

DEVELOPMENT OF CERTAIN SUMMATION FORMULAE ASSOCIATED TO HYPERGEOMETRIC FUNCTION

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ABSTRACT

The main objective of the present paper is to obtain certain new results in the aspects of Hypergeometric function. The results presented here are presumably new.

Key words: Gaussian Hypergeometric function, Recurrence relation, Bailey summation theorem, Contiguous relation.

2000 MSC NO: 33C05, 33C20, 33C45, 33C60, 33C70.

I INTRODUCTION

Generalized Gaussian Hypergeometric function of one variable is defined by

$${}_A F_B(a_1, a_2, \dots, a_A; b_1, b_2, \dots, b_B; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_A)_k z^k}{(b_1)_k (b_2)_k \dots (b_B)_k k!} \quad (1)$$

or

$${}_A F_B((a_A); (b_B); z) \equiv {}_A F_B((a_j)_{j=1}^A; (b_j)_{j=1}^B; z) = \sum_{k=0}^{\infty} \frac{((a_A)_k z^k}{((b_B)_k k!} \quad (2)$$

where the parameters b_1, b_2, \dots, b_B are neither zero nor negative integers and A, B are non negative integers.

Contiguous Relation is defined by

[Andrews p.363(9.16) , E.D. p.51(10), H.T.F.I. p.103(32)]

$$(a-b) {}_2 F_1(a, b; c; z) = a {}_2 F_1(a+1, b; c; z) - b {}_2 F_1(a, b+1; c; z) \quad (3)$$

Recurrence relation is defined by

$$\Gamma(\xi+1) = \xi \Gamma(\xi) \quad (4)$$

Bailey summation theorem[Prud, p.491(7.3,7.8)]

$${}_2 F_1(a, 1-a; c; \frac{1}{2}) = \frac{\Gamma(\frac{c}{2}) \Gamma(\frac{c+1}{2})}{\Gamma(\frac{c+a}{2}) \Gamma(\frac{c+1-a}{2})} = \frac{\sqrt{\pi} \Gamma(c)}{2^{c-1} \Gamma(\frac{c+2}{2}) \Gamma(\frac{c+1-a}{2})} \quad (5)$$

II. MAIN SUMMATION FORMULAE

$$\begin{aligned} {}_2 F_1(a, -a-25; c; \frac{1}{2}) &= \frac{\sqrt{\pi} \Gamma(c)}{2^{c+25}} \left[\frac{-8036312828544000a + 6383931501018240a^2 - 1168556166367200a^3}{\Gamma(\frac{c+a+26}{2}) \Gamma(\frac{c-a+1}{2})} \right. \\ &+ \frac{-33400168327344a^4 + 12326752815000a^5 + 436584731700a^6 - 32410396200a^7 - 1980385212a^8 - 17355000a^9}{\Gamma(\frac{c+a+26}{2}) \Gamma(\frac{c-a+1}{2})} \\ &+ \frac{933660a^{10} + 23400a^{11} + 156a^{12} + 8036313307545600c - 18853116649286400ac + 8311465180328544a^2 c}{\Gamma(\frac{c+a+26}{2}) \Gamma(\frac{c-a+1}{2})} \\ &+ \frac{-750403860921000a^3 c - 76964911768420a^4 c + 5160077122850a^5 c + 439747337263a^6 c - 588501550a^7 c}{\Gamma(\frac{c+a+26}{2}) \Gamma(\frac{c-a+1}{2})} \end{aligned}$$

