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$\theta\omega$-CLOSED SETS IN TOPOLOGICAL SPACES

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ABSTRACT

In this paper, we offer a new class of sets called $\theta\omega$-closed sets in topological spaces and we study some of its basic properties. The family of $\theta\omega$-closed sets of a topological space forms a topology and is denoted by $\tau\theta\omega$. Notice that this class of sets lies between the class of θ-closed sets and the class of θg-closed sets. Using these sets, we obtain a decomposition of θ-continuity and we introduce new spaces called $T\theta\omega$ and $_gT\theta\omega$. Using these spaces we obtain another decomposition of $T_{1/2}$-spaces.

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Key words and Phrases: *Topological space, θg-closed set, $\theta\omega$-closed set, $\theta\omega$-open set, ω-closed set, θ-continuous function, $\theta\omega$ lc*-continuous function, $T\theta\omega$-space, $_gT\theta\omega$-space.*

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1. INTRODUCTION

In 1963 Levine [15] introduced the notion of semi-open sets. Velicko [25] introduced the notion of θ-closed sets and it is well known that the collection of all θ-closed sets of a topological space forms a topology and is denoted by $\tau\theta$. Levine [14] also introduced the notion of g-closed sets and investigated its fundamental properties. This notion was shown to be productive and very useful. Dontchev and Maki [10] introduced the notion of θ-generalized closed sets.

After the advent of g-closed sets, Arya and Nour [4], Sheik John [21] and Dontchev [9] introduced gs-closed sets, ω-closed sets and gsp-closed sets respectively.

In this paper, we introduce a new class of sets called $\theta\omega$-closed sets in topological spaces. This class lies between the class of θ-closed sets and the class of θg-closed sets. We study some of its basic properties and characterizations. Interestingly it turns out that the family of $\theta\omega$-closed sets of a topological space forms a topology. This collection is denoted by $\tau\theta\omega$. From the definitions, it follows immediately that $\tau_\theta \subseteq \tau\theta\omega \subseteq \tau$. Using these sets, we obtain a decomposition of θ-continuity and we introduce new type of spaces called $T\theta\omega$-spaces and $_gT\theta\omega$-spaces. Using these spaces, we obtain another decomposition of $T_{1/2}$-spaces.

2. PRELIMINARIES

Throughout this paper (X, τ) and (Y, σ) (or X and Y) represents topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and A^c or X | A denote the closure of A, the interior of A and the complement of A respectively.

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We recall the following definitions which are useful in the sequel.

Definition: 2.1 A subset A of a space (X, τ) is called:
(i) semi-open set [15] if A $\subseteq$ cl(int(A));
(ii) preopen set [17] if A $\subseteq$ int(cl(A));
(iii) α-open set [18] if A $\subseteq$ int(cl(int(A)));
(iv) β-open set [1] (= semi-preopen [2]) if A $\subseteq$ cl(int(cl(A)));
(v) regular open set [22] if A = int(cl(A)).

The complements of the above mentioned open sets are called their respective closed sets.

The preclosure [19] (resp. semi-closure [7], α-closure [18], semi-pre-closure [2]) of a subset A of X, denoted by pcl(A) (resp. scl(A), αcl(A), spcl(A)), is defined to be the intersection of all preclosed (resp. semi-closed, α-closed, semi-preclosed) sets of (X, τ) containing A. It is known that pcl(A) (resp. scl(A), αcl(A), spcl(A)) is a preclosed (resp. semi-closed, α-closed, semi-preclosed) set.

Definition: 2.2 [25] A point x of a space X is called a θ-adherent point of a subset A of X if cl(U) ∩ A ≠ ϕ, for every open set U containing x. The set of all θ-adherent points of A is called the θ-closure of A and is denoted by cl_θ(A). A subset A of a space X is called θ-closed if and only if A = cl_θ(A). The complement of a θ-closed set is called θ-open. Similarly, the θ-interior of a set A in X, written int_θ(A), consists of those points x of A such that for some open set U containing x, cl(U) $\subseteq$ A. A set A is θ-open if and only if A = int_θ(A), or equivalently, X \ A is θ-closed.

