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SIMILARITY SOLUTIONS OF LAMINAR INCOMPRESSIBLE BOUNDARY LAYER EQUATIONS OF NON-NEWTONIAN VISCOINELASTIC FLUIDS

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ABSTRACT

Using deductive group theoretic method, similarity solutions for steady, two dimensional laminar boundary layer flows of incompressible non-Newtonian viscoinelastic fluids are derived. The important conclusion drawn from the present analysis is that for non-Newtonian viscoinelastic fluids of any model, similarity solutions exist only for the flows past 90° wedge. The only limitation for the existence of similarity solutions of present flow geometry is that the stress and the rate of strain can be related by arbitrary continuous function. Numerical solution for the Prandtl-Eyring model is presented in dimensionless form as an engineering application of the present analysis.

Keywords: *Deductive group theoretic method, Similarity solutions, Non-Newtonian fluids, Viscoinelastic fluids, Prandtl-Eyring model,*

AMS Subject classification: *76A05, 76M55, 54H15*

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1. NOMENCLATURE:

F	Transformed dependent variable
G	Transformed dependent variable
L	Reference length
Re	Renold number
u	Velocity component in the x-direction
v	Velocity component in the y-direction
U_∞	Velocity of the mainstream
U_e	Velocity at the edge of boundary layer
x, y	Rectangular coordinates

Greek symbols

ρ	Density
ν	Kinematics viscosity
μ	Viscosity
ψ	Stream function
τ	Shearing stress
Ω	Functional relation of τ and velocity gradient
η	Transformed independent variable

Subscripts

∞	Conditions at infinity in the y-direction
x	Local
0	Reference condition

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2. INTRODUCTION:

Today there has been remarkable interest in the boundary layer theory for the flow of non-Newtonian fluids, Rajagopal and Gupta [13], Rajagopal [14], Timol and Kalthia [17], Wafo [4], Patel and Timol [22]. To investigate the non-Newtonian effects, the class of solutions known as similarity solutions place an important role. This is because that is the only class of the exact solutions for the governing equations which are usually non-linear partial differential equations (PDEs) of the boundary layer type. Further this also serves as a reference to check approximate solutions.

It is well known that similarity solutions for the PDEs governing the flow of Newtonian and non-Newtonian fluids exist only for limited classes of main stream velocities at the edge of the boundary layer. For example, for two dimensional laminar boundary layer flow of Newtonian fluids, similarity solutions are limited to the well known Falkner-Skan solutions Rajagopal et al [15]. Most of the generalization of the Falkner-Skan solutions and approximate solutions in the literature are limited to the power law fluids; this is because they are mathematically the easiest to be treated among most of the non-Newtonian fluids.

The classical theory of Newtonian fluid depends upon the hypothesis of linear relationship between stress tensor and strain tensor, rate of strain tensor and even rate of stress tensor. The fluids which do not follow such a linear relationship are called non-Newtonian fluids. Non-Newtonian fluids are generally divided in to two categories like viscoinelastic fluids and viscoelastic fluids. The common feature of viscoinelastic fluids is that when at the rest they are isotropic and homogeneous and when they are subjected to a shear the resultant stress depends only on the rate of shear. However, such types of fluids show diverse behavior in response to applied stress. Numbers of rheological models have been proposed to explain such a diverse behavior. Some of this models are; Power-law fluids, Sisko fluids, Ellis fluids, Prandtl fluids Williamson fluids, Sutterby fluids, Reiner-Rivlin fluids, Bingham plastic, Prandtl-Eyring fluids, Powell-Eyring fluids, Reiner-Philippoff etc.

On the other hand viscoelastic fluids are those which posses a certain degree of elasticity in addition to viscosity. For these fluids stress tensor is related to both instantaneous strain and the past strain history. These fluids, when in motion, a certain amount of energy is stored up in the material as strain energy whiles some energy loss due to viscous dissipation. In this class of fluids unlike the viscoinelastic fluids, one cannot neglect the strain, however small it may be, as it is responsible for the recovery to the original state and for the possible reverse flow of the stress. Some of the well known viscoelastic fluids are Rivlin-Ericksen fluids, Walter fluids, Oldroyd fluids and certain second, third and forth order fluids.