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$$\begin{aligned}
& + \frac{-635308453a^8c - 14550250a^9c - 27027a^{10}c + 1950a^{11}c + 13a^{12}c + 12469186634711040c^2}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
& + \frac{-15244455037593600ac^2 + 4111481146496256a^2c^2 - 117688601160000a^3c^2 - 35000831923200a^4c^2}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
& + \frac{196862374800a^5c^2 + 104863331664a^6c^2 + 2276820000a^7c^2 - 38656800a^8c^2 - 1638000a^9c^2 - 13104a^{10}c^2}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
& + \frac{8101547955191808c^3 - 6293079853632000ac^3 + 1039604633254720a^2c^3 + 18015042942000a^3c^3}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
& - \frac{6607598716160a^4c^3 - 163157367100a^5c^3 + 8818126772a^6c^3 + 342797000a^7c^3 + 1721720a^8c^3 - 45500a^9c^3}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
& - \frac{364a^{10}c^3 + 2965404150398976c^4 - 1546408531200000ac^4 + 147164252352000a^2c^4 + 8477047488000a^3c^4}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
& - \frac{586669050240a^4c^4 - 28654080000a^5c^4 + 127545600a^6c^4 + 17472000a^7c^4 + 174720a^8c^4}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
& + \frac{689131315265536c^5 - 243815025958400ac^5 + 11317691841664a^2c^5 + 1233717430400a^3c^5}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
& - \frac{19378751392a^4c^5 - 1987148800a^5c^5 - 18001984a^6c^5 + 291200a^7c^5 + 2912a^8c^5 + 108040106803200c^6}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
& - \frac{25585214054400ac^6 + 313199437824a^2c^6 + 93824640000a^3c^6 + 566092800a^4c^6 - 62899200a^5c^6}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
& - \frac{838656a^6c^6 + 11793919115264c^7 - 1807265740800ac^7 - 20436229632a^2c^7 + 3992352000a^3c^7}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
& + \frac{64247040a^4c^7 - 748800a^5c^7 - 9984a^6c^7 + 907887771648c - 84913920000ac^8 - 2273356800a^2c^8}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
& + \frac{89856000a^3c^8 + 1797120a^4c^8 + 49116807168c^9 - 2543424000ac^9 - 91336960a^2c^9 + 832000a^3c^9}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
& + \frac{16640a^4c^9 + 1827471360c^{10} - 43929600ac^{10} - 1757184a^2c^{10} + 44515328c^{11} - 332800ac^{11}}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
& - \frac{13312a^2c^{11} + 638976c^{12} + 4096c^{13}}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} + \frac{2(12952950506496001295295050649600 - 1953217026028800a)}{\Gamma(\frac{c+a+25}{2})\Gamma(\frac{c-a}{2})} \\
& + \frac{2(720182075218848a^2 - 56242392874200a^3 - 6430140382484a^4 + 384225527550a^5 + 34503650159a^6)}{\Gamma(\frac{c+a+25}{2})\Gamma(\frac{c-a}{2})} \\
& + \frac{2(-18877950a^7 - 48737217a^8 - 1122750a^9 - 2107a^{10} + 150a^{11} + a^{12} + 2881193222799360c)}{\Gamma(\frac{c+a+25}{2})\Gamma(\frac{c-a}{2})} \\
& + \frac{2(-2832979633459200ac + 677960022661632a^2c - 15454257360000a^3c - 5564994547200a^4c)}{\Gamma(\frac{c+a+25}{2})\Gamma(\frac{c-a}{2})} \\
& + \frac{2(21440848800a^5c + 16171677984a^6c + 355656000a^7c - 5893440a^8c - 252000a^9c - 2016a^{10}c)}{\Gamma(\frac{c+a+25}{2})\Gamma(\frac{c-a}{2})} \\
& + \frac{2(2418876921028608c^2 - 1642725690969600ac^2 + 248581412961216a^2c^2 + 5112782498000a^3c^2)}{\Gamma(\frac{c+a+25}{2})\Gamma(\frac{c-a}{2})} \\
& + \frac{2(-1539633375040a^4c^2 - 39181485700a^5c^2 + 2021088524a^6c^2 + 79331000a^7c^2 + 399560a^8c^2)}{\Gamma(\frac{c+a+25}{2})\Gamma(\frac{c-a}{2})} \\
& + \frac{2(-10500a^9c^2 - 84a^{10}c^2 + 1082575376154624c^3 - 515546065920000ac^3 + 45753344563200a^2c^3)}{\Gamma(\frac{c+a+25}{2})\Gamma(\frac{c-a}{2})} \\
& + \frac{2(2753054976000a^3c^3 - 179634900480a^4c^3 - 8913408000a^5c^3 + 37954560a^6c^3 + 5376000a^7c^3)}{\Gamma(\frac{c+a+25}{2})\Gamma(\frac{c-a}{2})} \\
& + \frac{2(53760a^8c^3 + 297280550465536c^4 - 986653855360000ac^4 + 4297052178560a^2c^4 + 485263408000a^3c^4)}{\Gamma(\frac{c+a+25}{2})\Gamma(\frac{c-a}{2})} \\
& + \frac{2(-7280231840a^4c^4 - 766304000a^5c^4 - 6950720a^6c^4 + 112000a^7c^4 + 1120a^8c^4 + 53811029606400c^5)}{\Gamma(\frac{c+a+25}{2})\Gamma(\frac{c-a}{2})} \\
& + \frac{2(-12175194931200ac^5 + 134726602752a^2c^5 + 43690752000a^3c^5 + 269015040a^4c^5 - 29030400a^5c^5)}{\Gamma(\frac{c+a+25}{2})\Gamma(\frac{c-a}{2})} \\
& + \frac{2(-387072a^6c^5 + 6669524516864c^6 - 989589350400ac^6 - 11594774016a^2c^6 + 2155104000a^3c^6)}{\Gamma(\frac{c+a+25}{2})\Gamma(\frac{c-a}{2})} \\
& + \frac{2(34702080a^4c^6 - 403200a^5c^6 - 5376a^6c^6 + 575608061952c^7 - 52660224000ac^7 - 1415208960a^2c^7)}{\Gamma(\frac{c+a+25}{2})\Gamma(\frac{c-a}{2})} \\
& + \frac{2(55296000a^3c^7 + 1105920a^4c^7 + 34569351168c^8 - 1765056000ac^8 - 63402240a^2c^8 + 576000a^3c^8)}{\Gamma(\frac{c+a+25}{2})\Gamma(\frac{c-a}{2})}
\end{aligned}$$