A point x of a space X is called a δ-adherent point of a subset A of X if int(cl(U)) ∩ A ≠ ϕ, for every open set U containing x. The set of all δ-adherent points of A is called the δ-closure of A and is denoted by cl_δ(A). A subset A of a space X is called δ-closed if and only if A = cl_δ(A). The complement of a δ-closed set is called δ-open. Similarly, the δ-interior of a set A in X, written int_δ(A), consists of those points x of A such that for some regularly open set U containing x, U $\subseteq$ A. A set A is δ-open if and only if A = int_δ(A), or equivalently, X \ A is δ-closed.

The family of all θ-open (resp. δ-open) subsets of (X,τ) forms a topology on X and is denoted by τ_θ (resp. τ_δ). From the definitions it follows immediately that $\tau_\theta \subseteq \tau_\delta \subseteq \tau$. [6].

Definition: 2.3 A point x ∈ X is called a semi θ-cluster [8] point of A if A ∩ scl(U) ≠ ϕ for each semi-open set U containing x.

The set of all semi θ-cluster points of A is called the semi-θ-cluster of A and is denoted by sclθ(A). Hence, a subset A is called semi-θ-closed if sclθ(A) = A. The complement of a semi-θ-closed set is called semi-θ-open set.

Recall that a subset A of a space (X, τ) is said to be δ-semi-open [20] if A $\subseteq$ cl(intδ(A)).

Definition: 2.4 A subset A of a space (X, τ) is called:
(i) a generalized closed (briefly g-closed) set [14] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ).

(ii) a generalized semi-closed (briefly gs-closed) set [4] if scl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ).

(iii) an α-generalized closed (briefly αg-closed) set [16] if αcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ).

(iv) a generalized semi-preclosed (briefly gsp-closed) set [9] if spcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ).

(v) a generalized preclosed (briefly gp-closed) set [19] if pcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ).

(vi) a ĝ-closed set [23] (= ω-closed set [21]) if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open in (X, τ).

(vii) a θ-generalized closed set (briefly θg-closed) [10] if cl_θ(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ).

Remark: 2.5 The collection of all θg-closed (resp. ω-closed, g-closed, θ-closed, α-closed, semi-closed) sets of X is denoted by θG C(X) (resp. ω C(X), G C(X), θC(X), α C(X), S C(X)).

We denote the power set of X by P(X).

Remark: 2.6 [5] We have the following diagram in which the converses of the implications need not be true.

$$\begin{tikzcd}
\theta\text{-open} \arrow[r] \arrow[d] & \delta\text{-open} \arrow[r] \arrow[d] & \text{open} \arrow[d] \\
\text{semi-}\theta\text{-open} \arrow[r] & \delta\text{-semi-open} \arrow[r] & \text{semi-open}
\end{tikzcd}$$

Remark: 2.7 [21]
(1) Every θ-closed set is θg-closed.
(2) θg-closed sets and ω-closed sets are independent.

Remark: 2.8 [6] (X,τ) is regular if and only if $\tau_\theta = \tau$.

Remark: 2.9 [21] A space X is called $\tau\omega$ if ω-closed set in X is closed.

Definition 2.10 A topological space (X,τ) is called a R_1-space [11] if every two different points with distinct closures have disjoint neighborhoods.

Proposition 2.11 [6] Let (X,τ) be a space. Then,
(i) if A ⊆ X is preopen then cl(A) = cl_θ(A).
(ii) (X,τ) is R_1 if and only if cl({x}) = cl_θ({x}) for each x ∈ X.

Proposition 2.12 [11, 12] Let (X,τ) be a space. If A ⊆ X is preopen then cl(A) = α cl(A) = cl_δ(A).

Definition 2.13 [14] A space (X,τ) is called $T_{1/2}$-space if every g-closed set is closed.

3. $\theta\omega$-CLOSED SETS

We introduce the following definition.