For the derivation of the constitutive equations governing the motion of non-Newtonian fluids, the mathematical structure of stress-strain relationship, which is non linear, is important in functional form. This relationship may be implicit or explicit. In the present paper we have consider such relationship in the form of general arbitrary continuous function of the type

$$\Omega\left(\tau, \frac{\partial u}{\partial y}\right) = 0 \qquad (1)$$

Here τ is the shearing stress and $\frac{\partial u}{\partial y}$ is the rate of the strain of the fluids. Some empirical models of non-Newtonian fluids which belong to this class are listed in Table-1.

Hansen and Na [1] were probably first to derive similarity analysis of laminar incompressible boundary layer equations of the non-Newtonian fluids whose stress and rate of strain are related by arbitrary continuous function, while extending the work of Lee and Ames [26]. They have used linear group of transformations to derive their similarity equations which includes many of the non-Newtonian viscoinelastic fluids. Timol and Kalthia [18] are probably first to derive general non-Newtonian viscoinelastic fluids for two dimensional natural convection flows and three dimensional boundary layer flow respectively. Na [27] has studied both similarity and similarity cases, as an extension of previous analysis by Hansen and Na [1] for the case of Reiner-Philippoff flow. He has shown that similarity solution exist for the case of flow past 90° wedge only (see figure-1) otherwise non similar solution is available. Recently, Patel and Timol [21] have derived numerical solution of Powell-Eyring fluid flow past 90° wedge.

Now days for similarity analysis many techniques are available, among them the similarity methods which invoke the invariance under the group of transformations are known as group theoretic methods. These methods are more recent and are mathematically elegant and hence they are widely used in different fields. The group theoretic methods involve mainly two different types of groups of transformations, namely, assumed group of transformations and deductive group of transformations. The linear group transformations, scaling group transformations, spiral group transformations are the assumed group of transformations and are mainly due to Birkhoff [6] and Morgan [3]. Whereas the deductive group of transformations can be further classify in to two groups: finite group of transformations (Moran and Gaggioli

[19]) and infinitesimal group of transformation (Bluman and Cole [9], Bluman and Kumai [10]). The main drawback of the similarity methods based on the assumed group of transformation at the outset of the analysis is that the resulting similarity solutions are restrictive and hence some time leads to wrong conclusion that similarity transformations does not exist. On the other hand, the similarity methods based on general group of transformation are more systematic and lead to number of similarity solutions. Out of these the deductive group theoretic method provides a powerful tool because they are not based on linear operators, superposition, or any other aspects of linear solution techniques. Therefore, this method is applicable to nonlinear differential models.

In the present paper the deductive group method based on general group transformation is, probably first time, applied to derive similarity solutions for steady; two dimensional laminar boundary layer flows of incompressible non-Newtonian all those viscoinelastic fluids whose shearing stress related to rate strain by arbitrary continuous function given by equation (1). The similarity equations obtained are more general and systematic along with auxiliary conditions. Recently this method has been successfully applied to various non-linear problems by Abd-el- Malek et al [16], Parmar and Timol [11], Darji and Timol [25] and Adnan et al [12].

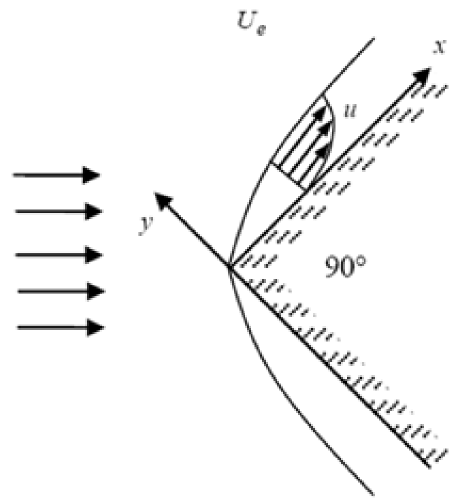

Figure 1: Schematic Diagram of flow past 90° wedge

3. GOVERNING EQUATIONS:

The governing differential equations for the boundary layer flow of the generalized non-Newtonian fluid can be written as:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2}$$

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y}\right)=\frac{\partial \tau}{\partial y}+\rho U_e\frac{dU_e}{dx} \tag{3}$$

With the stress-strain relationship given by (1)

