\( \theta\omega \)-CLOSED SETS IN TOPOLOGICAL SPACES

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ABSTRACT

In this paper, we offer a new class of sets called \( \theta\omega \)-closed sets in topological spaces and we study some of its basic properties. The family of \( \theta\omega \)-closed sets of a topological space forms a topology and is denoted by \( \tau_{\theta\omega} \). Notice that this class of sets lies between the class of \( \theta \)-closed sets and the class of \( g\theta \)-closed sets. Using these sets, we obtain a decomposition of \( \theta \)-continuity and we introduce new spaces called \( T_{\theta\omega} \) and \( \mathcal{T}_{\theta\omega} \). Using these spaces we obtain another decomposition of \( T^{1/2} \)-spaces.

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1. INTRODUCTION

In 1963 Levine [15] introduced the notion of semi-open sets. Velicko [25] introduced the notion of \( \theta \)-closed sets and it is well known that the collection of all \( \theta \)-closed sets of a topological space forms a topology and is denoted by \( \tau_{\theta} \). Levine [14] also introduced the notion of g-closed sets and investigated its fundamental properties. This notion was shown to be productive and very useful. Dontchev and Maki [10] introduced the notion of \( \theta \)-generalized closed sets.

After the advent of g-closed sets, Arya and Nour [4], Sheik John [21] and Dontchev [9] introduced gs-closed sets, \( \omega \)-closed sets and gsp-closed sets respectively.

In this paper, we introduce a new class of sets called \( \theta\omega \)-closed sets in topological spaces. This class lies between the class of \( \theta \)-closed sets and the class of \( \theta\gamma \)-closed sets. We study some of its basic properties and characterizations. Interestingly it turns out that the family of \( \theta\omega \)-closed sets of a topological space forms a topology. This collection is denoted by \( \tau_{\theta\omega} \). From the definitions, it follows immediately that \( \tau_{\theta} \subseteq \tau_{\theta\omega} \subseteq \tau \). Using these sets, we obtain a decomposition of \( \theta \)-continuity and we introduce new type of spaces called \( T_{\theta\omega} \)-spaces and \( \mathcal{T}_{\theta\omega} \)-spaces. Using these spaces, we obtain another decomposition of \( T^{1/2} \)-spaces.

2. PRELIMINARIES

Throughout this paper \((X, \tau)\) and \((Y, \sigma)\) (or \(X\) and \(Y\)) represents topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset \( A \) of a space \((X, \tau)\), \( \text{cl}(A) \), \( \text{int}(A) \) and \( A^c \) or \( X \setminus A \) denote the closure of \( A \), the interior of \( A \) and the complement of \( A \) respectively.

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We recall the following definitions which are useful in the sequel.

**Definition: 2.1** A subset $A$ of a space $(X, \tau)$ is called:
(i) semi-open set [15] if $A \subseteq \text{cl}(\text{int}(A))$;
(ii) preopen set [17] if $A \subseteq \text{int}(\text{cl}(A))$;
(iii) $\alpha$-open set [18] if $A \subseteq \text{int}(\text{cl}(A))$;
(iv) $\beta$-open set [1] if $A \subseteq \text{cl}(\text{int}(A))$;
(v) regular open set [22] if $A = \text{int}(\text{cl}(A))$.

The complements of the above mentioned open sets are called their respective closed sets.

The preclosure [19] (resp. semi-closure [7], $\alpha$-closure [18], semi-pre-closure [2]) of a subset $A$ of $X$, denoted by $\text{pcl}(A)$ (resp. $\text{scl}(A)$, $\text{cl}(A)$, $\text{spcl}(A)$), is defined to be the intersection of all preclosed (resp. semi-closed, $\alpha$-closed, semi-preclosed) sets of $(X, \tau)$ containing $A$. It is known that $\text{pcl}(A)$ (resp. $\text{scl}(A)$, $\text{cl}(A)$, $\text{spcl}(A)$) is a preclosed (resp. semi-closed, $\alpha$-closed, semi-preclosed) set.

