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COMMON FIXED POINT THEOREMS FOR FOUR SELF MAPS ON A MULTIPLICATIVE METRIC SPACE

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ABSTRACT

In this paper we improve and extend theorem 3.1 of Khan and Imdad[9] replacing λ *by* φ, *where* $\varphi : \mathrm{R}_+ \to \left[0, \frac{1}{2}\right)$ *is an increasing and continuous function. We also show that the example 4.1 mentioned in Khan and Imdad does not illustrate theorem 3.1 of Khan and Imdad. However an example to illustrate theorem 3.1 of Khan and Imdad is given.*

Key words: *Multiplicative metric space, multiplicative contraction, Coincidence point, Coincidentally Commuting mappings, Common fixed point.*

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1. INTRODUCTION AND PRELIMINARIES

The study of fixed points of mappings satisfying certain contractions has many applications and has been at the center of various research activities. In the past years many authors generalized Banach's fixed point theorem in various spaces such as Quasi metric space, Fuzzy metric space, Z-metric space, Partial metric space, probabilistic metric space and generalized metric spaces. On the other hand, in 2008 Bashirov *et al.* [1] defined a new distance so called a multiplicative distance by using the concept of multiplicative absolute value. After than, in 2012, by using the same idea of multiplicative distance, M.Ozavsar and A.C.Civikel [7] investigate multiplicative metric space and introduced the concept of a multiplicative contraction mappings and proved some fixed point theorems for multiplicative contraction, such mappings on a complete multiplicative metric spaces. In 2014, He *et al.* [12] proved a Common fixed point theorem for four self mappings in complete multiplicative metric space.

Recently motivated by the concept of compatible mappings, compatible mapping of type(A), and R-Weak commutativity given by G.Jungck *et al.* [4,5,6], B.C.Dhage [2] termed a pair of self mappings to be Coincidentally Commuting(Weakly commuting), and proved some fixed point theorems for these mappings.

In this paper we improve the result 3.1 of Q.H.Khan and M.Imdad in [9]. Also the example 4.1 mentioned in [9] does not illustrate the result 3.1 of Q.H.Khan and M.Imdad [9], however, we gave an example to illustrates the result 3.1 of Q.H.Khan and M.Imdad [9].

The following one way implications are obviously true but their converse are not true.

Commuting maps $\Rightarrow$ Weakly Commuting maps $\Rightarrow$ Compatible maps $\Rightarrow$ Coincidentally Commuting maps.

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Definition 1.1 (A.E.Bashirov, E.M.Kurplnara, A.Ozyapici [1]): Let X be a nonempty set. A multiplicative metric is a mapping $d : X \times X \to \mathbb{R}^+$ satisfying the following conditions:
(i) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$, if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(iii) $d(x, y) \leq d(x, z).d(z, y)$ for all $x, y, z \in X$. (Multiplicative triangle inequality)

Definition 1.2 (M.Ozavsar, A.C.Civikel [7]): Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. the sequence $\{x_n\}$ is called a multiplicative Cauchy sequence if, for each $\varepsilon > 1$, there exists $N \in \mathbf{N}$ such that $d(x_n, x_m) < \varepsilon$, for all $m, n \geq N$.

Definition 1.3 (M.Ozavsar, A.C.Civikel [7]): A multiplicative metric space X is complete, if every multiplicative Cauchy sequence in it is multiplicative convergent to $x \in X$.

Definition 1.4 (F.Gu, L.M.Cui, Y.H.Wu [3]): Suppose that S and T are two self mappings of a multiplicative metric space (X, d); S, T are called commutative mappings if, $STx = TSx$ for all $x \in X$.

Definition 1.5 (F.Gu, L.M.Cui, Y.H.Wu [3]): Suppose that S and T are two self mappings of a multiplicative metric space (X, d); The pair (S, T) are called weak commutative mappings if $d(STx, TSx) \leq d(Tx, Sx)$ for all $x \in X$.

Definition 1.6 (F.Gu, L.M.Cui, Y.H.Wu [3]): Suppose that S and T are two self mappings of a multiplicative metric space (X, d); A point $x \in X$ is said to be a coincident point of S and T if $d(Sx, Tx) = 1$.

Definition 1.7 (M.Ozavsar, A.C.Civikel [7]): Let (X, d) be a multiplicative metric space. A mapping $f : X \to X$ is called a multiplicative contraction if there exists a real constant $\lambda \in [0,1)$ such that $d(fx, fy) \leq d(x, y)^{\lambda}$ for all $x, y \in X$.

Definition 1.8 (M.Ozavsar, A.C.Civikel [7])(Multiplicative continuity): Let (X, d_X) and (Y, d_Y) be two multiplicative metric spaces and $f : X \to Y$ be a function. If for every $\varepsilon > 1$, there exists $\delta > 1$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$, then we call f multiplicative continuous at $x \in X$.

Definition 1.9 (M.Ozavsar, A.C.Civikel [7])(Semi-Multiplicative continuity): Let (X, d) be a multiplicative metric space, and (Y, d) be a metric space and $f : X \to Y$ be a function. If for all $\varepsilon > 0$, there exists $\delta > 1$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$, then we call f Semi multiplicative continuous at $x \in X$. Similarly a function $g : Y \to X$ is also said to be semi multiplicative continuous at $y \in Y$ if it satisfies a similar requirement.

Definition 1.10 (M.Ozavsar, A.C.Civikel [7])(Multiplicative convergence): Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. If for every multiplicative open ball $B_\varepsilon(x) = \{y / d(x, y) < \varepsilon\}, \varepsilon > 1$ there exists a natural number N such that $n \geq N$, $x_n \in B_\varepsilon(x)$. The sequence $\{x_n\}$ is said to be multiplicative converging to x, denoted by $x_n \to x \ (n \to \infty)$.

