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# RELATION BETWEEN GROUPS, TOPOLOGICAL GROUPS AND LIE GROUPS

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***ABSTRACT***

*In this paper, we have studied the relation between Topological Groups and Lie Groups. Also we established the relation between locally Euclidean group and lie group and its properties.*

***Keywords:*** *Lie Groups, Lie Algebra.*

## 1. INTRODUCTION

Lie algebras are vector spaces endowed with a special non-associative multiplication called a Lie bracket. They arise naturally in the study of mathematical objects called Lie groups, which serve as groups of transformations on spaces with certain symmetries. An example of a Lie group is the group O (3) of rotations of the unit sphere $x^2 + y^2 + z^2 = 1$ in $R^3$. While the study of Lie algebras without Lie groups deprives the subject of much of its motivation, Lie algebra theory is nonetheless a rich and beautiful subject which will reward the physics and mathematics student wishing to study the structure of such objects, and who expects to pursue further studies in Geometry, Algebra or Analysis.

## 2. PRELIMINARIES

**Definition 2.1:** A pair $(G,*)$ is called a group if the following are satisfied:

1. $G$ is a nonempty set and * is a binary operation on $G$.
2. $a*(b*c)=(a*b)*c \quad for \quad all \quad a,b \quad c \in G$
3. There exists $e \in G$ such that $a*e=e*a=a$ for all $a \in G$.
4. For each $a \in G$, there exists $a' \in G$ such that $a*a'=a'*a=e$.

**Definition 2.2:** A topological space is a pair $(X,\tau)$ where $X$ a set and F is a family of subsets of $X$ satisfying:

1. $\phi \in$ F and $X \in$ F,
2. F is a closed under arbitrary unions,
3. F is closed under finite intersection.

**Definition 2.3:** A manifold is a topological Space that locally resembles Euclidean space near each point. More precisely, each point of an n-dimensional manifold has a neighborhood that is homeomorphic to the Euclidean space of dimension n.

**Definition 2.4:** A smooth manifold or $C^\infty$-manifold is a differentiable manifold for which all the transition maps are smooth. That is, derivatives of all orders exist; so it is a $C^k$-manifold for all k.

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**Definition 2.5:** Let X be a topological space. Most commonly X is called locally compact, if every point of X has a compact neighborhood.

## 3. GROUPS, TOPOLOGICAL GROUPS AND LIE GROUPS

**Definition 3.1:** Topological Group $G$ is a topological space and group such that the group operations of product $G*G \to G:(x,y) \mapsto xy$ and taking inverse $G \to G: x \mapsto x^{-1}$ are continues functions with respect to the topology.

**Example 3.2:** The real numbers $R$, together with addition as operations and its usual topology form a topological group. More generally, Euclidean $n$ - space $R^n$ with addition and standard topology is a topological group.

**Example 3.3:** General linear group $GL(n,R)$ of all invertible matrices with real entries can be viewed as a topological group with the topology defined by viewing $GL(n,R)$ as a subset of Euclidean space $R^{n*n}$ .

**Proposition 3.4:** Every group is a topological group.

**Proof:** Let $G$ be a group and consider $G$ with discrete topology.

We know that any function from a discrete topological space to anther topological space is continues. Thus group operations of product and inverse functions are continues with respect to discrete topology. Hence $G$ is a topological group.

An important special type of topological group is Lie Group.

**Definition 3.5:** A Lie group is a set $G$ with two structures: $G$ is a group and $G$ is a smooth manifold.

**Definition 3.6:** A locally compact group is a topological group $G$ for which the underlying topology is locally compact and Hausdorff.

**Definition 3.7:** A topological space $X$ is called locally Euclidean if there is a non-negative integer $n$ such that every point in $X$ has a neighborhood which is homeomorphic to the Euclidean space $E^n$ (or, equivalently to the real $n$-space $R^n$).

**3.8. Topological properties of Lie groups**

1. Locally Compact Hausdorff.
2. Locally Euclidean.

**Lemma 3.9:** Every locally compact group need not be a lie group.

**Examples:**

1. The infinite dimensional torus $(R/Z)^Z$
2. The $p$ -adic $Z_p = \lim_{n\to\infty} \dfrac{Z}{p^n Z}$
3. The $p$ -adic field $Q_p = \bigcup_n p^{-n} Z_p$
4. The semi-direct product $Z*Q_p$ (with Z acting on $Q_p$ by multiplication by $p$ ).

**Gleason-Yamabe Theorem**: **3.10**

Every locally compact group is locally the inverse limit of Lie groups (related to Hilbert's fifth problem).

 

**Examples:**

1. $(R/Z)^Z$ is the inverse limit of the lie groups $(R/Z)^n$
2. $Z_p$ is the inverse limit of the lie groups $\frac{Z}{p^n Z}$
3. $Q_p$ Contains $Z_p$ as an open subgroup.
4. $Z*Q_p$ Contains $Z_p$ as an open subgroup.

**Proposition 3.11:** Every locally Euclidean group is a lie group.

**Proof:** Let $G$ be a locally Euclidean group. From the definition it is clear that $G$ is a group and smooth manifold. Hence $G$ is a lie group.

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