**Definition: 2.2** [25] A point $x$ of a space $X$ is called a $\theta$-adherent point of a subset $A$ of $X$ if $\text{cl}(U) \cap A \neq \emptyset$, for every open set $U$ containing $x$. The set of all $\theta$-adherent points of $A$ is called the $\theta$-closure of $A$ and is denoted by $\text{cl}_\theta(A)$. A subset $A$ of a space $X$ is called $\theta$-closed if and only if $A = \text{cl}_\theta(A)$. The complement of a $\theta$-closed set is called $\theta$-open. Similarly, the $\theta$-interior of a set $A$ in $X$, written $\text{int}_\theta(A)$, consists of those points $x$ of $A$ such that for some open set $U$ containing $x$, $\text{cl}(U) \subseteq A$. A set $A$ is $\theta$-open if and only if $A = \text{int}_\theta(A)$, or equivalently, $X \setminus A$ is $\theta$-closed.

A point $x$ of a space $X$ is called a $\delta$-adherent point of a subset $A$ of $X$ if $\text{int}(\text{cl}(U)) \cap A \neq \emptyset$, for every open set $U$ containing $x$. The set of all $\delta$-adherent points of $A$ is called the $\delta$-closure of $A$ and is denoted by $\text{cl}_\delta(A)$. A subset $A$ of a space $X$ is called $\delta$-closed if and only if $A = \text{cl}_\delta(A)$. The complement of a $\delta$-closed set is called $\delta$-open. Similarly, the $\delta$-interior of a set $A$ in $X$, written $\text{int}_\delta(A)$, consists of those points $x$ of $A$ such that for some regularly open set $U$ containing $x$, $U \subseteq A$. A set $A$ is $\delta$-open if and only if $A = \text{int}_\delta(A)$, or equivalently, $X \setminus A$ is $\delta$-closed.

The family of all $\theta$-open (resp. $\delta$-open) subsets of $(X, \tau)$ forms a topology on $X$ and is denoted by $\tau_\theta$ (resp. $\tau_\delta$).

From the definitions it follows immediately that $\tau_\theta \subseteq \tau_\delta \subseteq \tau$ [6].

**Definition: 2.3** A point $x \in X$ is called a semi-$\theta$-cluster [8] point of $A$ if $A \cap \text{scl}(U) \neq \emptyset$ for each semi-open set $U$ containing $x$.

The set of all semi-$\theta$-cluster points of $A$ is called the semi-$\theta$-cluster of $A$ and is denoted by $\text{scl}(A)$. Hence, a subset $A$ is called semi-$\theta$-closed if $\text{scl}(A) = A$. The complement of a semi-$\theta$-closed set is called semi-$\theta$-open set.

Recall that a subset $A$ of a space $(X, \tau)$ is said to be $\delta$-semi-open [20] if $A \subseteq \text{cl}(\text{int} \delta(A))$.

**Definition: 2.4** A subset $A$ of a space $(X, \tau)$ is called:
(i) a generalized closed (briefly g-closed) set [14] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
(ii) a generalized semi-closed (briefly gs-closed) set [4] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
(iii) an $\alpha$-generalized closed (briefly $\alpha$ g-closed) set [16] if $\alpha \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
(iv) a generalized semi-preclosed (briefly gsp-closed) set [9] if $\text{spcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
(v) a generalized preclosed (briefly gp-closed) set [19] if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
(vi) a $\tilde{g}$-closed set [23] (= $\omega$-closed set [21]) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$.
(vii) a $\theta$-generalized closed set (briefly $\theta g$-closed) [10] if $\text{cl}_\theta(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$. 

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Remark: 2.5 The collection of all $\theta g$-closed (resp. $\omega$-closed, $g$-closed, $\theta$-closed, $\alpha$-closed, semi-closed) sets of $X$ is denoted by $\theta G C(X)$ (resp. $\omega C(X)$, $G C(X)$, $\theta C(X)$, $\alpha C(X)$, $S C(X)$).

We denote the power set of $X$ by $P(X)$.

Remark: 2.6 [5] We have the following diagram in which the converses of the implications need not be true.

 Remark: 2.7 [21]

(1) Every $\theta$-closed set is $\theta g$-closed.
(2) $\theta g$-closed sets and $\omega$-closed sets are independent.

Remark: 2.8 [6] $(X, \tau)$ is regular if and only if $\theta \tau = \tau$.

Remark: 2.9 [21] A space $X$ is called $\tau \omega$ if $\omega$-closed set in $X$ is closed.

Definition 2.10 A topological space $(X, \tau)$ is called a $R_1$-space [11] if every two different points with distinct closures have disjoint neighborhoods.