Lemma 1.11 (M.Ozavsar, A.C.Civikel [7]): Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$ then $x_n \to x \ (n \to \infty)$ if and only if $d(x_n, x) \to 1 \ (n \to \infty)$

Lemma 1.12 (M.Ozavsar, A.C.Civikel [7]): Let (X, d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X. If the sequence $\{x_n\}$ is multiplicative convergent, then the multiplicative limit point is unique.

Lemma 1.13 (M.Ozavsar, A.C.Civikel [7]): Let (X,d_X) and (Y,d_Y) be two multiplicative metric spaces and $f:X\to Y$ be a mapping and $\{x_n\}$ be any sequence in X. Then f is multiplicative continuous at the point $x\in X$ if and only if $f(x_n)\to f(x)$ for every sequence $\{x_n\}$ with $x_n\to x\ (n\to\infty)$.

Theorem 1.14 (M.Ozavsar, A.C.Civikel [7]): Let (X,d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. If the sequence is multiplicative convergent, then it is multiplicative Cauchy sequence.

Lemma 1.15 (M.Ozavsar, A.C.Civikel [7]): Let (X,d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is multiplicative Cauchy sequence if and only if $d(x_m,x_n)\to 1\ (m,n\to\infty)$

Theorem 1.16 (M.Ozavsar, A.C.Civikel [7]): Let (X,d) be a multiplicative metric space and let $f:X\to X$ be a multiplicative contraction. If (X,d) is complete, then f has a unique fixed point.

In 2014, X.He, M.Song and D.Chen [12] proved the fixed point theorem for two pairs of of weak commutative mappings on a multiplicative metric space.

Theorem 1.17 (X.He, M.Song, D.Chen [12]): Let S,T,A and B be self mappings of a complete multiplicative metric space X, satisfying the following conditions:
(i) $S(X)\subset B(X),T(X)\subset A(X),$
(ii) A and S are weak commutative, B and T also are weak commutative,
(iii) One of S,T,A and B is continuous,
(iv) $d(Sx,Ty)\le\{max\{d(Ax,By),d(Ax,Sx),d(Ty,By),d(Sx,By),d(Ax,Ty)\}\}^{\lambda}\ \ \lambda\in(0,\frac{1}{2})\ \ \forall x,y\in X$.
Then S,T,A and B have a unique common fixed point.

Now we given below the examples of multiplicative metrics.

Example 1.18 (M.Ozavsar, A.C.Civikel [7]): Let $X=[1,\infty)$, define $d:X\times X\to\mathbb{R}^+$ by $d(x,y)=max\left\{\frac{x}{y},\frac{y}{x}\right\}$ then d is a multiplicative metric on X.

Example 1.19 (M.Sarwar and R.Badsahah-e [8]): Let (X,d) be a metric space, let a > 1, define $d':X\times X\to\mathbb{R}^+$ by $d'(x,y)=a^{d(x,y)}=\begin{cases}1 & \text{if } x=y.\\ 0 & \text{if } x\ne \text{y.}\end{cases}$ for all $x,y\in X$. Then d' is a multiplicative metric on X and (X,d') is known as the discrete multiplicative metric space .

Example 1.20: (M.Sarwar and R.Badsahah-e [8]): Let $d:\mathbb{R}\times\mathbb{R}\to[1,\infty)$ be defined as $d(x,y)=a^{|x-y|}$, where $x,y\in\mathbb{R}$ and a>1. Then d is a multiplicative metric and $(\mathbb{R},d)$ is a multiplicative metric space. We may call it usual multiplicative metric space.
Recently, Q.H.Khan and M.Imdad improved theorem 1.17 by relaxing the continuity requirement of the maps completely and to reduce the commutativity requirement of the maps to coincidence point only.

Theorem 1.21: (Q.H.Khan, M.Imdad [9]). Let A,B,I and J be self mappings of a multiplicative metric space (X,d), satisfying $A(X)\subset J(X)$, $B(X)\subset I(X)$ and

$$d(Ax,By)\le max\{d(Ix,Jy),d(Ix,Ax),d(By,Jy),d(Ax,Jy),d(Ix,By)\}^{\lambda}\ \ \lambda\in(0,\frac{1}{2})\forall x,y,z\in X.$$

If one of $A(X),B(X),I(X),J(X)$ is a complete subspace of X, then the following conclusions hold.
(i) (A,I) has a coincidence point;
(ii) (B,J) has a coincidence point;

Further if the pairs (A,I) and (B,J) are coincidentally commuting, then A,I,B and J have a unique common fixed point.

2. MAIN RESULT

In this section we mainly improve and extend the theorem 1.21, in [9], replacing λ by φ, where $\varphi:\mathrm{R}_{+}\to\left[0,\frac{1}{2}\right)$ is an increasing and continuous function.

Theorem 2.1: Let A,B,I and J be self mappings of a multiplicative metric space (X,d), satisfying $A(X)\subset J(X)$, $B(X)\subset I(X)$ and define $\varphi:\mathrm{R}_{+}\to\left[0,\frac{1}{2}\right)$ such that φ is continuous and increasing and

$$d(Ax,By)\le[M(x,y)]^{\varphi(M(x,y))} \tag{2.1.1}$$

where $M(x,y)=max\{d(Ix,Jy),d(Ix,Ax),d(By,Jy),d(Ax,Jy),d(Ix,By)\}\quad \forall\, x,y,z\in X$.
If one of $A(X),B(X),I(X),J(X)$ is a complete subspace of X, then the following conclusions hold.
(i) (A,I) has a coincidence point;
(ii) (B,J) has a coincidence point;

Further if the pairs (A,I) and (B,J) are coincidentally commuting, then A,B,I and J have a unique common fixed point.

Proof: Let $x_0\in X$, since $A(X)\subset J(X)$, then $\exists$ a point $x_1\in X$ such that $Ax_0=Jx_1$.

Also since $B(X)\subset I(X)$, $\exists$ $x_2\in X$ such that $Bx_1=Ix_2$.