Definition: 3.1 A subset A of X is called a $\theta\omega$-closed set if cl_θ(A) ⊆ U whenever A ⊆ U and U is semi-open in (X, τ). The complement of $\theta\omega$-closed set is called $\theta\omega$-open set.

The collection of all $\theta\omega$-closed sets of X is denoted by $\theta\omega$C(X).

Proposition: 3.2 Every θ-closed set is $\theta\omega$-closed.

Proof: Let A be an θ-closed set and G be any semi-open set containing A in (X, τ). Since A is θ-closed, cl_θ(A) = A for every subset A of X. Therefore cl_θ(A) ⊆ G and hence A is $\theta\omega$-closed set.

The converse of Proposition 3.2 need not be true as seen from the following example.

Example: 3.3 Let X = {a, b, c} with τ = {ϕ, {a}, X}. Then $\theta\omega$C(X) = {ϕ, {b, c}, X} and θC(X) = {ϕ, X}. Here, A = {b, c} is $\theta\omega$-closed but not θ-closed set in (X, τ).

Proposition: 3.4 Every $\theta\omega$-closed set is g-closed.

Proof: Let A be an $\theta\omega$-closed set and G be any open set containing A in (X, τ). Since every open set is semi-open and A is $\theta\omega$-closed, cl_θ(A) ⊆ G. Since cl(A) ⊆ cl_θ(A) ⊆ G , cl(A) ⊆ G and hence A is g-closed.
The converse of Proposition 3.4 need not be true as seen from the following example.

Example: 3.5 Let X and τ be as in the Example 3.3. Then $\theta\omega$C(X) = {ϕ, {b, c}, X} and GC(X) = {ϕ, {b}, {c}, {a, b}, {a, c}, {b, c}, X}. Here, A = {a, b} is g-closed but not $\theta\omega$-closed set in (X, τ).

Proposition: 3.6 Every $\theta\omega$-closed set is ω-closed.

Proof: Let A be an $\theta\omega$-closed subset of (X, τ) and G be any semi-open set containing A. Since cl(A) ⊆ cl_θ(A) ⊆ G and hence A is ω-closed.

The converse of Proposition 3.6 need not be true as seen from the following example.

Example: 3.7 Let X = {a, b, c} with τ = {ϕ, {a}, {a, b}, X}. Then $\theta\omega$C(X) = {ϕ, {b, c}, X} and ωC(X) = {ϕ, {c}, {b, c}, X}. Here, A = {c} is ω-closed but not $\theta\omega$-closed set in (X, τ).

Proposition: 3.8 Every θg-closed set is g-closed.

Proof: Let A be an θg-closed subset of (X, τ) and G be any open set containing A. Since cl(A) ⊆ cl_θ(A) ⊆ G and hence A is g-closed.

The converse of Proposition 3.8 need not be true as seen from the following example.

Example: 3.9 Let X = {a, b, c} with τ = {ϕ, {a}, {a, b}, {a, c}, X}. Then θG C(X) = {ϕ, {b, c}, X} and G C(X) = {ϕ, {b}, {c}, {b, c}, X}. Here, A = {b} is g-closed but not θg-closed set in (X, τ).

Proposition: 3.10 Every $\theta\omega$-closed set is θg-closed.

Proof: Let A be an $\theta\omega$-closed set and G be any open set containing A in (X, τ). Since every open set is semi-open and A is $\theta\omega$-closed, cl_θ(A) ⊆ G. Therefore cl_θ(A) ⊆ G and G is open. Hence A is θg-closed.

The converse of Proposition 3.10 need not be true as seen from the following example.

Example: 3.11 Let X and τ be as in the Example 3.3. Then $\theta\omega$C(X) = {ϕ, {b, c}, X} and θGC(X) = {ϕ, {b}, {c}, {a, b}, {a, c}, {b, c}, X}. Here, A = {a, c} is θg-closed but not $\theta\omega$-closed set in (X, τ).

Remark: 3.12 The following examples show that $\theta\omega$-closedness is independent of closedness, semi-closedness and α-closedness.