Proposition 2.11 [6] Let $(X, \tau)$ be a space. Then,
(i) if $A \subseteq X$ is preopen then $cl(A) = \alpha cl(A)$.
(ii) $(X, \tau)$ is $R_1$ if and only if $cl(\{x\}) = \delta cl(\{x\})$ for each $x \in X$.

Proposition 2.12 [11, 12] Let $(X, \tau)$ be a space. If $A \subseteq X$ is preopen then $cl(A) = \alpha cl(A) = \delta cl(A)$.

Definition 2.13 [14] A space $(X, \tau)$ is called $T_{1/2}$-space if every g-closed set is closed.

3. $\theta \omega$-CLOSED SETS

We introduce the following definition.

Definition: 3.1 A subset $A$ of $X$ is called a $\theta \omega$-closed set if $cl_\theta(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open in $(X, \tau)$. The complement of $\theta \omega$-closed set is called $\theta \omega$-open set.

The collection of all $\theta \omega$-closed sets of $X$ is denoted by $\theta \omega C(X)$.

Proposition: 3.2 Every $\theta$-closed set is $\theta \omega$-closed.

Proof: Let $A$ be an $\theta$-closed set and $G$ be any semi-open set containing $A$ in $(X, \tau)$. Since $A$ is $\theta$-closed, $cl_\theta(A) = A$ for every subset $A$ of $X$. Therefore $cl_\theta(A) \subseteq G$ and hence $A$ is $\theta \omega$-closed set.

The converse of Proposition 3.2 need not be true as seen from the following example.

Example: 3.3 Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, X\}$. Then $\theta \omega C(X) = \{\phi, \{b, c\}, X\}$ and $\theta C(X) = \{\phi, X\}$. Here, $A = \{b, c\}$ is $\theta \omega$-closed but not $\theta$-closed set in $(X, \tau)$.

Proposition: 3.4 Every $\theta \omega$-closed set is $g$-closed.
Proof: Let A be an $\theta\omega$-closed set and G be any open set containing A in $(X, \tau)$. Since every open set is semi-open and A is $\theta\omega$-closed, $cl_G(A) \subseteq G$. Since $cl(A) \subseteq cl_G(A) \subseteq G$, $cl(A) \subseteq G$ and hence A is g-closed.

The converse of Proposition 3.4 need not be true as seen from the following example.

Example: 3.5 Let X and $\tau$ be as in the Example 3.3. Then $\theta\omega C(X) = \{\emptyset, [b, c], X\}$ and $G C(X) = \{\emptyset, [b], [c], [a, b], [a, c], [b, c], X\}$. Here, $A = \{a, b\}$ is g-closed but not $\theta\omega$-closed set in $(X, \tau)$.

Proposition: 3.6 Every $\theta\omega$-closed set is $\omega$-closed.

Proof: Let A be an $\theta\omega$-closed subset of $(X, \tau)$ and G be any semi-open set containing A. Since $cl(A) \subseteq cl_G(A) \subseteq G$ and hence A is $\omega$-closed.

The converse of Proposition 3.6 need not be true as seen from the following example.

Example: 3.7 Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, [a], [a, b], X\}$. Then $\theta\omega C(X) = \{\emptyset, [b, c], X\}$ and $\omega C(X) = \{\emptyset, [c], [b, c], X\}$. Here, $A = \{c\}$ is $\omega$-closed but not $\theta\omega$-closed set in $(X, \tau)$.

Proposition: 3.8 Every $g\theta$-closed set is g-closed.

Proof: Let A be an $g\theta$-closed subset of $(X, \tau)$ and G be any open set containing A. Since $cl(A) \subseteq cl_G(A) \subseteq G$ and hence A is g-closed.

The converse of Proposition 3.8 need not be true as seen from the following example.

Example: 3.9 Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, [a], [a, b], [a, c], X\}$. Then $G\theta C(X) = \{\emptyset, [b, c], X\}$ and $G C(X) = \{\emptyset, [b], [c], [b, c], X\}$. Here, $A = \{b\}$ is g-closed but not $G\theta$-closed set in $(X, \tau)$.

Proposition: 3.10 Every $\theta\omega$-closed set is $G\theta$-closed.