$$\begin{aligned}
 & + \frac{2(11520a^4c^8 + 1416560640c^9 - 33792000ac^9 - 1351680a^2c^9 + 37756928c^{10} - 281600ac^{10})}{\Gamma(\frac{c+a+25}{2})\Gamma(\frac{c-a}{2})} \\
 & + \frac{2(11264a^2c^{10} + 589824c^{11} + 4096c^{12})}{\Gamma(\frac{c+a+25}{2})\Gamma(\frac{c-a}{2})}] \quad (6)
 \end{aligned}$$

$$\begin{aligned}
 {}_2F_1(a, -a-26; c; \frac{1}{2}) = & \frac{\sqrt{\pi}\Gamma(c)}{2^{c+26}} [\frac{-16072626136089600a + 13440284546567040a^2}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
 & - 2865451725805248a^3 + 29899479296856a^4 + 26048131417484a^5 + 79808986650a^6 - 85137424259a^7 \\
 & + \frac{-2401982262a^8 + 36821217a^9 + 2318910a^{10} + 27607a^{11} + 6a^{12} - a^{13} + 16072626615091200c}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
 & - 38378656329546240ac + 18135288054787392a^2c - 2143868237944368a^3c - 95889592630904a^4c \\
 & + \frac{14727578672700a^5c + 579349499122a^6c - 20464852716a^7c - 1220378670a^8c - 11006940a^9c + 241766a^{10}c}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
 & + \frac{4284a^{11}c + 14a^{12}c + 24938373269422080c^2 - 31472930307806208ac^2 + 9360754382865024a^2c^2}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
 & - 525250859221216a^3c^2 - 60965497026000a^4c^2 + 2228972480040a^5c^2 + 195977165580a^6c^2 \\
 & + \frac{718417476a^7c^2 - 139716360a^8c^2 - 2478560a^9c^2 - 4788a^{10}c^2 + 84a^{11}c^2 + 16203095910383616c^3}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
 & - 13177592511000576ac^3 + 2506066361255936a^2c^3 - 29234743852800a^3c^3 - 13902669779200a^4c^3 \\
 & + \frac{-28640969280a^5c^3 + 23184415296a^6c^3 + 432754560a^7c^3 - 3534720a^8c^3 - 114240a^9c^3 - 448a^{10}c^3}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
 & + \frac{5930808300797952c^4 - 3289236014145536ac^4 + 387524157580800a^2c^4 + 8910512877440a^3c^4}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
 & + \frac{-1530020741760a^4c^4 - 35555960160a^5c^4 + 1051223040a^6c^4 + 32262720a^7c^4 + 120960a^8c^4 - 1120a^9c^4}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
 & + \frac{1378262630531072c^5 - 528211311897600ac^5 + 35382079905024a^2c^5 + 1940021943552a^3c^5}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
 & - 82568711232a^4c^5 - 3405749760a^5c^5 + 4480896a^6c^5 + 822528a^7c^5 + 4032a^8c^5 + 216080213606400c^6 \\
 & - 56700900798464ac^6 + 1755444483072a^2c^6 + 175031894016a^3c^6 - 1491114240a^4c^6 - 137114880a^5c^6 \\
 & - 854784a^6c^6 + 5376a^7c^6 + 23587838230528c^7 - 4125683662848ac^7 + 24202340352a^2c^7 \\
 & + \frac{8523095040a^3c^7 + 47109120a^4c^7 - 2350080a^5c^7 - 15360a^6c^7 + 1815775543296c^8 - 202061703168ac^8}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
 & - 2163939840a^2c^8 + 225834240a^3c^8 + 2419200a^4c^8 - 11520a^5c^8 + 98233614336c^9 - 6452751360ac^9 \\
 & - 130803200a^2c^9 + 2872320a^3c^9 + 28160a^4c^9 + 3654942720c^{10} - 125052928ac^{10} - 2939904a^2c^{10} \\
 & + \frac{11264a^3c^{10} + 89030656c^{11} - 1253376ac^{11} - 24576a^2c^{11} + 1277952c^{12} - 4096ac^{12} + 8192c^{13}}{\Gamma(\frac{c+a+26}{2})\Gamma(\frac{c-a+1}{2})} \\
 & + \frac{64764752532480000 - 97779798507686400a + 36056589656477760a^2 - 2694702659543952a^3}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})} \\
 & - 369574828819456a^4 + 20665327875016a^5 + 2236939003900a^6 + 3670183859a^7 - 3889759488a^8 \\