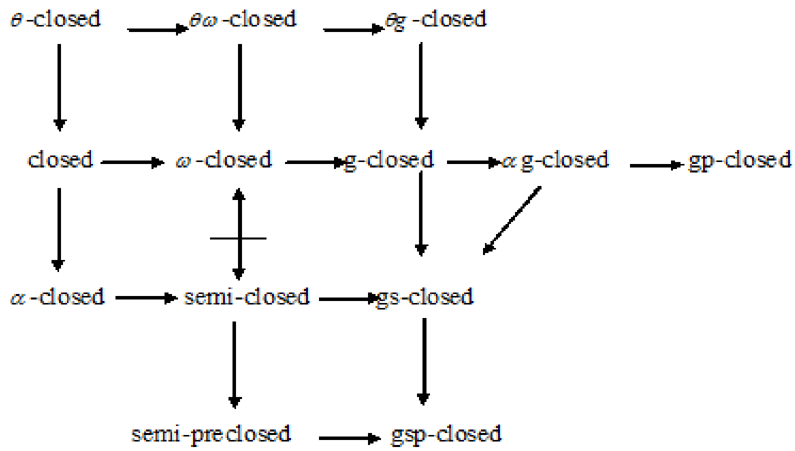
Example: 3.13 Let X and τ be as in the Example 3.3. Then $\theta\omega$C(X) = {ϕ, {b, c}, X} and α C(X) = S C(X) = {ϕ, {b}, {c}, {b, c}, X}. Here, A = {b} is α-closed as well as semi-closed in (X, τ) but it is not $\theta\omega$-closed in (X, τ).

Example: 3.14 Let X = {a, b, c} with τ = {ϕ, {a, b}, X}. Then $\theta\omega$C(X) = {ϕ, {c}, {a, c}, {b, c}, X} and α C(X) = S C(X) = {ϕ, {c}, X}. Here, A = {a, c} is $\theta\omega$-closed but it is neither α-closed nor semi-closed in (X,τ).

Example: 3.15 In Example 3.7, {c} is closed set but not $\theta\omega$-closed.

In Example 3.14, {b, c} is $\theta\omega$-closed set but not closed.

Remark: 3.16 From the above discussions and known results in [9, 11, 21, 24], we obtain the following diagram, where A → B (resp. A $\longleftrightarrow\!\!\!\!\!\!\!\!\;\;\;|\;\;$ B) represents A implies B but not conversely (resp. A and B are independent of each other).



None of the above implications is reversible as shown in the above examples and in the related papers [9, 11, 21, 24].

4. PROPERTIES OF $\theta\omega$-CLOSED SETS

Definition: 4.1 [21] The intersection of all semi-open subsets of (X,τ) containing A is called the semi-kernel of A and is denoted by s-ker (A).

Lemma: 4.2 A subset A of (X,τ) is $\theta\omega$-closed if and only if cl_θ(A) ⊆ s-ker(A).

Proof: Suppose that A is $\theta\omega$-closed. Then cl_θ(A) ⊆ U whenever A ⊆ U and U is semi-open. Let x ∈ cl_θ(A). If x ∉ s-ker (A), then there is a semi-open set U containing A such that x ∉ U. Since U is a semi-open set containing A, we have x ∉ cl_θ(A) and this is a contradiction.

Conversely, let cl_θ(A) ⊆ s-ker(A). If U is any semi-open set containing A, then cl_θ(A) ⊆ s-ker(A) ⊆ U. Therefore, A is $\theta\omega$-closed.

Remark: 4.3 The collection of all $\theta\omega$-closed sets of a topological space forms a topology and is denoted by $\tau\theta\omega$.

Remark: 4.4 If A is a $\theta\omega$-closed set and F is a θ-closed set, then A ∩ F is a $\theta\omega$-closed set.

Proof: Since F is θ-closed, it is $\theta\omega$-closed. Therefore by Remark 4.3, A ∩ F is also a $\theta\omega$-closed set.