Together with boundary conditions

$$u(x,0)=v(x,0)=0,\quad u(x,\infty)=U_e(x) \tag{4}$$

4. FORMULATION OF THE PROBLEM:

The above equations can be made dimensionless using the following quantities,

$$\left.\begin{aligned} & x^* = \frac{x}{L}; \quad y^* = \frac{y}{L}\sqrt{Re}; \quad u^* = \frac{u}{U_\infty}; \quad v^* = \frac{v}{U_\infty}\sqrt{Re}; \\ & \tau^* = \frac{\tau}{\rho U_\infty^2}\sqrt{Re}; \quad \mu^* = \frac{\mu_0}{\mu_\infty}; \quad Re = \frac{U_\infty L}{\nu} \end{aligned}\right\}$$

And a stream function $\psi^*\left(x^*, y^*\right)$, such that

$$u^* = \frac{\partial \psi^*}{\partial y^*}; \qquad v^* = -\frac{\partial \psi^*}{\partial x^*}$$

Substitute the values in equations (1)-(4) and dropping the asterisks (for simplicity), we get

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial \tau}{\partial y} + U_e \frac{dU_e}{dx} \tag{5}$$

$$\Omega\left(\tau, \frac{\partial^2 \psi}{\partial y^2}\right) = 0 \tag{6}$$

With the boundary conditions,

$$\frac{\partial \psi}{\partial y}(x,0) = \frac{\partial \psi}{\partial x}(x,0) = 0, \qquad \frac{\partial \psi}{\partial y}(x,\infty) = U_e(x) \tag{7}$$

5. METHODOLOGY AND SOLUTION OF THE PROBLEM:

Our method of solution depends on the application of a one-parameter deductive group of transformation to the partial differential equation (5) and (6). Under this transformation the two independent variables will be reduced by one and the differential equations (5) will transforms into the ordinary differential equation.

5.1. The group systematic formulation:

The procedure is initiated with the group *G*, a class of transformation of one-parameter 'a'of the form:

$$G: \quad T_a(Q) = \aleph^Q(a)s + \Re^Q(a) = \bar{Q} \tag{8}$$

Where Q stands for x, y, ψ, U_e, τ whereas $\aleph' s$ and $\Re' s$ are real-valued and are at least differentiable in the real argument a.

5.2. The invariance analysis:

To transform the differential equation, transformations of the derivatives of ψ are obtained from *G* via chain-rule operations:

$$\left.\begin{aligned} & \bar{s}_{\bar{i}} = \left(\frac{\aleph^Q}{\aleph^i}\right) Q_i \\ & \bar{Q}_{\bar{i}\,\bar{j}} = \left(\frac{\aleph^Q}{\aleph^i \aleph^j}\right) Q_{ij} \end{aligned}\right\} \quad Q = \psi, U_e, \tau; \quad i, j = x, y \tag{9}$$

Equation (5) and (6) are said to be invariantly transformed, for some function $\xi(a)$ whenever,

$$\left.\begin{aligned}&\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial^2\bar{\psi}}{\partial\bar{y}\partial\bar{x}}-\frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2}-\frac{\partial\bar{\tau}}{\partial\bar{y}}-\bar{U}_{\bar{e}}\frac{d\bar{U}_{\bar{e}}}{d\bar{x}}=\xi(a)\left[\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial\tau}{\partial y}-U_e\frac{dU_e}{dx}\right]\\&\Omega\left(\bar{\tau},\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2}\right)=\Omega\left(\tau,\frac{\partial^2\psi}{\partial y^2}\right)\end{aligned}\right\}$$

Substituting the values from the equation (8) and (9) in above system of equation, yields

$$\begin{aligned}&\frac{(\aleph^{\psi})^2}{\aleph^{x}(\aleph^{y})^2}\left[\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\right]-\frac{\aleph^{\tau}}{\aleph^{y}}\frac{\partial\tau}{\partial y}-\left(\aleph^{U_e}U_e+\Re^{U_e}\right)\frac{\aleph^{U_e}}{\aleph^{x}}\frac{dU_e}{dx}\\&\qquad=\xi(a)\left[\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial\tau}{\partial y}-\frac{dU_e}{dx}\right]\end{aligned}\tag{10}$$

$$\Omega\left(\aleph^{\tau}\tau+\Re^{\tau},\frac{\aleph^{\psi}}{(\aleph^{y})^2}\frac{\partial^2\psi}{\partial y^2}\right)=\Omega\left(\tau,\frac{\partial^2\psi}{\partial y^2}\right)\tag{11}$$

