON THE SUBSET GRAPH OF A NEAR-RING

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ABSTRACT

Let N be a near-ring. Let F be the collection of all non-empty subsets of N. We define a new graph, the subset graph F_R as the graph with all the members of F as vertices and any two distinct vertices A, B are adjacent if and only if $A+B = \{a + b: a \in A, b \in B\}$ is a right N - subset of N. In this paper we discuss about the connectivity, diameter and girth of the graph F_R. We also discuss about some induced subgraphs of F_R and some graphical parameters of these subgraphs viz .diameter, girth etc.

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1. INTRODUCTION

One of the generalized structures of rings are the near-rings. A near-ring is actually what is required to describe the formation of the endomorphism of various mathematical structures adequately. Let M (G) be the set of all maps of an additive (not necessarily abelian) group G into itself. The concept of near-ring arises when we define addition and multiplication on the set M(G) as $(f + g)(a) = f(a) + g(a)$ and $(fg)(a) = f(g(a))$, for all $a \in G$; $f, g \in M(G)$. An algebraic system (N, +, .) is called a near-ring if (N, +) is a group (not necessarily abelian), (N, .) is a semigroup and $(a + b).c = a.c + b.c$ for all $a, b, c \in N$. This near-ring is termed as right near-ring. The additive identity of the group (N, +) of a near-ring N is called the zero element and it is denoted by 0. In a near-ring N, $0a = 0$, $\forall a \in N$. A near-ring N is called zero symmetric near-ring if $a0 = 0, \forall a \in N$. A normal subgroup I of (N, +) is called a right ideal of N if $IN \subseteq I$. A non empty subset A of N is known as (i) a right N - subset of N if $AN \subseteq A$, (ii) a left N - subset of N if $NA \subseteq A$ and (iii) an invariant subset of N if $AN \subseteq A$, $NA \subseteq A$. It is clear that an invariant subset of a near-ring N is a left as well as right N -subset of N. Moreover, every right (left) N -subset contains the zero element of N. Throughout this paper by a near-ring N we mean a zero symmetric right abelian near-ring unless otherwise stated.

Let us consider a near-ring N where (N, +) is an abelian group. Also let F be the set of all non-empty subsets of N. We define the subset graph F_R as the graph with all the members of F as vertices and any two distinct vertices A, B are adjacent if and only if $A + B = \{a + b : a \in A, b \in B\}$ is a right N - subset of N.

Let G be a graph. The graph G is said to be connected if there is a path between any two distinct vertices of G. On the other side, the graph G is called totally disconnected if no two vertices of G are adjacent. For vertices x and y of G, the distance between x and y denoted by d(x, y) is defined as the length of the shortest path from x to y; $d(x, y) = \infty$, if there is no such path. The diameter of G is $\text{diam}(G) = \sup\{d(x, y): x, y \text{ are vertices of } G\}$. The girth of G, denoted by $gr(G)$, is the length of a shortest cycle in G; $gr(G) = \infty$ if G contains no cycle.

For usual graph-theoretic terms and definitions, one can look at [1]. General references for the algebraic part of this paper are [2], [3], [4], [5], [6].

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Example 1.1: Let us consider the near-ring $N = \{0, a, b\}$ under the operations defined by the following tables.

+	0	a	b
0	0	a	b
A	a	0	a
B	b	a	0

.	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

Here we can see that N, $\{0, a\}$, $\{0, b\}$ are right N-subsets of N. The graph F_R is given below:

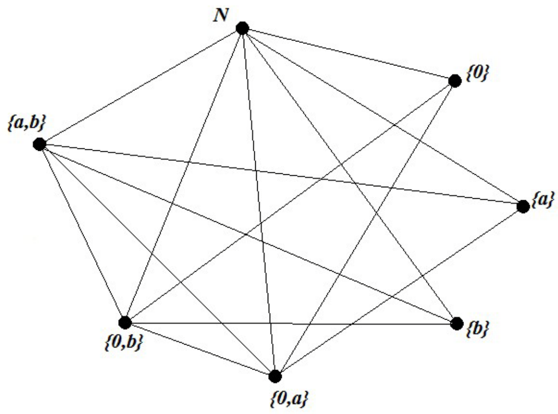

Figure-1.1: The subset graph F_R

2. MAIN RESULTS

Let N be a right-near ring and we denote cardinality of N by α, i.e. $|N| = \alpha$. α may be of infinite cardinality. We start our section with the following lemma.

Lemma 2.1 [Lemma 1.1.17[3]]: The sum $A + B = \{a + b | a \in A \text{ and } b \in B\}$ of two right N-subsets of N is also a right N-subset of N.

Theorem 2.2: For any right near-ring N with $|N| \leq 2$, the graph F_R is always connected.

Proof: If $|N| = 1$, then the proof is clear. Let $|N| = 2$ and $N = \{0, a\}$. The vertices $\{0\}$ and N are always connected. The vertices $\{a\}$ and $\{0\}$ are never adjacent, since $\{a\} + \{0\} = \{a\}$ which cannot be a right N-subset of N. Now $\{a\} + N = N$ and clearly this is a right N-subset of N. Hence the graph F_R is connected.

Remark 2.3: In theorem 2.2, we have considered the near-ring as a zero-symmetric near-ring. It is interesting to note that if N is not zero-symmetric then with the same adjacency relation for $|N| = 2$, the graph F_R is a complete graph K_3 with diameter 1 and girth 3, e.g. consider the near-ring $Z_2 = \{0, 1\}$ under addition modulo 2 and '.' on Z_2 is defined as $a.b = a, \forall a, b \in Z_2$.