Proposition: 4.5 If a set A is $\theta\omega$-closed in (X, τ), then cl_θ(A) – A contains no nonempty semi-closed set in (X, τ).

Proof: Suppose that A is $\theta\omega$-closed. Let F be a semi-closed subset of cl_θ(A) – A. Then A ⊆ F^c. Therefore cl_θ(A) ⊆ F^c. Consequently, F ⊆ (cl_θ(A))^c. We already have F ⊆ cl_θ(A). Thus F ⊆ cl_θ(A) ∩ (cl_θ(A))^c and F is empty.

The converse of Proposition 4.5 need not be true as seen from the following example.

Example: 4.6 Let X and τ be as in the Example 3.14. Then $\theta\omega$C(X) = {ϕ, {c}, {a, c}, {b, c}, X} and SC(X) = {ϕ, {c}, X}. If A = {c}, then cl_θ(A) – A = {a, b} does not contain any nonempty semi-closed set. But A is not $\theta\omega$-closed in (X, τ).

Proposition: 4.7 Let A ⊆ Y ⊆ X where Y is open and suppose that A is $\theta\omega$-closed in (X, τ). Then A is $\theta\omega$-closed relative to Y.

Proof: Let A ⊆ Y ∩ G, where G is semi-open in (X, τ). Then A ⊆ G and hence cl_θ(A) ⊆ G. This implies that Y ∩ cl_θ(A) ⊆ Y ∩ G. Thus A is $\theta\omega$-closed relative to Y since the intersection of open and semi-open is semi-open [6].

Proposition: 4.8 If A is a semi-open and $\theta\omega$-closed in (X,τ), then A is θ-closed in (X,τ).

Proof: Since A is semi-open and $\theta\omega$-closed, cl_θ(A) ⊆ A and hence A is θ-closed in (X,τ).

Theorem: 4.9 Let A be a subset of a regular space (X,τ). Then,

(i) A is $\theta\omega$-closed if and only if A is ω-closed.

(ii) if (X, τ) is τ_ω, then A is $\theta\omega$-closed if and only if A is closed.

Proof:
(i) It follows from Remark 2.8.
(ii) It follows from Remark 2.9.

Theorem: 4.10 Let A be a preopen subset of a topological space (X,τ). Then the following conditions are equivalent.
(i) A is $\theta\omega$-closed.
(ii) A is θg -closed (or ω-closed).
(iii) A is g-closed.
(iv) A is α g-closed.

Proof:
(i) ⇒ (ii) ⇒ (iii) ⇒ (iv). It is obvious from Remark 3.16.
(iv) ⇒ (i). It follows from Propositions 2.11 and 2.12.

Recall that a partition space [11] is a topological space where every open set is closed.

Corollary: 4.11 Let A be a subset of the partition space (X,τ). Then the following conditions are equivalent.
(i) A is $\theta\omega$-closed.
(ii) A is θg -closed (or ω-closed).
(iii) A is g-closed.
(iv) A is α g-closed.

Proof: A topological space is a partition space if and only if every subset is preopen. Then the claim follows straight from Theorem 4.10.

Theorem: 4.12 For a singleton subset A of an R_1 topological space (X,τ), the following conditions are equivalent.
(i) A is $\theta\omega$-closed.
(ii) A is ω-closed.

Proof:
(i) ⇒ (ii) is clear.
(ii) ⇒ (i). Note that in R_1-spaces, the concepts of closure and θ-closure coincide for singleton sets: see Proposition 2.11.

Theorem: 4.13 For a subset A of a topological space (X,τ), the following conditions are equivalent.
(i) A is clopen.
(ii) A is $\theta\omega$-closed, preopen and semi-closed.
(iii) A is $\theta\omega$-closed and (regular) open.
(iv) A is α g-closed and (regular) open.

Proof:
(i) ⇒ (ii) ⇒ (iii) ⇒ (iv) are obvious.
(iv) ⇒ (i). It follows from Theorem 3.13 [11].

Lemma: 4.14 In any space, if a singleton is θ-open then it is regular open.