The invariance of equations (10)-(11) together with boundary conditions, implies that

$$\left.\begin{aligned}&\Re^{U_e}=\Re^{\tau}=\Re^{y}=\Re^{\psi}=0\\&\frac{(\aleph^{\psi})^2}{\aleph^{x}(\aleph^{y})^2}=\frac{\aleph^{\tau}}{\aleph^{y}}=\frac{(\aleph^{U_e})^2}{\aleph^{x}}=\xi(a)\\&\aleph^{\tau}=\frac{\aleph^{\psi}}{(\aleph^{y})^2}=1\end{aligned}\right\}\tag{12}$$

These yields,

$$\aleph^{x}=(\aleph^{y})^3,\quad \aleph^{\psi}=(\aleph^{y})^2,\quad \aleph^{U_e}=\aleph^{y},\quad \aleph^{\tau}=1$$

Finally, we get the one-parameter group G, which transforms invariantly the differential equations (5) and the auxiliary conditions (7).

The group G is of the form:

$$G:\left\{\begin{aligned}&G_S:\ \begin{cases}\bar{x}=(\aleph^{y})^3x+\Re^{x}\\ \bar{y}=\aleph^{y}y\end{cases}\\&\bar{\psi}=(\aleph^{y})^2\psi\\&\bar{U}_{\bar{e}}=\aleph^{y}U_e\\&\bar{\tau}=\tau\end{aligned}\right.$$

5.4. The complete set of absolute invariants:

Our aim is to make use of group methods to represent the problem in the form of an ordinary differential equation. Now we have proceeded in our analysis to obtain a complete set of absolute invariants. If $\eta = \eta(x, y)$ is the absolute invariant of the independent variables then,

$$g_j(x, y, \psi, U_e, \tau) = F_j(\eta), \quad j = 1,2,3 \tag{13}$$

is the absolute invariant for the dependent variables ψ, U_e and τ.

The application of a basic theorem in group theory, Moran and Gaggioli [19], states that: A function $g(x, y, \psi, U_e, \tau)$ is an absolute invariant of a one-parameter group if it satisfies the following first-order linear partial differential equation,

$$\sum_{i=1}^{5}(\alpha_i Q_i + \beta_i)\frac{\partial g}{\partial Q_i} = 0, \qquad Q_i = x, y, \psi, U_e, \tau \tag{14}$$

Where

$$\alpha_i = \left.\frac{\partial \aleph^i}{\partial a}\right|_{a=a^0} \quad \text{and} \quad \beta_i = \left.\frac{\partial \Re^i}{\partial a}\right|_{a=a^0} \qquad i = 1,2,3,4,5 \tag{15}$$

and 'a^0' denotes the value of 'a' which yields the identity element of the group *G*.

Since $\Re^{U_e} = \Re^{\tau} = \Re^{\psi} = \Re^{y} = 0$ implies that $\beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$ and from (15) we get $\alpha_1 = 3\alpha_2 = \frac{3}{2}\alpha_3 = \alpha_4,\ \alpha_5 = 0$

The equation (14) reduces to

$$(\alpha_1 x + \beta_1)\frac{\partial g}{\partial x} + \left(\frac{\alpha_1 y}{3}\right)\frac{\partial g}{\partial y} + \left(\frac{2\alpha_1 \psi}{3}\right)\frac{\partial g}{\partial \psi} + \left(\frac{\alpha_1 U_e}{3}\right)\frac{\partial g}{\partial U_e} + (0)\frac{\partial g}{\partial \tau} = 0. \tag{16}$$

The absolute invariant of independent variable owing the equation (16) is $\eta = \eta(x, y)$ if it will satisfies the first order linear partial differential equation

$$(x + \beta)\frac{\partial \eta}{\partial x} + \frac{y}{3}\frac{\partial \eta}{\partial y} = 0, \quad \text{where } \beta = \frac{\beta_1}{\alpha_1} \tag{17}$$

Applying the variable separable method one can obtain

$$\eta(x, y) = y(x + \beta)^{-\frac{1}{3}} \tag{18}$$

Further the absolute invariants of dependent variables owing the equation (16) are followed by

$$g_3(x, y, \psi) = \frac{\psi}{(x+\beta)^{\frac{2}{3}}} = F(\eta), \qquad g_4(x, y, U_e) = \frac{U_e}{(x+\beta)^{\frac{1}{3}}} = G(\eta)$$

$$g_5(x, y, \tau) = \tau = H(\eta)$$

Since $U_e(x)$ is independent of *y*, $G(\eta)$ must be constant. Without loss of generality we assume unity. Whence

$$\psi(x, y) = (x+\beta)^{\frac{2}{3}} F(\eta), \qquad U_e(x) = (x+\beta)^{\frac{1}{3}}, \qquad \tau(x, y) = H(\eta) \tag{19}$$