Proof: Let A be an $\theta\omega$-closed set and G be any open set containing A in $(X, \tau)$. Since every open set is semi-open and A is $\theta\omega$-closed, $cl_G(A) \subseteq G$. Therefore $cl(A) \subseteq G$ and G is open. Hence A is $G\theta$-closed.

The converse of Proposition 3.10 need not be true as seen from the following example.

Example: 3.11 Let X and $\tau$ be as in the Example 3.3. Then $\theta\omega C(X) = \{\emptyset, [b, c], X\}$ and $G\theta C(X) = \{\emptyset, [b], [c], [a, b], [a, c], [b, c], X\}$. Here, $A = \{a, c\}$ is $G\theta$-closed but not $\theta\omega$-closed set in $(X, \tau)$.

Remark: 3.12 The following examples show that $\theta\omega$-closedness is independent of closedness, semi-closedness and $\alpha$-closedness.

Example: 3.13 Let X and $\tau$ be as in the Example 3.3. Then $\theta\omega C(X) = \{\emptyset, [b, c], X\}$ and $\alpha C(X) = \{\emptyset, [b], [c], [a, b], [a, c], [b, c], X\}$. Here, $A = \{b\}$ is $\alpha$-closed as well as semi-closed in $(X, \tau)$ but it is not $\theta\omega$-closed in $(X, \tau)$.

Example: 3.14 Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, [a, b], X\}$. Then $\theta\omega C(X) = \{\emptyset, [c], [a, c], [b, c], X\}$ and $\alpha C(X) = \{\emptyset, [c], X\}$. Here, $A = \{a, c\}$ is $\theta\omega$-closed but it is neither $\alpha$-closed nor semi-closed in $(X, \tau)$.

Example: 3.15 In Example 3.7, $\{c\}$ is closed set but not $\theta\omega$-closed.

In Example 3.14, $\{b, c\}$ is $\theta\omega$-closed set but not closed.

Remark: 3.16 From the above discussions and known results in [9, 11, 21, 24], we obtain the following diagram, where $A \rightarrow B$ (resp. $A \leftrightarrow B$) represents $A$ implies $B$ but not conversely (resp. A and B are independent of each other).
None of the above implications is reversible as shown in the above examples and in the related papers [9, 11, 21, 24].

4. PROPERTIES OF $\theta\omega$-CLOSED SETS

Definition: 4.1 [21] The intersection of all semi-open subsets of $(X, \tau)$ containing $A$ is called the semi-kernel of $A$ and is denoted by $s$-ker ($A$).

Lemma: 4.2 A subset $A$ of $(X, \tau)$ is $\theta\omega$-closed if and only if $\theta$cl ($A$) $\subseteq$ $s$-ker ($A$).

Proof: Suppose that $A$ is $\theta\omega$-closed. Then $cl_\theta$ ($A$) $\subseteq$ $U$ whenever $A \subseteq U$ and $U$ is semi-open. Let $x \in cl_\theta$ ($A$). If $x \notin s$-ker ($A$), then there is a semi-open set $U$ containing $A$ such that $x \notin U$. Since $U$ is a semi-open set containing $A$, we have $x \notin cl_\theta$ ($A$) and this is a contradiction.

Conversely, let $cl_\theta$ ($A$) $\subseteq$ $s$-ker ($A$). If $U$ is any semi-open set containing $A$, then $cl_\theta$ ($A$) $\subseteq$ $s$-ker ($A$) $\subseteq$ $U$. Therefore, $A$ is $\theta\omega$-closed.

Remark: 4.3 The collection of all $\theta\omega$-closed sets of a topological space forms a topology and is denoted by $\tau_{\theta\omega}$.

Remark: 4.4 If $A$ is a $\theta\omega$-closed set and $F$ is a $\theta$-closed set, then $A \cap F$ is a $\theta\omega$-closed set.

Proof: Since $F$ is $\theta$-closed, it is $\theta\omega$-closed. Therefore by Remark 4.3, $A \cap F$ is also a $\theta\omega$-closed set.

Proposition: 4.5 If a set $A$ is $\theta\omega$-closed in $(X, \tau)$, then $\theta$cl ($A$) $\cap$ $A$ contains no nonempty semi-closed set in $(X, \tau)$.