Proof: It follows from the fact that, in any space, a singleton is δ-open if and only if it is regular open [11].

Lemma: 4.15 In a regular space, singleton is θ-open if and only if it is regular open.

Lemma: 4.16 If A is both closed and preopen of a topological space X, then the following are equivalent.
(i) A is θ-closed.
(ii) A is δ-closed.
(iii) A is α-closed.

Proof: It is obvious from the fact that A = cl(A) = cl_δ(A) = cl_θ(A) = α cl(A) (see. Propositions 2.11 and 2.12)

Lemma: 4.17 If a subset A of a space (X,τ) is clopen, then the following are equivalent.
(i) A is θ-closed.
(ii) A is δ-closed.
(iii) A is α-closed.
(iv) A is regular closed.

Definition: 4.18 A space (X,τ) is called locally s-θ-indiscrete space if every semi-open set is θ-closed.

Theorem: 4.19 For a topological space (X,τ), the following conditions are equivalent.
(i) X is locally s-θ-indiscrete.
(ii) Every subset of X is $\theta\omega$-closed.

Proof:
(i) ⇒ (ii). Let A ⊆ U, where U is semi-open and A is an arbitrary subset of X. Since X is locally s-θ-indiscrete, then U is θ-closed. We have cl_θ(A) ⊆ cl_θ(U) = U. Thus A is $\theta\omega$-closed.

(ii) ⇒ (i). If U ⊆ X is semi-open, then by (ii) cl_θ(U) ⊆ U or equivalently U is θ-closed. Hence X is locally s-θ-indiscrete.

5. DECOMPOSITION OF θ-CONTINUITY

In this section, we obtain a decomposition of continuity called θ-continuity in topological spaces.

To obtain a decomposition of θ-continuity, we first introduce the notion of $\theta\omega$lc*-continuous functions in topological spaces and by using $\theta\omega$-continuity, prove that a function is θ-continuous if and only if it is both $\theta\omega$-continuous and $\theta\omega$lc*-continuous.

We introduce the following definition.

Definition: 5.1 A subset A of a space (X, τ) is called $\theta\omega$lc*-set if A = M ∩ N, where M is semi-open and N is θ-closed in (X, τ).

Example: 5.2 Let X and τ be as in the Example 3.3. Then {a, b} is $\theta\omega$lc*-set in (X, τ).

Remark: 5.3 Every θ-closed set is $\theta\omega$lc*-set but not conversely.

Example: 5.4 Let X = {a, b, c} with τ = {ϕ, {b}, X}. Then {b, c} is $\theta\omega$lc*-set but not θ-closed in (X,τ).

Remark: 5.5 $\theta\omega$-closed sets and $\theta\omega$lc*-sets are independent of each other.

Example: 5.6 Let X and τ be as in the Example 3.14. Then {a, c} is an $\theta\omega$-closed set but not $\theta\omega$lc*-set in (X,τ).

Example: 5.7 Let X and τ be as in the Example 5.4. Then {a, b} is an $\theta\omega$lc*-set but not $\theta\omega$-closed set in (X,τ).

Proposition: 5.8 Let (X,τ) be a topological space. Then a subset A of (X,τ) is θ-closed if and only if it is both $\theta\omega$-closed and $\theta\omega$lc*-set.

Proof: Necessity is trivial. To prove the sufficiency, assume that A is both $\theta\omega$-closed and $\theta\omega$lc*-set.

Then A = M $\cap$ N, where M is semi-open and N is θ-closed in (X, τ). Therefore, A $\subseteq$ M and A $\subseteq$ N and so by hypothesis, cl_θ(A) $\subseteq$ M and cl_θ(A) $\subseteq$ N. Thus cl_θ(A) $\subseteq$ M $\cap$ N = A and hence cl_θ(A) = A i.e., A is θ-closed in (X, τ).

We introduce the following definition

Definition: 5.9 A function f: (X,τ) → (Y,σ) is said to be $\theta\omega$ lc*-continuous if for each closed set V of (Y,σ), f^{-1}(V) is a $\theta\omega$ lc*-set in (X, τ).