Using this argument repeatedly, we can construct a sequence $\{z_n\}$ in X such that
$z_{2n}=Ax_{2n}=Jx_{2n+1}$ and $z_{2n+1}=Bx_{2n+1}=Ix_{2n+2}$

Suppose $z_{2n+1}=z_{2n}$ for some n.

$$d(z_{2n+1},z_{2n+2}) = d(Bx_{2n+1},Ax_{2n+2}) = d(Ax_{2n+2},Bx_{2n+1})$$

$$d(z_{2n+1},z_{2n+2}) = d(Ax_{2n+2},Bx_{2n+1}) \le \{M(x_{2n+2},x_{2n+1})\}^{\varphi(M(x_{2n+2},x_{2n+1}))} \tag{2.1.2}$$

where

$$\begin{aligned}M(x_{2n+2,}x_{2n+1}) &= max\ \{d(Ix_{2n+2},Jx_{2n+1}),d(Ix_{2n+2},Ax_{2n+2}),d(Bx_{2n+1},Jx_{2n+1}),d(Ax_{2n+2},Jx_{2n+1}),d(Ix_{2n+2},Bx_{2n+1})\}\\ &= max\ \{d(z_{2n+1},z_{2n}),d(z_{2n+1},z_{2n+2}),d(z_{2n+2},z_{2n}),d(z_{2n+2},z_{2n}),d(z_{2n+1},z_{2n+1})\}\\ &= max\{1,d(z_{2n+1},z_{2n+2})\}=d(z_{2n+1},z_{2n+2})\end{aligned}$$

$$\begin{aligned}\therefore\ d(z_{2n+1},z_{2n+2}) &\le \{d(z_{2n+1},z_{2n+2})\}^{\varphi(d(z_{2n+1},z_{2n+2}))}\\ &< d(z_{2n+1},z_{2n+2})\ \text{, if } z_{2n+1}\ne z_{2n+2}\text{. a contradiction.}\end{aligned}$$

$\therefore z_{2n+1}=z_{2n+2}$.

Again $d(z_{2n+2},z_{2n+3}) = d(Ax_{2n+2},Bx_{2n+3}) \le \{M(x_{2n+2},x_{2n+3})\}^{\varphi(M(x_{2n+2},x_{2n+3}))}$ (2.1.3)

where

$$\begin{aligned}M(x_{2n+2,}x_{2n+3}) &= max\{d(Ix_{2n+2},Jx_{2n+3}),d(Ix_{2n+2},Ax_{2n+2}),d(Bx_{2n+3},Jx_{2n+3}),d(Ax_{2n+2},Jx_{2n+3}),d(Ix_{2n+2},Bx_{2n+3})\}\\ &= max\ \{d(z_{2n+1},z_{2n}),d(z_{2n+1},z_{2n+2}),d(z_{2n+3},z_{2n+2}),d(z_{2n+2},z_{2n+2}),d(z_{2n+1},z_{2n+3})\}\\ &= max\{1,d(z_{2n+2},z_{2n+3})\}=d(z_{2n+2},z_{2n+3})\end{aligned}$$

$$\begin{aligned}\therefore\ d(z_{2n+2},z_{2n+3}) &\le \{d(z_{2n+2},z_{2n+3})\}^{\varphi(d(z_{2n+2},z_{2n+3}))}\\ &< d(z_{2n+2},z_{2n+3})\ \text{, if } z_{2n+2}\ne z_{2n+3}\text{. a contradiction.}\end{aligned}$$

$\therefore z_{2n+2} = z_{2n+3}$.

Similarly we get $z_{2n+3} = z_{2n+4} = ...$

$\therefore z = z_{2n} = z_{2n+1} = z_{2n+2} = z_{2n+3} = ...$ (2.1.4)

Since $z_{2n} = Ax_{2n} = Jx_{2n+1}$

$z_{2n+1} = Bx_{2n+1} = Ix_{2n+2}$

Also $z_{2n+2} = Ax_{2n+2} = Jx_{2n+3}.....$

$\therefore\ Bx_{2n+1} = Jx_{2n+1} \Rightarrow BJx_{2n+1} = JBx_{2n+1} \quad (\because (B, J)\, is\, coincidently\, commuting)$

$\Rightarrow Bz_{2n} = Jz_{2n}$ i.e., $Bz = Jz$

and $Ax_{2n+2} = Ix_{2n+2} \Rightarrow AIx_{2n+2} = IAx_{2n+2} \quad (\because (A, I)\, is\, coincidently\, commuting)$

$\Rightarrow Az_{2n} = Iz_{2n}$ i.e., $Az = Iz$

$\therefore Az = Iz = Bz = Jz$

Now $d(Az, B(AZ)) \le \{M(z, Az\}^{\varphi(M(z,Az))}$

$$M(z, Az) = max\{d(Iz, J(Az)), d(Iz, Az), d(B(Az), J(Az)), d(Az, J(Az)), d(Iz, B(Az))\}$$
$$= max\{d(Az, B(Az)), d(Az, Az), d(B(Az), B(Az)), d(Az, B(Az)), d(Az, B(Az))\}$$
$$= d(Az, B(AZ))$$

$\therefore d(Az, B(AZ)) \le \{d(Az, B(Az))\}^{\varphi(d(Az,B(Az)))}$

$< d(Az, B(Az))$, if $Az \ne BAz$ .a contradiction.

$\therefore\ Az = B(Az)$.

$\therefore\ Az$ is a fixed point of B.

Also $d(A(Bz), BZ) \le \{M(Bz, z\}^{\varphi(M(Bz,z))}$

$$M(Bz, z) = max\{d(I(Bz), Jz), d(I(Bz), A(Bz)), d(Bz, Jz), d(A(Bz), Jz), d(I(Bz), Bz)\}$$
$$= max\{d(A(Bz), Bz), d(A(Bz), A(Bz)), d(Bz, Bz), d(A(Bz), Bz), d(A(Bz), Bz)\}$$
$$= d(A(Bz), Bz)$$

$\therefore\ d(A(Bz), Bz) \le \{d(A(Bz), Bz)\}^{\varphi(d(A(Bz),Bz))}$

$< d(A(Bz), Bz)$, if $ABz \ne Bz$ .a contradiction.