 & + \frac{-110394717a^9 - 282260a^{10} + 26993a^{11} + 344a^{12} + a^{13} + 146650251241267200c - 145689354969315840ac}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})} \\
 & + \frac{35136406966416192a^2c - 655706340883568a^3c - 330238342330904a^4c + 294647270700a^5c}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})} \\
 & + \frac{1136579900122a^6c + 30109073484a^7c - 547160670a^8c - 31278940a^9c - 358834a^{10}c + 84a^{11}c}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})} \\
 & + \frac{14a^{12}c + 126706232497029120c^2 - 87746742971476992ac^2 + 1348042894330176a^2c^2}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})} \\
 & + \frac{361996459861216a^3c^2 - 97470584814000a^4c^2 - 2983094576040a^5c^2 + 159407581620a^6c^2}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{7772974524a^7c^2 + 51012360a^8c^2 - 1889440a^9c^2 - 28812a^{10}c^2 - 84a^{11}c^2 + 58966522649788416c^3}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})} \\
 & + \frac{-28945598936446976ac^3 + 2623721049575936a^2c^3 + 185974235603200a^3c^3 - 12411840803200a^4c^3}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})} \\
 & + \frac{-740580953280a^5c^3 + 3187711296a^6c^3 + 667282560a^7c^3 + 9569280a^8c^3 - 2240a^9c^3 - 448a^{10}c^3}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})} \\
 & + \frac{17029178275586048c^4 - 5903690107454464ac^4 + 262078681747200a^2c^4 + 35380823666560a^3c^4}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})} \\
 & + \frac{-549072162240a^4c^4 - 74910759840a^5c^4 - 877511040a^6c^4 + 20153280a^7c^4 + 383040a^8c^4 + 1120a^9c^4}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})} \\
 & + \frac{3285112581251072c^5 - 789599995008000ac^5 + 8118484564224a^2c^5 + 3597900241152a^3c^5}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})} \\
 & + \frac{28605624768a^4c^5 - 3603317760a^5c^5 - 68901504a^6c^5 + 16128a^7c^5 + 4032a^8c^5 + 441098285056000c^6}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})} \\
 & + \frac{-71082094759936ac^6 - 1046287721472a^2c^6 + 210330537984a^3c^6 + 4359210240a^4c^6 - 72549120a^5c^6}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})} \\
 & + \frac{-1833216a^6c^6 - 5376a^7c^6 + 42119452131328c^7 - 4314008629248ac^7 - 141761691648a^2c^7 + 6863447040a^3c^7}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})} \\
 & + \frac{196869120a^4c^7 - 46080a^5c^7 - 15360a^6c^7 + 2879683682304c^8 - 172053752832ac^8 - 7941980160a^2c^8}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})} \\
 & + \frac{103637760a^3c^8 + 3916800a^4c^8 + 11520a^5c^8 + 139966734336c^9 - 4194319360ac^9 - 240627200a^2c^9}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})} \\
 & + \frac{56320a^3c^9 + 28160a^4c^9 + 4720967680c^{10} - 50665472ac^{10} - 3818496a^2c^{10} - 11264a^3c^{10} + 105005056c^{11}}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})} \\
 & + \frac{-24576ac^{11} - 24576a^2c^{11} + 1384448c^{12} + 4096ac^{12} + 8192c^{13}}{\Gamma(\frac{c+a+27}{2})\Gamma(\frac{c-a}{2})}] \quad (7)
 \end{aligned}$$

III. DERIVATIONS OF THE FORMULAE:

Putting $b = -a-25$, $z = \frac{1}{2}$ in known formula (3), we get

$$(2a+25) {}_2F_1(a, -a-25; c; \frac{1}{2}) = a {}_2F_1(a+1, -a-25; c; \frac{1}{2}) + (a+25) {}_2F_1(a, -a-24; c; \frac{1}{2})$$

Now applying the same method of Ref [2], we get both the formulae.

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