Example: 5.10 Let X = Y = {a, b, c} with τ = {ϕ, {c}, X} and σ = {ϕ, {a}, {a, b}, Y}. Let f: (X,τ) → (Y,σ) be the identity function. Then f is $\theta\omega$ lc*-continuous function.

Definition: 5.11 A function f: (X,τ) → (Y,σ) is called
(i) θ-continuous [3] if for each closed set V of Y, f^{-1}(V) is θ-closed in X.
(ii) $\theta\omega$-continuous if for each closed set V of Y, f^{-1}(V) is $\theta\omega$-closed in X.

Proposition: 5.12 Every θ-continuous function is $\theta\omega$-continuous but not conversely.

Proof: It follows from Proposition 3.2.

Example: 5.13 Let X = Y = {a, b, c}, τ = {ϕ, {a, b}, X} and σ = {ϕ, {b}, Y}. We have θC(X) = {ϕ, X} and $\theta\omega$C(X) = {ϕ, {c}, {a, c}, {b, c}, X}. Define f: (X,τ) → (Y,σ) be the identity map. Then f is $\theta\omega$-continuous but not θ-continuous, since f^{-1}({a, c}) = {a, c} is not θ-closed in (X, τ).

Remark: 5.14 Every θ-continuous function is $\theta\omega$ lc*-continuous but not conversely.

Example: 5.15 Let X = Y = {a, b, c} with τ = {ϕ, {b}, X} and σ = {ϕ, {a}, {a, c}, Y}. Let f: (X,τ) → (Y,σ) be the identity function. Then f is $\theta\omega$ lc*-continuous function but not θ-continuous since for the closed set {b} in (Y,σ), f^{-1}({b}) = {b}, which is not θ-closed in (X,τ).

Remark: 5.16 $\theta\omega$-continuity and $\theta\omega$ lc*-continuity are independent of each other.

Example: 5.17 Let X = Y = {a, b, c} with τ = {ϕ, {a, b}, X} and σ = {ϕ, {b}, Y}. Let f: (X, τ) → (Y, σ) be the identity function. Then f is $\theta\omega$-continuous but not $\theta\omega$ lc*-continuous.

Example: 5.18 Let X = Y = {a, b, c} with τ = {ϕ, {a}, X} and σ = {ϕ, {b, c}, Y}. Let f: (X,τ) → (Y, σ) be the identity function. Then f is $\theta\omega$ lc*-continuous function but not $\theta\omega$-continuous.

We have the following decomposition for continuity.

Theorem: 5.19 A function f: (X,τ) → (Y,σ) is θ-continuous if and only if it is both $\theta\omega$-continuous and $\theta\omega$ lc*-continuous.

Proof: Assume that f is θ-continuous. Then by Proposition 5.12 and Remark 5.14, f is both $\theta\omega$-continuous and $\theta\omega$ lc*-continuous.

Conversely, assume that f is both $\theta\omega$-continuous and $\theta\omega$ lc*-continuous. Let V be a closed subset of (Y,σ). Then f^{-1}(V) is both $\theta\omega$-closed and $\theta\omega$ lc*-set. By Proposition 5.8, f^{-1}(V) is a θ-closed set in (X,τ) and so f is θ-continuous.

6. DECOMPOSITION OF $T_{1/2}$-SPACES

We introduce the following definition:

Definition: 6.1 A space (X, τ) is called a T$\theta\omega$-space if every $\theta\omega$-closed set in it is closed.

Example: 6.2 Let X and τ be as in the Example 3.3. Then $\theta\omega$C(X) = {ϕ, {b, c}, X} and the sets in {ϕ, {b, c}, X} are closed. Thus (X,τ) is a T$\theta\omega$-space.

Example: 6.3 Let X and τ be as in the Example 3.14. Then $\theta\omega$C(X) = {ϕ, {c}, {a, c}, {b, c}, X} and the sets in {ϕ, {c}, X} are closed. Thus (X,τ) is not a T$\theta\omega$-space.