For $|N| > 2$, the graph F_R is never totally disconnected. However for any near-ring N, it is a tough job to check whether there is an isolated vertex or not in the graph F_R. In the following theorem we develop some conditions under which the graph F_R never contains an isolated vertex.

Theorem 2.4: For $|N| > 2$, the graph F_R is always connected if one of the following holds:

(i) for any proper subset A, $A + N = N$.
(ii) for any two subsets A, B, $A + B = N$.
(iii) every subset of N is a right N- subset of N.

Proof:

(i) Let $A \subset N$. Now $A + N \subseteq N$ and $(A + N)N \subseteq NN \subseteq N$. It is clear that $(A + N)$ is a right N subset of N only if $N \subseteq (A + N)$, i.e. $A + N = N$. Thus N is adjacent to every other vertex and hence the graph F_R is connected.

(ii) Let A, B be any two proper subsets of N such that $A + B = N$. Now $(A + B)N = NN \subseteq N = A + B$ and thus A and B are adjacent. If either $A = N$ or $B = N$, then from above (i) clearly A and B are adjacent and thus F_R is connected.
(iii) The proof is clear from lemma 2.1.

Remark 2.5: In the theorem 2.4, if N satisfies either condition (ii) or (iii), then the resulting graph F_R will be a complete graph $K_{2^{\alpha}-1}$. However If N satisfies condition (i) in theorem 2.5, then the graph F_R contains a spanning subgraph isomorphic to the star graph $K_{1,2^{\alpha}-2}$.

Next, let us find the girth of the graph F_R.

Theorem 2.6: For $|N| \leq 2, gr(F_R) = \infty$.

Proof: The proof is clear from the proof of theorem 2.2.

Theorem 2.7: For any right near-ring N with $|N| \geq 3$, we have $gr(F_R) = 3$ if one of the following conditions holds.

(i) If for some $n_1(= 0) \in N$, there exists a nonzero element $n_2 (= n_1) \in N$ such that $n_1 + n_2 = 0$.
(ii) If for any two subsets $A, B, A + B = N$.
(iii) If every subset of N is a right N-subset of N.
(iv) If there exists an element $n_1 \in N$ such that $n_1N = 0, \forall n \in N$.

Proof: The proofs of (i), (ii) and (iii) are clear from the theorem 2.4.

(iv) Let $n_1 \in N$ such that $n_1 n = 0, \forall n \in N$. There may be two cases. In the first case let $n_1 + n_1 = 0$. Then clearly there exists a 3-cycle in $\{N\} - \{n_1\} - \{0, n_1\} - \{N\}$ in the graph F_R. Next, let $-n_1 (= n_1)$ be the additive inverse of n_1 in the near-ring N. In this case also we can construct a 3-cycle as $\{N\}-\{n_1\}-\{-n_1\}-\{N\}$ and hence $gr(FR)=3$.

3. ON SOME SUBGRAPHS OF THE GRAPH F_R

In this section we discuss about some induced subgraphs of the graph F_R. We also try to find some graphical parameters like diameter, girth, chromatic number etc. of these subgraphs.

I. Let us consider the subclass of the family of subsets of N which consists of all the right N-subsets of N. Let us denote this induced subgraph of F_R by R_R. We have the following results:

Theorem 3.1: The subgraph R_R is a complete subgraph of F_R.

Proof: We know that for any two right N-subsets of N say A, B, their sum $A + B = \{a + b : a \in A \text{ and } b \in B\}$ is also a right N-subset of N. Hence the statement is clear.

Corollary 3.2: For any near-ring N we have $diam(R_R) \leq 1$.

Corollary 3.3: $gr(R_R) = 3$ *or* ∞.

II. For any near-ring N, an element $x \in N$ is said to be nilpotent if $x^t = 0$, for some $t \in \mathbb{Z}^+$. A subset S of N is called a nilpotent subset of N if there exists a $k \in \mathbb{Z}^+$ such that $S^k = 0$ which means $s_1 s_2 s_3 ... s_k = 0$ for each $s_i \in S$, i = 1, 2, 3, ..., k. A near-ring N is called a strongly semi-prime near-ring if N has no non-zero nilpotent invariant subset. Let us consider an induced subgraph of F_R whose vertex set consists of all the nilpotent subsets of N. Let us denote this subgraph by Nil_R.

Lemma 3.4[Lemma 2.1.32,[3]]: A strongly semi-prime near-ring N has no non-zero nilpotent left(right) N-subsets of N.

Theorem 3.5: If N is strongly semi-prime then the subgraph Nil_R is the disjoint union of K_2 and K_1 's.

Proof: Let N be a strongly semi-prime near-ring. By lemma 3.4, it is clear that N has no non- zero nilpotent right N-subsets of N. Thus the subgraph contains a line $\{0\} - N$ and other vertices (if any) are isolated. Hence the result.

Remark 3.6: A near-ring N is called regular if $\forall n \in N$, there exists $x \in N$ such that $nxn = n$. A near-ring N is called weakly regular if for any ideal I of N, each left I- subgroup A of I, $A^2 = A$. It can be seen that if N is a weakly regular near-ring, then also theorem 3.5 holds [Lemma 5.2.5,[4]]. As all regular near-rings are weakly regular [Lemma 5.2.7, [4]] so we can finally state that the subgraph Nil_R is disjoint union of K_2 and K_1's for a regular near-ring N.

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