$\therefore\ A(Bz) = Bz$.

$\therefore\ Bz$ is a fixed point of A.

Similarly we can proved I and J.

$\therefore\ A, B, I$ and J have unique common fixed point.

Suppose $z_{2n+1} \ne z_{2n}$ for every n.

$d(z_{2n+1}, z_{2n+2}) = d(Bx_{2n+1}, Ax_{2n+2}) = d(Ax_{2n+2}, Bx_{2n+1})$

$d(z_{2n+1}, z_{2n+2}) = d(Ax_{2n+2}, Bx_{2n+1}) \le \{M(x_{2n+2}, x_{2n+1})\}^{\varphi(M(x_{2n+2},x_{2n+1}))}$ (2.1.5)

Where

$$M(x_{2n+2}, x_{2n+1}) = \max\{d(Ix_{2n+2}, Jx_{2n+1}), d(Ix_{2n+2}, Ax_{2n+2}), d(Bx_{2n+1}, Jx_{2n+1}), d(Ax_{2n+2}, Jx_{2n+1}), d(Ix_{2n+2}, Bx_{2n+1})\}$$
$$= max\{d(z_{2n+1}, z_{2n}), d(z_{2n+1}, z_{2n+2}), d(z_{2n+1}, z_{2n}), d(z_{2n+2}, z_{2n}), d(z_{2n+1}, z_{2n+1})\}$$
$$= max\{d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2}), d(z_{2n}, z_{2n+2})\}$$

From (2.1.5)
$d(z_{2n+1},z_{2n+2}) \le \{max\ \{d(z_{2n},z_{2n+1}),d(z_{2n+1},z_{2n+2}),d(z_{2n},z_{2n+2})\}\}^{\varphi\{max\{d(z_{2n},z_{2n+1}),d(z_{2n+1},z_{2n+2}),d(z_{2n},z_{2n+2})\}\}}$

Claim: $d(z_{2n+1},z_{2n+2}) < (d(z_{2n},z_{2n+1})$

(i) If $d(z_{2n+1},z_{2n+2})$ is maximum of $d(z_{2n},z_{2n+1})$, $d(z_{2n+1},z_{2n+2})$ and $d(z_{2n},z_{2n+2})$.
then $d(z_{2n+1},z_{2n+2}) \le d[z_{2n+1},z_{2n+2}]^{\phi(d(z_{2n+1},z_{2n+2}))}$
$< d(z_{2n+1},z_{2n+2})$. a contradiction.

Hence $d(z_{2n+1},z_{2n+2})$ is not a maximum.

(ii) If $d(z_{2n},z_{2n+1})$ is maximum of $(d(z_{2n},z_{2n+1})$, $d(z_{2n+1},z_{2n+2})$ and $d(z_{2n},z_{2n+2})$ .then
$d(z_{2n+1},z_{2n+2}) \le [d(z_{2n},z_{2n+1})]^{\varphi(d(z_{2n},z_{2n+1}))} < (d(z_{2n},z_{2n+1})$. Hence the claim established.

(iii) if $d(z_{2n},z_{2n+2})$ is maximum of $(d(z_{2n},z_{2n+1})$, $d(z_{2n+1},z_{2n+2})$ and $d(z_{2n},z_{2n+2})$ .then
$d(z_{2n+1},z_{2n+2}) \le [d(z_{2n},z_{2n+2})]^{\varphi(d(z_{2n},z_{2n+2}))}$.

We have $(d(z_{2n},z_{2n+2}) \le d(z_{2n},z_{2n+1}).d(z_{2n+1},z_{2n+2})$

Now $d(z_{2n+1},z_{2n+2}) \le [d(z_{2n},z_{2n+2})]^{\varphi(d(z_{2n},z_{2n+2}))}$
$\le [d(z_{2n},z_{2n+1}).d(z_{2n+1},z_{2n+2})]^{\varphi(d(z_{2n},z_{2n+2}))}$
$\le [d(z_{2n+1},z_{2n+2}).d(z_{2n+1},z_{2n+2})]^{\varphi d(z_{2n},z_{2n+2})}$ $(Suppose\, d(z_{2n},z_{2n+1}) \le d(z_{2n+1},z_{2n+2}))$

$\therefore\ d(z_{2n+1},z_{2n+2}) \le [d(z_{2n+1},z_{2n+2})]^{2.\varphi d(z_{2n},z_{2n+2})}$
$< d(z_{2n+1},z_{2n+2})$. a contradiction.

$\therefore\ d(z_{2n+1},z_{2n+2}) < d(z_{2n},z_{2n+1})$ (2.1.7)

Again $d(z_{2n+2},z_{2n+1}) = d(Ax_{2n+2},Bx_{2n+1})$
$d(z_{2n+2},z_{2n+1}) \le \{M(x_{2n+2},x_{2n+1})\}^{\varphi(M(x_{2n+2},x_{2n+1}))}$ (2.1.8)
Where
$M(x_{2n+2},x_{2n+1}) = max\ \{d(Ix_{2n+2},Jx_{2n+1}),d(Ix_{2n+2},Ax_{2n+2}),d(Bx_{2n+1},Jx_{2n+1}),d(Ax_{2n+2},Jx_{2n+1}),d(Ix_{2n+2},Bx_{2n+1})\}$
$= max\ \{d(z_{2n+1},z_{2n}),d(z_{2n+1},z_{2n+2}),d(z_{2n+1},z_{2n}),d(z_{2n+2},z_{2n}),d(z_{2n+1},z_{2n+1})\}$
$= max\ \{d(z_{2n+1},z_{2n}),d(z_{2n+1},z_{2n+2}),d(z_{2n},z_{2n+2})\}$

From (2.1.8) $d(z_{2n+2},z_{2n+1}) \le \{max\ \{d(z_{2n+1},z_{2n}),d(z_{2n+1},z_{2n+2}),d(z_{2n},z_{2n+2})\}\}^{\varphi\{max\{d(z_{2n+1},z_{2n}),d(z_{2n+1},z_{2n+2}),d(z_{2n},z_{2n+2})\}\}}$
(2.1.9)

Claim: $d(z_{2n+2},z_{2n+1}) < (d(z_{2n+1},z_{2n})$

(i) If $d(z_{2n+1},z_{2n+2})$ is maximum of $d(z_{2n+1},z_{2n})$, $d(z_{2n+1},z_{2n+2})$ and $d(z_{2n},z_{2n+2})$. then
$d(z_{2n+2},z_{2n+1}) \le d[z_{2n+2},z_{2n+1}]^{\varphi(d(z_{2n+2},z_{2n+1}))}$
$< d(z_{2n+2},z_{2n+1})$. a contradiction.