Proof: Suppose that $A$ is $\theta\omega$-closed. Let $A \subseteq Y \cap G$, where $G$ is semi-open in $(X, \tau)$. Then $A \subseteq F$. Therefore $cl_\theta$ ($A$) $\subseteq F$. Consequently, $F \subseteq (cl_\theta$ ($A$))$. We already have $F \subseteq cl_\theta$ ($A$). Thus $F \subseteq cl_\theta$ ($A$) $\cap$ ($cl_\theta$ ($A$)) and $F$ is empty.

The converse of Proposition 4.5 need not be true as seen from the following example.

Example: 4.6 Let $X$ and $\tau$ be as in the Example 3.14. Then $\theta\omega$C($X$) = $\{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$ and $S$C($X$) = $\{\phi, \{c\}, X\}$. If $A = \{c\}$, then $cl_\theta$ ($A$) $\cap$ $A$ = $\{a, b\}$ does not contain any nonempty semi-closed set. But $A$ is not $\theta\omega$-closed in $(X, \tau)$.

Proposition: 4.7 Let $A \subseteq Y \subseteq X$ where $Y$ is open and suppose that $A$ is $\theta\omega$-closed in $(X, \tau)$. Then $A$ is $\theta\omega$-closed relative to $Y$.

Proof: Let $A \subseteq Y \cap G$, where $G$ is semi-open in $(X, \tau)$. Then $A \subseteq G$ and hence $cl_\theta$ ($A$) $\subseteq G$. This implies that $Y \cap cl_\theta$ ($A$) $\subseteq Y \cap G$. Thus $A$ is $\theta\omega$-closed relative to $Y$ since the intersection of open and semi-open is semi-open [6].
Proposition: 4.8 If A is a semi-open and $\theta\omega$-closed in $(X, \tau)$, then A is $\theta$-closed in $(X, \tau)$.

Proof: Since A is semi-open and $\theta\omega$-closed, $cl_\theta(A) \subseteq A$ and hence A is $\theta$-closed in $(X, \tau)$.

Theorem: 4.9 Let A be a subset of a regular space $(X, \tau)$. Then,

(i) A is $\theta\omega$-closed if and only if A is $\omega$-closed.

(ii) if $(X, \tau)$ is $\tau\omega$, then A is $\theta\omega$-closed if and only if A is closed.

Proof:
(i) It follows from Remark 2.8.
(ii) It follows from Remark 2.9.

Theorem: 4.10 Let A be a preopen subset of a topological space $(X, \tau)$. Then the following conditions are equivalent.
(i) A is $\theta\omega$-closed.
(ii) A is $g\theta$-closed (or $\omega$-closed).
(iii) A is g-closed.
(iv) A is $\alpha$ g-closed.

Proof:
(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). It is obvious from Remark 3.16.
(iv) $\Rightarrow$ (i). It follows from Propositions 2.11 and 2.12.

Recall that a partition space [11] is a topological space where every open set is closed.

Corollary: 4.11 Let A be a subset of the partition space $(X, \tau)$. Then the following conditions are equivalent.
(i) A is $\theta\omega$-closed.
(ii) A is $g\theta$-closed (or $\omega$-closed).
(iii) A is g-closed.
(iv) A is $\alpha$ g-closed.

Proof: A topological space is a partition space if and only if every subset is preopen. Then the claim follows straight from Theorem 4.10.

Theorem: 4.12 For a singleton subset A of an $R_1$ topological space $(X, \tau)$, the following conditions are equivalent.
(i) A is $\theta\omega$-closed.
(ii) A is $\omega$-closed.

Proof:
(i) $\Rightarrow$ (ii) is clear.
(ii) $\Rightarrow$ (i). Note that in $R_1$-spaces, the concepts of closure and $\theta$-closure coincide for singleton sets: see Proposition 2.11.

Theorem: 4.13 For a subset A of a topological space $(X, \tau)$, the following conditions are equivalent.
(i) A is clopen.
(ii) A is $\theta\omega$-closed, preopen and semi-closed.
(iii) A is $\theta\omega$-closed and (regular) open.
(iv) A is $\alpha$ g-closed and (regular) open.

Proof:
(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious.
(iv) $\Rightarrow$ (i). It follows from Theorem 3.13 [11].

Lemma: 4.14 In any space, if a singleton is $\theta$-open then it is regular open.

Proof: It follows from the fact that, in any space, a singleton is $\delta$-open if and only if it is regular open [11].