5.5. The reduction to an ordinary differential equation:

Using the similarity transformation (19) in equations (6) and (7), yields to following non-linear ordinary differential equations

$$\left(F'\right)^2 - 2FF'' - 3H' - 1 = 0 \tag{20}$$

Together with boundary conditions,

$$F(0) = F'(0) = 0, \qquad F'(\infty) = 1 \tag{21}$$

And the Stress-Strain functional relationship is given by $\Omega\left(H, F''\right) = 0$

This will be different for different fluid models listed in Table-1 and the primes denote the ordinary derivatives with respect to η.

6. NUMERICAL SOLUTION FOR PRANDTL-EYRING MODEL:

Non-Newtonian fluid models based on functional relationship between shear-stress and rte of the strain, shown by equation (1) are listed in Table-1. Among these models most research work is so far carried out on power-law fluid model (Model-4 in Table-1), this is because of its mathematical simplicity. On the other hand fluid models given in Table-1 are mathematically more complex and the natures of partial differential equations governing these flows are too non liner boundary value type and hence their analytical or numerical solution is bit difficult. For the present study the partial differential equation model, although mathematically more complex, is chosen mainly due to two reasons. Firstly, it can be deduced from kinetic theory of liquids rather than the empirical relation as in power-law model. Secondly, it correctly reduces to Newtonian behavior for both low and high shear rate. This reason is somewhat opposite to pseudo plastic system whereas the power-law model has infinite effective viscosity for low shear rate and thus limiting its range of applicability.

Mathematically, the Prandtl-Eyring model can be written as (Bird et al [24], Skelland [2])

$$\tau = A \sinh^{-1}\left(\frac{1}{B}\frac{\partial u}{\partial y}\right) \tag{22}$$

Introducing the dimensionless quantities (defined in Section-4) into equation (22) and then substituting it into the equation (20), we get

$$F''' = \frac{1}{\alpha}\left(F'^2 - 2FF'' - 1\right)\left(1 + \beta F''^2\right)^{1/2} \tag{23}$$

Where $\alpha = \dfrac{A}{\mu B};\quad \beta = \dfrac{\rho U_\infty^3}{\mu L B^2}$ are dimensionless numbers and can be referred as a flow parameters.

A numerical solution of (23) is obtained using Method of Satisfaction of Asymptotic Boundary Condition (MSABC) due to Nachtsheim and Swigert [23] with the boundary conditions (21). The detail of this technique is recently presented by Patel and Timol [21] and hence same is not repeated here. Controlling the non-dimensional numbers $\alpha = 10$ and then for $\beta = 5\times10^3;\ \beta = 5\times10^4;\ \beta = 5\times10^5$ the velocity profile and the slope of velocity profile are generated. (See figure-2 and figure- 3).

From figure (2), it is clear that both α and β have great influence on the velocity of the Prandtl-Eyring fluids. Here for fix value of α the velocity of fluid is increases rapidly and approaches to one as β increases. The slope of velocity profile in figure (3) is found always decreases fast an approaches to zero as β increases. Figures (2) and (3) are plotted in terms of dimensionless parameters and hence they represent behavior of all Prandtl-Eyring fluids

7. CONCLUSION:

The similarity solutions laminar incompressible boundary layer equations of all non-Newtonian viscoinelastic fluids that are characterized by the property that its shearing stress is related to the rate of strain by some arbitrary continuous function as shown in equation (1), is derived. It is interesting to note that the deductive group theoretic method based on general group of transformation is applied to derive proper similarity transformations for the non linear partial differential equation with the stress-strain functional relationship condition, governing the flow under consideration. It is to be observing that similarity solutions for all non-Newtonian fluids exist only for the flow past 90° wedge only. The present similarity equation is solved by MSABC for the case of Prandtl-Eyring fluid model. In similar manner one can discuss the similarity analysis numerical solutions other non-Newtonian fluid models listed in Table-1.