Hence $d(z_{2n+2},z_{2n+1})$ is not a maximum.

(ii) If $d(z_{2n+1},z_{2n})$ is maximum of $(d(z_{2n+1},z_{2n})$, $d(z_{2n+1},z_{2n+2})$ and $d(z_{2n},z_{2n+2})$. then
$d(z_{2n+2},z_{2n+1}) \le [d(z_{2n+1},z_{2n})]^{\varphi(d(z_{2n+1},z_{2n}))} < (d(z_{2n+1},z_{2n})$.

Hence the claim established.

(iii) if $d(z_{2n}, z_{2n+2})$ is maximum of $(d(z_{2n+1}, z_{2n})$, $d(z_{2n+1}, z_{2n+2})$ and $d(z_{2n}, z_{2n+2})$. Then
$d(z_{2n+2}, z_{2n+1}) \leq [d(z_{2n}, z_{2n+2})]^{\varphi(d(z_{2n}, z_{2n+2}))}$.

We have $(d(z_{2n}, z_{2n+2}) \leq d(z_{2n}, z_{2n+1}).d(z_{2n+1}, z_{2n+2})$

Now
$$d(z_{2n+2}, z_{2n+1}) \leq [d(z_{2n}, z_{2n+2})]^{\varphi(d(z_{2n}, z_{2n+2}))}$$
$$\leq [d(z_{2n}, z_{2n+1}).d(z_{2n+1}, z_{2n+2})]^{\varphi(d(z_{2n}, z_{2n+2}))}$$
$$\leq [d(z_{2n+1}, z_{2n+2}).d(z_{2n+1}, z_{2n+2})]^{\varphi d(z_{2n}, z_{2n+2})} \quad (Suppose\, d(z_{2n}, z_{2n+1}) \leq d(z_{2n+1}, z_{2n+2}))$$

$$\therefore \quad d(z_{2n+2}, z_{2n+1}) \leq [d(z_{2n+1}, z_{2n+2})]^{2.\varphi d(z_{2n}, z_{2n+2})}$$
$< d(z_{2n+1}, z_{2n+2})$. a contradiction.

$$\therefore \quad d(z_{2n+2}, z_{2n+1}) < d(z_{2n+1}, z_{2n}) \tag{2.1.10}$$

From (2.1.7) and (2.1.10) $d(z_{n+1}, z_n) < d(z_n, z_{n-1})$ for $n =$ 1, 2, 3...

$$\therefore \quad d(z_{n+1}, z_{n+2}) < d(z_n, z_{n+1}) \text{ for } n = 1,2,3... \tag{2.1.11}$$

$\therefore \quad \{d(z_n, z_{n+1})\}$ is a strictly decreasing sequence of positive numbers.

Suppose $d(z_{n+1}, z_{n+2}) \to r \geq 1$

We show that $r = 1$.

From (2.1.6) and (2.1.9)
$$d(z_{2n+2}, z_{2n+1}) \leq \{max\, \{d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2}), d(z_{2n}, z_{2n+2})\}\}^{\varphi\{max\, \{d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2}), d(z_{2n}, z_{2n+2})\}\}}$$

As $n \to \infty$,
$$r \leq \{max\, \{r, r, r^2\}\}^{\varphi\{max\, \{r, r, r^2\}\}}$$
$r \leq \{r^2\}^{\varphi(r^2)} = r^{2.\phi(r^2)} < r$, if $r > 1$, *a contradiction.*

$\therefore \quad r = 1.$

$\therefore \quad \lim_{n\to\infty} d(z_{2n+2}, z_{2n+1}) = 1.$ i.e., $\lim_{n\to\infty} d(z_{2n+1}, z_{2n+2}) = 1.$

Now we show that $\{z_n\}$ is a Cauchy sequence.

Given $\varepsilon > 1,\ \exists$ a natural number N such that $n \geq N$
$$d(z_n, z_{n+k}) \leq d(z_n, z_{n+1}).d(z_{n+1}, z_{n+2})..........d(z_{n+k-1}, z_{n+k}) \tag{2.1.12}$$

We have
$$d(z_{2n+1}, z_{2n+2}) \leq \{max\, \{d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2}), d(z_{2n}, z_{2n+2})\}\}^{\varphi\{max\, \{d(z_{2n}, z_{2n+1}), d(z_{2n+1}, z_{2n+2}), d(z_{2n}, z_{2n+2})\}\}}$$
$$= \{max\, \{d(z_{2n}, z_{2n+1}), d(z_{2n}, z_{2n+2})\}\}^{\varphi\{max\, \{d(z_{2n}, z_{2n+1}), d(z_{2n}, z_{2n+2})\}\}}$$
$(\because d \text{ is decreasing sequence})$

$$d(z_{2n+1}, z_{2n+2}) \leq [d(z_{2n}, z_{2n+1})^2]^{\varphi[d(z_{2n}, z_{2n+1})^2]}$$

$$\therefore \quad d(z_{2n+1}, z_{2n+2}) \leq [d(z_{2n}, z_{2n+1})]^{2.\varphi[d(z_{2n}, z_{2n+1})^2)]}$$

Similarly $d(z_{2n}, z_{2n+1}) \leq [d(z_{2n-1}, z_{2n})]^{2.\varphi[d(z_{2n-1}, z_{2n})^2]}$ and
$d(z_{n+1}, z_{n+2}) \leq [d(z_n, z_{n+1})]^{2.\varphi[d(z_n, z_{n+1})^2]}$.