Lemma: 4.15 In a regular space, singleton is $\theta$-open if and only if it is regular open.
Lemma: 4.16 If A is both closed and preopen of a topological space X, then the following are equivalent.
(i) A is $\theta$-closed.
(ii) A is $\delta$-closed.
(iii) A is $\alpha$-closed.

Proof: It is obvious from the fact that $A = \text{cl}(A) = \delta\text{cl}(A) = \theta\text{cl}(A) = \alpha\text{cl}(A)$ (see. Propositions 2.11 and 2.12)

Lemma: 4.17 If a subset A of a space $(X, \tau)$ is clopen, then the following are equivalent.
(i) A is $\theta$-closed.
(ii) A is $\delta$-closed.
(iii) A is $\alpha$-closed.
(iv) A is regular closed.

Definition: 4.18 A space $(X, \tau)$ is called locally $s$-$\theta$-indiscrete space if every semi-open set is $\theta$-closed.

Theorem: 4.19 For a topological space $(X, \tau)$, the following conditions are equivalent.
(i) X is locally $s$-$\theta$-indiscrete.
(ii) Every subset of X is $\theta\omega$-closed.

Proof:
(i) $\Rightarrow$ (ii). Let $A \subseteq U$, where U is semi-open and A is an arbitrary subset of X. Since X is locally $s$-$\theta$-indiscrete, then U is $\theta$-closed. We have $\delta\text{cl}(A) \subseteq \delta\text{cl}(U) = U$. Thus A is $\theta\omega$-closed.

(ii) $\Rightarrow$ (i). If $U \subseteq X$ is semi-open, then by (ii) $\delta\text{cl}(U) \subseteq U$ or equivalently U is $\theta$-closed. Hence X is locally $s$-$\theta$-indiscrete.

5. DECOMPOSITION OF $\theta$-CONTINUITY

In this section, we obtain a decomposition of continuity called $\theta$-continuity in topological spaces.

To obtain a decomposition of $\theta$-continuity, we first introduce the notion of $\theta\omega$lc*-continuous functions in topological spaces and by using $\theta\omega$-continuity, prove that a function is $\theta$-continuous if and only if it is both $\theta\omega$-continuous and $\theta\omega$lc*-continuous.

We introduce the following definition.

Definition: 5.1 A subset A of a space $(X, \tau)$ is called $\theta\omega$lc*-set if $A = M \cap N$, where M is semi-open and N is $\theta$-closed in $(X, \tau)$.

Example: 5.2 Let X and $\tau$ be as in the Example 3.3. Then $\{a, b\}$ is $\theta\omega$lc*-set in $(X, \tau)$.

Remark: 5.3 Every $\theta$-closed set is $\theta\omega$lc*-set but not conversely.

Example: 5.4 Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{b\}, X\}$. Then $\{b, c\}$ is $\theta\omega$lc*-set but not $\theta$-closed in $(X, \tau)$.

Remark: 5.5 $\theta\omega$-closed sets and $\theta\omega$lc*-sets are independent of each other.

Example: 5.6 Let X and $\tau$ be as in the Example 3.14. Then $\{a, c\}$ is an $\theta\omega$-closed set but not $\theta\omega$lc*-set in $(X, \tau)$.

Example: 5.7 Let X and $\tau$ be as in the Example 5.4. Then $\{a, b\}$ is an $\theta\omega$lc*-set but not $\theta\omega$-closed set in $(X, \tau)$.

Proposition: 5.8 Let $(X, \tau)$ be a topological space. Then a subset A of $(X, \tau)$ is $\theta$-closed if and only if it is both $\theta\omega$-closed and $\theta\omega$lc*-set.

Proof: Necessity is trivial. To prove the sufficiency, assume that A is both $\theta\omega$-closed and $\theta\omega$lc*-set.
Then $A = M \cap N$, where $M$ is semi-open and $N$ is $\theta$-closed in $(X, \tau)$. Therefore, $A \subseteq M$ and $A \subseteq N$ and so by hypothesis, $cl_{\theta}(A) \subseteq M$ and $cl_{\theta}(A) \subseteq N$. Thus $cl_{\theta}(A) \subseteq M \cap N = A$ and hence $cl_{\theta}(A) = A$ i.e., $A$ is $\theta$-closed in $(X, \tau)$.