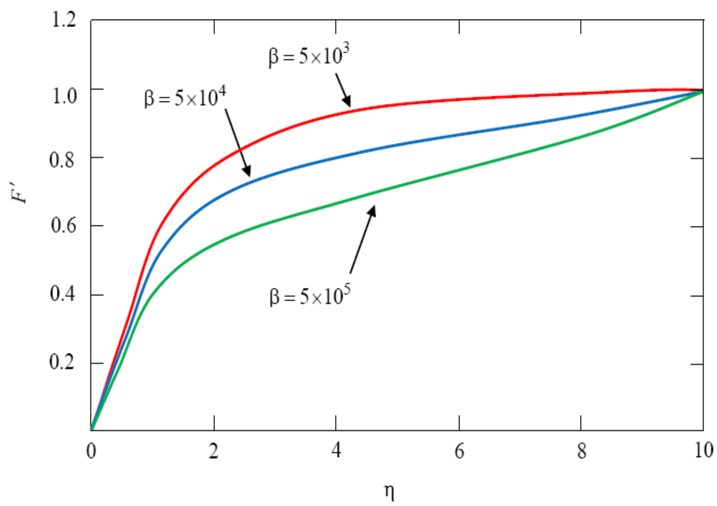

Figure-2: Velocity profile

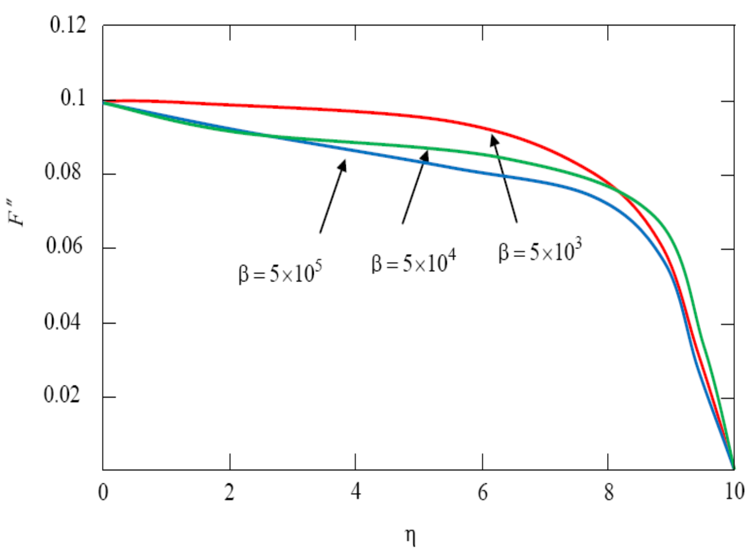

Figure-3: Slope of velocity profiles

	Fluid Model	Stress Vs Rate of Strain
1	Prandtl	$\tau = A\,sin^{-1}\left(\frac{1}{C}\frac{\partial u}{\partial y}\right)$
2	Powell Eyring	$\tau = \mu\frac{\partial u}{\partial y} + \frac{1}{B}sinh^{-1}\left(\frac{1}{C}\frac{\partial u}{\partial y}\right)$
3	Williamson	$\tau = \left(\frac{A}{B + \frac{\partial u}{\partial y}} + \mu_\infty\right)\frac{\partial u}{\partial y}$
4	Power-law	$\tau = m\left\lvert\frac{\partial u}{\partial y}\right\rvert^{n-1}\frac{\partial u}{\partial y}$
5	Eyring	$\tau = \frac{1}{B}\frac{\partial u}{\partial y} + C\,sin\left(\frac{\tau}{A}\right)$
6	Ellis	$-\frac{\partial u}{\partial y} = \left(A + B\lvert\tau\rvert^{\alpha-1}\right)\tau$
7	Prandtl-Eyring	$\tau = A\,sinh^{-1}\left(\frac{1}{B}\frac{\partial u}{\partial y}\right)$
8	Sisko	$\tau = A\frac{\partial u}{\partial y} + B\left\lvert\frac{\partial u}{\partial y}\right\rvert^{n}$
9	Reiner-Philippoff	$\tau = \left[\mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + (\tau/\tau_s)^2}\right]\frac{\partial u}{\partial y}$
10	Sutterby	$\tau = \mu_0\left[\left(B\frac{\partial u}{\partial y}\right)^{-1} sinh^{-1}\left(B\frac{\partial u}{\partial y}\right)\right]^{A}\frac{\partial u}{\partial y}$

Table 1: Stress-Strain relationship of different non-Newtonian fluids

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