Put $\lambda = 2.\varphi[d(z_n, z_{n+1})^2]$.

$$d(z_{n+2}, z_{n+3}) \le [d(z_{n+1}, z_{n+2})]^{2.\varphi[d(z_{n+1}, z_{n+2})^2]}$$
$$\le [[d(z_n, z_{n+1})]^{\lambda}]^{2.\varphi[d(z_{n+1}, z_{n+2})^2]} \quad (\because d \text{ is decreasing and } \varphi \text{ is increasing})$$
$$\le [[d(z_n, z_{n+1})]^{\lambda}]^{\lambda}$$
$$\le [d(z_n, z_{n+1})]^{\lambda^2}$$

Similarly $d(z_{n+3}, z_{n+4}) \le [d(z_{n+2}, z_{n+3})]^{2.\varphi[d(z_{n+2}, z_{n+3})^2]}$

$$\le [[d(z_n, z_{n+1})]^{\lambda^2}]^{2.\varphi[d(z_{n+2}, z_{n+3})^2]}$$
$$\le [[d(z_n, z_{n+1})]^{\lambda^2}]^{\lambda}.$$
$$\le [d(z_n, z_{n+1})]^{\lambda^3}.$$

....

....

$$From(2.1.12)\ d(z_n, z_{n+k}) \le [d(z_n, z_{n+1})].[d(z_n, z_{n+1})]^{\lambda}.[d(z_n, z_{n+1})]^{\lambda^2}.[d(z_n, z_{n+1})]^{\lambda^3}......[d(z_n, z_{n+1})]^{\lambda^{n+k-1}}$$
$$= [d(z_n, z_{n+1})]^{1+\lambda+\lambda^2+\lambda^3.......+\lambda^{n+k-1}}$$
$$= [d(z_n, z_{n+1})]^{\frac{1}{1-\lambda}} < (\delta)^{\frac{1}{1-\lambda}} < \varepsilon \quad \text{if } n \ge N$$

Given $\varepsilon > 1$, choose δ such that $\delta^{\frac{1}{1-\lambda}} < \varepsilon$

$$i.e., \delta < \varepsilon^{1-\lambda}$$

$$\therefore\ 1 < \delta < \varepsilon^{1-\lambda} \qquad \therefore\ \delta^{\frac{1}{1-\lambda}} < \varepsilon$$

$\therefore\ d(z_n, z_{n+k}) \to 1, (n, k \to \infty)$.

$\therefore\ \{z_n\}$ is a multiplicative Cauchy sequence.

Now suppose that $I(X)$ is a complete subspace of X, then we observe that the subsequence $\{z_{2n+1}\}$ which is contained in $I(X)$ has a limit point z in $I(X)$.

$\therefore\ \exists u \in X$ such that $Iu = z$.

Now we prove that $Au = z.$ Take $x = u, and\ y = x_{2n+1}$

$$d(Au, Bx_{2n+1}) \le \{M(u, x_{2n+1})\}^{\varphi(M(u, x_{2n+1}))}$$

where $M(u, x_{2n+1}) = max\ \{d(Iu, Jx_{2n+1}), d(Iu, Au), d(Bx_{2n+1}, Jx_{2n+1}), d(Au, Jx_{2n+1}), d(Iu, Bx_{2n+1})\}$

$$= max\ \{d(z, z_{2n}), d(z, Au), d(z_{2n+1}, z_{2n}), d(Au, z_{2n}), d(z, z_{2n+1})\}$$

$$\therefore d(Au, z_{2n+1}) \le \{max\ \{d(z, z_{2n}), d(z, Au), d(z_{2n+1}, z_{2n}), d(Au, z_{2n}), d(z, z_{2n+1})\}\}^{\varphi(max\{d(z,z_{2n}),d(z,Au),d(z_{2n+1},z_{2n}),d(Au,z_{2n}),d(z,z_{2n+1})\})}$$

On letting $n \to \infty$.

$$d(Au, z) \le \{max\ \{d(z,z), d(z, Au), d(z,z), d(Au, z), d(z,z)\}\}^{\varphi(max\{d(z,z),d(z,Au),d(z,z),d(Au,z),d(z,z)\})}$$

$d(Au, z) \le \{d(z, Au)\}^{\phi(d(z,Au))} < d(z, Au)$, if $z \ne Au$, a contradiction.

$\therefore\ Au = z$

$\therefore\ Au = z = Iu$

$\therefore\ u$ is a coincident point of A and I.

Since $z = Au \in A(X) \subset J(X)$, implies that $z \in J(X)$, $Jv = z$ for some $v \in X$.

Now we prove that $Bv = z.$ Take $x = x_{2n+2}$ and $y = v$

$d(Ax_{2n+2}, Bv) \leq \{x_{2n+2}, v)\}^{\varphi(M(x_{2n+2},v))}$

where $M(x_{2n+2}, v) = \max \{d(Ix_{2n+2}, Jv), d(Ix_{2n+2}, Ax_{2n+2}), d(Bv, Jv), d(Ax_{2n+2}, Jv), d(Ix_{2n+2}, Bv)\}$

$= max \{d(z_{2n+1}, z), d(z_{2n+1}, z_{2n+2}), d(Bv, z), d(z_{2n+2}, z), d(z_{2n+1}, Bv)\}$

$$\therefore d(z_{2n+2}, Bv) \leq \{max \{d(z_{2n+1}, z), d(z_{2n+1}, z_{2n+2}), d(Bv, z), d(z_{2n+2}, z), d(z_{2n+1}, Bv)\}\}^{\varphi(max\{d(z_{2n+1},z),d(z_{2n+1},z_{2n+2}),d(Bv,z),d(z_{2n+2},z),d(z_{2n+1},Bv)\}}$$

On letting $n \to \infty$.