We introduce the following definition

**Definition: 5.9** A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\theta\omega$lc*-continuous if for each closed set $V$ of $(Y, \sigma)$, $f^{-1}(V)$ is a $\theta\omega$lc*-set in $(X, \tau)$.

**Example: 5.10** Let $X = Y = \{a, b, c\}$ with $\tau = \{\phi, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then $f$ is $\theta\omega$lc*-continuous function.

**Definition: 5.11** A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called
(i) $\theta$-continuous [3] if for each closed set $V$ of $Y$, $f^{-1}(V)$ is $\theta$-closed in $X$.
(ii) $\theta\omega$-continuous if for each closed set $V$ of $Y$, $f^{-1}(V)$ is $\theta\omega$-closed in $X$.

**Proposition: 5.12** Every $\theta$-continuous function is $\theta\omega$-continuous but not conversely.

**Proof:** It follows from Proposition 3.2.

**Example: 5.13** Let $X = Y = \{a, b, c\}$ with $\tau = \{\phi, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$. We have $\theta C(X) = \{\phi, X\}$ and $\theta\omega C(X) = \{\phi, \{a\}, \{a, b\}, X\}$. Define $f: (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then $f$ is $\theta\omega$-continuous but not $\theta$-continuous, since $f^{-1}(\{a\}) = \{a, c\}$ is not $\theta$-closed in $(X, \tau)$.

**Remark: 5.14** Every $\theta$-continuous function is $\theta\omega$lc*-continuous but not conversely.

**Example: 5.15** Let $X = Y = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then $f$ is $\theta\omega$lc*-continuous function but not $\theta$-continuous since for the closed set $\{b\}$ in $(Y, \sigma)$, $f^{-1}(\{b\}) = \{b\}$, which is not $\theta$-closed in $(X, \tau)$.

**Remark: 5.16** $\theta\omega$-continuity and $\theta\omega$lc*-continuity are independent of each other.

**Example: 5.17** Let $X = Y = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{b\}, Y\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then $f$ is $\theta\omega$-continuous but not $\theta\omega$lc*-continuous.

**Example: 5.18** Let $X = Y = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, X\}$ and $\sigma = \{\phi, \{b, c\}, Y\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then $f$ is $\theta\omega$lc*-continuous function but not $\theta\omega$-continuous.

We have the following decomposition for continuity.

**Theorem: 5.19** A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is $\theta$-continuous if and only if it is both $\theta\omega$-continuous and $\theta\omega$lc*-continuous.

**Proof:** Assume that $f$ is $\theta$-continuous. Then by Proposition 5.12 and Remark 5.14, $f$ is both $\theta\omega$-continuous and $\theta\omega$lc*-continuous.

Conversely, assume that $f$ is both $\theta\omega$-continuous and $\theta\omega$lc*-continuous. Let $V$ be a closed subset of $(Y, \sigma)$. Then $f^{-1}(V)$ is both $\theta\omega$-closed and $\theta\omega$lc*-set. By Proposition 5.8, $f^{-1}(V)$ is a $\theta$-closed set in $(X, \tau)$ and so $f$ is $\theta$-continuous.

6. **DECOMPOSITION OF T_{\theta\omega\omega}\$-SPACES**

We introduce the following definition:

**Definition: 6.1** A space $(X, \tau)$ is called a $T_{\theta\omega\omega}$-space if every $\theta\omega$-closed set in it is closed.

**Example: 6.2** Let $X$ and $\tau$ be as in the Example 3.3. Then $\theta\omega C(X) = \{\phi, \{b, c\}, X\}$ and the sets in $\{\phi, \{b, c\}, X\}$ are closed. Thus $(X, \tau)$ is a $T_{\theta\omega\omega}$-space.
Example: 6.3 Let X and $\tau$ be as in the Example 3.14. Then $\theta_\omega C(X) = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$ and the sets in $\{\phi, \{c\}, X\}$ are closed. Thus $(X, \tau)$ is not a $T_{\theta_\omega}$-space.

Theorem: 6.4 For a topological space $(X, \tau)$, the following properties are equivalent:
(i) $(X, \tau)$ is a $T_{\theta_\omega}$-space.
(ii) Every singleton of $(X, \tau)$ is either open or semi-closed.