Theorem: 6.4 For a topological space (X,τ), the following properties are equivalent:
(i) (X, τ) is a T$\theta\omega$-space.
(ii) Every singleton of (X, τ) is either open or semi-closed.

Proof:
(i) → (ii). If {x} is not semi-closed, then X − {x} is not semi-open. Hence X is only semi-open set containing X − {x}. Therefore cl_θ(X − {x}) ⊆ X. Thus X − {x} is $\theta\omega$-closed. By (i) X − {x} is closed, i.e. {x} is open.

(ii) → (i). Let A ⊆ X be a $\theta\omega$-closed. Let x ∈ cl_θ(A). We consider the following two cases:

Case (a) Let {x} be open. Since x belongs to the closure of A, then {x} ∩ A ≠ ϕ. This shows that x ∈ A.

Case (b) Let {x} be semi-closed. If we assume that x ∉ A, then we would have x ∈ cl_θ(A) −A which cannot happen according to Proposition 4.5. Hence x ∈ A.

So in both cases we have cl_θ(A) ⊆ A. Since the reverse inclusion is trivial, then A = cl_θ(A) or equivalently A is θ-closed. It implies that A is closed.

Definition: 6.5 A space (X,τ) is called $_g$T$\theta\omega$-space if every g-closed set is $\theta\omega$-closed.

Example: 6.6 Let X and τ be as in the Example 3.14. Then GC(X) = $\theta\omega$C(X) = {ϕ, {c}, {a, c}, {b, c}, X}. Thus (X, τ) is a $_g$T$\theta\omega$-space.

Example: 6.7 Let X and τ be as in the Example 3.3. Then GC(X) = {ϕ, {b}, {c}, {a, b}, {a, c}, {b, c}, X} and $\theta\omega$C(X) = {ϕ, {b, c}, X}. Thus (X,τ) is not a $_g$T$\theta\omega$-space.

Proposition: 6.8 Every $T_{1/2}$-space is T$\theta\omega$-space but not conversely.

Proof: Follows from Proposition 3.4.

The converse of Proposition 6.8 need not be true as seen from the following example.

Example: 6.9 Let X and τ be as in the Example 3.3, Then GC(X) = {ϕ, {b}, {c}, {a, b}, {a, c}, {b, c}, X} and $\theta\omega$C(X) = {ϕ, {b, c}, X}. Thus (X,τ) is T$\theta\omega$-space but it is not a $T_{1/2}$-space.

Proposition: 6.10 Every $T_{1/2}$-space is $_g$T$\theta\omega$-space but not conversely.

Proof: Follows from Proposition 3.2.

The converse of Proposition 6.10 need not be true as seen from the following example.

Example: 6.11 Let X and τ be as in the Example 3.14. Then GC(X) = $\theta\omega$C(X) = {ϕ, {c}, {a, c}, {b, c}, X}. Thus (X,τ) is a $_g$T$\theta\omega$-space but not a $T_{1/2}$-space.

Remark: 6.12 T$\theta\omega$-spaces and $_g$T$\theta\omega$-spaces are independent.

Example: 6.13 Let X and τ be as in the Example 3.14, Thus (X,τ) is a $_g$T$\theta\omega$-space but it is not a T$\theta\omega$-space.

Example: 6.14 Let X and τ be as in the Example 3.3. Thus (X,τ) is a T$\theta\omega$-space but it is not a $_g$T$\theta\omega$-space.

Theorem: 6.15 A space (X, τ) is $T_{1/2}$ if and only if it is both T$\theta\omega$ and $_g$T$\theta\omega$.

Proof: Necessity. Follows from Propositions 6.8 and 6.10.
Sufficiency. Assume that (X,τ) is both T$\theta\omega$ and $_g$T$\theta\omega$. Let A be a g-closed set of (X,τ). Then A is $\theta\omega$-closed, since (X,τ) is $_g$T$\theta\omega$. Again since (X, τ) is a T$\theta\omega$, A is closed set in (X, τ) and so (X, τ) is $T_{1/2}$.

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