$d(z, Bv) \leq \{max \{d(z,z), d(z,z), d(Bv,z), d(z,z), d(z,Bv)\}\}^{\varphi(max\{d(z,z),d(z,z),d(Bv,z),d(z,z),d(z,Bv)\}}$

$d(z, Bv) \leq \{d(z, Bv)\}^{\varphi(d(z,Bv))} < d(z, Bv)$, if $z \neq Bv$, a contradiction.

$\therefore Bv = z$

$\therefore Bv = z = Jv$. $\therefore v$ is a coincident point of B and J.

$\therefore Au = Iu = Bv = Jv = z$.

Suppose (A, I) is coincidentally commuting. Then $AIu = IAu \Rightarrow Az = Iz$

Suppose (B, J) is coincidentally commuting. Then $BJv = JBv \Rightarrow Bz = Jz$

$\therefore d(Au, Bz) \leq \{M(u,z)\}^{\varphi(M(u,z))}$

where $M(u, z) = \max \{d(Iu, Jz), d(Iu, Au), d(Bz, Jz), d(Au, Jz), d(Iu, Bz)\}$

$= max\{d(z, Bz), d(z,z), d(Bz, Bz), d(z, Bz), d(z, Bz)\}$

$= max\{1, d(z, Bz)\} = d(z, Bz)$.

$d(z, Bz) \leq \{d(z, Bz)\}^{\varphi(d(z,Bz)} < d(z, Bz)$, if $z \neq Bz$, a contradiction.

$\therefore Bz = z = Jz$.

$\therefore z$ is a fixed point of B and J.

Now $d(Az, Bz) \leq \{M(z,z)\}^{\varphi(M(z,z))}$

where $M(z,z) = max\{d(Iz, Jz), d(Iz, Az), d(Bz, Jz), d(Az, Jz), d(Iz, Bz)\}$

$= max\{d(Az, Bz), d(Az, Az), d(Bz, Bz), d(Az, Bz), d(Az, Bz)\}$

$= max\{1, d(Az, Bz)\} = d(Az, Bz)$.

$\therefore d(Az, Bz) \leq \{d(Az, Bz)\}^{\varphi(d(Az,Bz)} < d(Az, Bz)$, if $Az \neq Bz$, a contradiction.

$\therefore Az = Bz = z$.

$\therefore Az = Bz = Jz = Iz = z$.

$\therefore z$ is a fixed point of A,B,I and J.

Suppose w is another fixed point of A, B, I and J.

$d(z, w) = d(Az, Bw) \leq \{M(z, w)\}^{\varphi(M(z,w))}$

where $M(z, w) = max\{d(Iz, Jw), d(Iz, Az), d(Bw, Jw), d(Az, Jw), d(Iz, Bw)\}$

$= max \{d(z, w), d(z,z), d(w,w), d(z,w), d(z,w)\}$

$= max \{1, d(z, w)\} = d(z, w)$.

$d(z, w) \leq \{d(z, w)\}^{\varphi(d(z,w)} < d(z, w)$, if $z \neq w$, a contradiction.

$\therefore z = w$.

$\therefore z$ is a unique common fixed point of A, B, I and J.

Corollary 2.2: Let A,B,I and J be self mappings of a multiplicative metric space (X,d), satisfying $A^p(X)=C(X)\subset J(X)$, $B^q(X)=D(X)\subset I(X)$ and define $\varphi:\mathrm{R}_+\to[0,\frac{1}{2})$ such that φ is continuous and increasing and $d(Cx,Dy)\le[M(x,y)]^{\varphi(M(x,y))}$ (2.2.1)

where $M(x,y)=max\,\{d(Ix,Jy),d(Ix,Cx),d(Dy,Jy),d(Cx,Jy),d(Ix,By)\}$

$\forall\, x,y,z\in X$, for some $p,q\in\mathbf{Z}^+$.

If one of $C(X),D(X),I(X),J(X)$ is a complete subspace of X, then the following conclusions hold.

(i) (C,I) has a coincidence point;

(ii) (D,J) has a coincidence point;

Further if the pairs (C,I) and (D,J) are coincidentally commuting, then C,D,I and J have a unique common fixed point.

Proof: Take $C=A^p, D=B^q$ in theorem 2.1.

Theorem 2.3: Let A,B,I and J be self mappings of a multiplicative metric space (X,d), satisfying

(i) $A(X)\subset J(X)$, $B(X)\subset I(X)$

(ii) I and A are commuting mappings and J and B also commuting mappings.

(iii) One of A,B,I and J is continuous.

(iv) $d(A^px,B^qy)\le max\{d(Ix,Jy),d(Ix,A^px),$

$$d(B^qy,Jy),d(A^px,Jy),d(Ix,B^qy)\}\}^{\varphi(max\{d(Ix,Jy),d(Ix,A^px),d(B^qy,Jy),d(A^px,Jy),d(Ix,B^qy)\})}$$

$\forall\, x,y,z\in X$, for some $p,q\in\mathbf{Z}^+$.

where $\varphi:\mathrm{R}_+\to\left[0,\frac{1}{2}\right)$ is continuous and increasing.

Then A,B,I and J have a unique common fixed point.

Proof: By corollary 2.2, C,D,I and J have a unique common fixed point say u.

$\therefore\ Cu=Du=Iu=Ju=u$

Now $Cu=A^pu=u$.

$A(A^pu)=Au$ i.e., $A^p(Au)=Au$

$\therefore$ Au is a fixed point of $A^p=C$, i.e., $C(Au)=Au$.

and $Du=B^qu=u$

i.e., $B^q(Bu)=Bu$

$\therefore$ Bu is a fixed point of $B^q=D$, i.e., $D(Bu)=Bu$.

Since A and I are commuting, $I(Au)=A(Iu)=Au$

$\therefore$ Au is a fixed point of I.

Since B and J are commuting, $J(Bu)=B(Ju)=Bu$

$\therefore$ Bu is a fixed point of J.