Proof:
(i) $\rightarrow$ (ii). If $\{x\}$ is not semi-closed, then $X - \{x\}$ is not semi-open. Hence $X$ is only semi-open set containing $X - \{x\}$. Therefore $cl_\theta (X - \{x\}) \subseteq X$. Thus $X - \{x\}$ is $\theta_\omega$-closed. By (i) $X - \{x\}$ is closed, i.e. $\{x\}$ is open.

(ii) $\rightarrow$ (i). Let $A \subseteq X$ be a $\theta_\omega$-closed. Let $x \in cl_\theta (A)$. We consider the following two cases:

Case (a) Let $\{x\}$ be open. Since $x$ belongs to the closure of $A$, then $\{x\} \cap A \neq \emptyset$. This shows that $x \in A$.

Case (b) Let $\{x\}$ be semi-closed. If we assume that $x \notin A$, then we would have $x \in cl_\theta (A) - A$ which cannot happen according to Proposition 4.5. Hence $x \in A$.

So in both cases we have $cl_\theta (A) \subseteq A$. Since the reverse inclusion is trivial, then $A = cl_\theta (A)$ or equivalently $A$ is $\theta$-closed. It implies that $A$ is closed.

Definition: 6.5 A space $(X, \tau)$ is called $g_{\ T_{\theta_\omega}}$-space if every g-closed set is $\theta_\omega$-closed.

Example: 6.6 Let X and $\tau$ be as in the Example 3.14. Then $G C(X) = \theta_\omega C(X) = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$. Thus $(X, \tau)$ is a $g_{\ T_{\theta_\omega}}$-space.

Example: 6.7 Let X and $\tau$ be as in the Example 3.3. Then $G C(X) = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ and $\theta_\omega C(X) = \{\emptyset, \{b, c\}, X\}$. Thus $(X, \tau)$ is not a $g_{\ T_{\theta_\omega}}$-space.

Proposition: 6.8 Every $T_{1/2}$-space is $T_{\theta_\omega}$-space but not conversely.

Proof: Follows from Proposition 3.4.

The converse of Proposition 6.8 need not be true as seen from the following example.

Example: 6.9 Let X and $\tau$ be as in the Example 3.3. Then $G C(X) = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ and $\theta_\omega C(X) = \{\emptyset, \{b, c\}, X\}$. Thus $(X, \tau)$ is a $T_{\theta_\omega}$-space but it is not a $T_{1/2}$-space.

Proposition: 6.10 Every $T_{1/2}$-space is $g_{\ T_{\theta_\omega}}$-space but not conversely.

Proof: Follows from Proposition 3.2.

The converse of Proposition 6.10 need not be true as seen from the following example.

Example: 6.11 Let X and $\tau$ be as in the Example 3.14. Then $G C(X) = \theta_\omega C(X) = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$. Thus $(X, \tau)$ is a $g_{\ T_{\theta_\omega}}$-space but it is not a $T_{1/2}$-space.

Remark: 6.12 $T_{\theta_\omega}$-spaces and $g_{\ T_{\theta_\omega}}$-spaces are independent.

Example: 6.13 Let X and $\tau$ be as in the Example 3.14. Thus $(X, \tau)$ is a $g_{\ T_{\theta_\omega}}$-space but it is not a $T_{\theta_\omega}$-space.

Example: 6.14 Let X and $\tau$ be as in the Example 3.3. Thus $(X, \tau)$ is a $T_{\theta_\omega}$-space but it is not a $g_{\ T_{\theta_\omega}}$-space.

Theorem: 6.15 A space $(X, \tau)$ is $T_{1/2}$ if and only if it is both $T_{\theta_\omega}$ and $g_{\ T_{\theta_\omega}}$.

Proof: Necessity. Follows from Propositions 6.8 and 6.10.

Sufficiency. Assume that $(X, \tau)$ is both $T_{\theta_\omega}$ and $g_{\ T_{\theta_\omega}}$. Let $A$ be a g-closed set of $(X, \tau)$. Then $A$ is $\theta_\omega$-closed, since $(X, \tau)$ is $g_{\ T_{\theta_\omega}}$. Again since $(X, \tau)$ is a $T_{\theta_\omega}$, $A$ is closed set in $(X, \tau)$ and so $(X, \tau)$ is $T_{1/2}$.
REFERENCES


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