Now we show that $Au=Bu$.

$$\begin{aligned}d(C(Au),D(Bu))&\le\{max\{d(I(Au),J(Bu)),d(I(Au),C(Au)),d(D(Bu),J(Bu)),d(C(Au),J(Bu)),\\&\quad d(I(Au),D(Bu))\}\}^{\varphi(max\{d(I(Au),J(Bu)),d(I(Au),C(Au)),d(D(Bu),J(Bu)),d(C(Au),J(Bu)),d(I(Au),D(Bu))\})}\\&\le\{max\{d(Au,Bu),d(Au,Au),d(Bu,Bu),d(Au,Bu),\\&\quad d(Au,Bu)\}\}^{\varphi(max\{d(Au,Bu),d(Au,Au),d(Bu,Bu),d(Au,Bu),d(Au,Bu)\})}\end{aligned}$$

$\therefore\ d(Au,Bu) \le \{d(Au,Bu)\}^{\varphi(d(Au,Bu))} < d(Au,Bu)$, if $Au \ne Bu$.
$\therefore\ Au = Bu$.

Now $CAu = Au = Bu$, $DBu = Bu = Au$, i.e., $DAu = Au$.
$\therefore\ IAu = Au$ and $JAu = Au.$
$\therefore\ Au$ is a fixed point of C, D, I and J.

But u is a fixed point of C, D, I and J.

By uniqueness $Au = u = Bu = Iu = Ju$.
A, B, I and J have unique common fixed point.
Theorem 1.21 is as a corollary of our main result, i.e., theorem 2.1.

Corollary 2.4: Theorem 1.21

Proof: Take $\varphi(t) = \lambda$, $\lambda = \left(0, \frac{1}{2}\right)$ in theorem 2.1.

We given below the following example in support of corollary 2.4. It may note that the example (Example 4.1 of [9]) which is stated as supporting theorem 1.21 does not support theorem 1.21.

Example 2.5: Let $X = \left[0, \frac{1}{2}\right]$ and (X, d) be a multiplicative metric space. Define the mapping $d : X \times X \to \mathbb{R}^+$ by $d(x, y) = e^{|x-y|}$ for all $x, y \in X$.

Define the self maps A, B, I and J on X by

$Ax = x^4,\ \ 0 \le x \le \frac{1}{2}$

$Bx = x^3, 0 \le x \le \frac{1}{2}$

$Ix = x,\ \ 0 \le x \le \frac{1}{2}$

$Jx = x,\ \ 0 \le x \le \frac{1}{2}$

Now $A(X) = \left[0, \frac{1}{16}\right] \subset \left[0, \frac{1}{2}\right] = J(X)$.

$B(X) = \left[0, \frac{1}{8}\right] \subset \left[0, \frac{1}{2}\right] = I(X)$.

Also $A(X), B(X), I(X), J(X)$ are complete.

By the inequality of theorem 1.21, we have

$$d(Ax, By) \le max\{d(Ix, Jy), d(Ix, Ax), d(By, Jy), d(Ax, Jy), d(Ix, By)\}^{\lambda}\quad \lambda \in \left(0, \frac{1}{2}\right) \forall\, x, y \in X.$$

$$e^{|x^4-y^3|} \le max\{e^{|x-y|}, e^{|x-x^4|}, e^{|y-y^3|}, e^{|x^4-y|}, e^{|x-y^3|}\}^{\lambda} \tag{2.5.1}$$

Case- (i): Suppose $y^3 \le x^4$
i.e., $y^3 \le x^4 \le x^3\ \therefore y \le x$.

$x^4 - y^3 \leq (x - x^4)\lambda$, holds for λ such that $\frac{1}{7} \leq \lambda < \frac{1}{2}$

$$x^4 - y^3 \leq \lambda x - \lambda x^4$$

$$x^4(1+\lambda) - y^3 \leq \lambda x$$

$$x^3(1+\lambda) \leq \lambda$$

$$x^3 \leq \frac{\lambda}{1+\lambda} \quad \forall x \in \left[0, \frac{1}{2}\right]$$

$$x^3 \leq \frac{1}{8}\frac{\lambda}{1+\lambda}$$

$$\therefore (1+\lambda) \leq 8\lambda$$

$$\therefore 1 \leq 7\lambda$$

$$\frac{1}{7} \leq \lambda < \frac{1}{2}.$$

Case-(ii): Suppose $x^4 \leq y^3$

$y^3 - x^4 \leq (y - x^4)\lambda$, holds for λ such that $\frac{1}{4} \leq \lambda < \frac{1}{2}$

$$y^3 \leq (y - x^4)\lambda + x^4.$$

$$y^3 \leq y\lambda + (1-\lambda)x^4$$

$$y^3 \leq y\lambda \leq y\lambda + (1-\lambda)x^4$$

$$\therefore y^2 \leq \lambda, \quad \forall y \in \left[0, \frac{1}{2}\right]$$

$$\therefore y^2 \leq \frac{1}{4} \leq \lambda < \frac{1}{2}.$$

$$\therefore \lambda = max\left\{\frac{1}{4}, \frac{1}{7}\right\} = \frac{1}{4}$$

Take $\lambda = \frac{1}{4}$

From (2.1.5) for $\lambda \in \left[\frac{1}{4}, \frac{1}{2}\right)$ we have that

$x^4 - y^3 \leq (x - x^4).\frac{1}{4}$ and $y^3 - x^4 \leq (y - x^4).\frac{1}{4}$.

$e^{x^4 - y^3} \leq e^{(x - x^4)\frac{1}{4}}$ and $e^{y^3 - x^4} \leq e^{(y - x^4)\frac{1}{4}}$.

i.e., $e^{|x^4 - y^3|} \leq max\{e^{|x-y|}, e^{|x-x^4|}, e^{|y-y^3|}, e^{|x^4-y|}, e^{|x-y^3|}\}^{\frac{1}{4}} \; \forall x, y \in X$.

Therefore all the conditions of theorem 1.21 are satisfied.

$\therefore$ A, B, I and J have a unique common fixed point 0.

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