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LINEAR MODEL WITH TWO PARAMETER DOUBLY TRUNCATED NEW SYMMETRIC DISTRIBUTED ERRORS

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ABSTRACT

Regression model is one of the most important statistical tool for analyzing data. In regression analysis it is customarily to consider that the error term follows Gaussian distribution which is mesokurtic and having infinite range. But in many data sets arising at places like agricultural experiments, biological experiments, financial analysis, space experiments, etc., the error term may not have mesokurtic in nature and its range is finite. As a result of it, the regression model with Gaussian errors may badly fit such data. To have accurate analysis for this type of data we develop and analyze a two variable regression model with doubly truncated new symmetric distributed errors. The model parameters are estimated by driving maximum likelihood estimators. The properties of this estimators are also discussed. Simulation study is conducted to compare the efficiency of proposed model with that of new symmetric distributed errors and Gaussian errors. It is observed that the proposed model performs much better than other two models when the variable are platykurtic and having constraints on tail ends.

Keywords: *Regression model; Doubly truncated new symmetric distribution; Simulation studies; Maximum likelihood estimators.*

1. INTRODUCTION

Applied Statisticians have to balance several things before analyzing data sets. In general in regression analysis it is assumed that the error term follows a normal (Gaussian) distribution. This assumption is valid only when the variable under study is mesokurtic and having infinite range. However, many scientific studies revealed that the distribution of the error term may not follow the Gaussian distribution, since the presence of skewness or heavy tails in the distribution of the error terms (Gabriele Soffritti and GiulianoGalimberti (2011), Fama (1965), Sutton (1997)). Recently much work has been reported in literature regarding regression analysis with different parametric distributions for error terms. Zellner(1976), Sutradhar and Ali (1986), Galea *et al.* (1997), Liu (2002), and Diaz-Garcia *et al.* (2003) have studied the regression models with a class of elliptic type distributions specially t- distribution for error term. Liu (1996) has studied regression analysis with missing values with the assumptions that the error term follows elliptic distribution family. Ferreira and Steel (2003, 2004) have considered the same problem under the Bayesian framework assuming skewed and heavy tailed distribution for error term. Zeckhauser and Thomson (1970) has studied regression model with power distributed error terms. Gabriele Soffritti and Giuliano Galimberti (2011) have studied multivariate regression model with mixture of multivariate Gaussian error distribution.

In all these papers the major consideration is on the estimation of the model parameters rather than the structure of error term distribution. If the structure of the error term is non-normal one cannot use the standard methodology of regression analysis for analyzing the data. Hence considering the peakedness (Kurtosis) of the variable Asrat Atsedeweyn and Srinivasa Rao (2014) have investigated linear regression model with new symmetric distributed errors. They assumed that in a regression model error term follows a new symmetric distribution given by Srinivasa Rao *et al.* (1997). The new symmetric distribution includes a family of platykurtic distributions and its probability density function is as a form

$$g(y;\mu,\sigma)=\frac{\left[2+((y-\mu)/\sigma)^2\right]e^{-(1/2)((y-\mu)/\sigma)^2}}{3\sigma\sqrt{2\pi}} \tag{1}$$

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However, for some data sets this regression model also may not serve the purpose since the range of the error term is assumed to have infinite range $(-\infty,\infty)$. But in many data sets the range of the error term or the response variable are constrained to have finite values and hence the range is finite. Ignoring the finite range for error term may leads to falsification in the model and the analysis may not be accurate. Hence, to have an accurate analysis of the data sets it is needed to develop and analyze regression model with doubly truncated new symmetric distributed errors. Very little work has been reported in literature regarding regression analysis with doubly truncated symmetric distributed errors which is useful for analyzing several data sets arising at agricultural experiments, space research, financial modeling and biological experiments etc. This article fills the gap in this area of research.

The rest of the paper is organized as follows: In Section 2, the doubly truncated new symmetric distribution and their properties are discussed. In Section 3, the Maximum likelihood estimation of the model are derived using Newton–Raphson (NR) iterative method. In Section 4, simulation study is carried for studying the properties of model parameters under Maximum likelihood method of estimation. In Section 5, the least square estimators of the model parameters are studied. In Section 6, a comparative study is carried to study the efficiency of the proposed model with that of Gaussian model and new symmetric distributed errors. Finally, the conclusion of the paper is given in Section 7.

2. DOUBLY TRUNCATED NEW SYMMETRIC DISTRIBUTION AND ITS PROPERTIES

Double truncation occurs when the values of a variable are observable only over finite range, i.e., only when they exceed a lower bound and are also smaller than some upper bound. For each observation, $a \le Y_i \le b,$ where a and b are fixed points of truncation.

A continuous random variable Y is said to have a two parameter Doubly Truncated New Symmetric distribution with parameters μ and σ if its probability density function is of the form,

$$f(y) = \frac{g(y)}{\int_a^b g(ydy}\text{'}a \le y \le b; a \le \mu \le b; a,b \in (-\infty,\infty)$$

Where, $g(y) = \dfrac{\left[2+((y-\mu)/\sigma)^2\right]e^{-(1/2)((y-\mu)/\sigma)^2}}{3\sigma\sqrt{2\pi}}$

and $\displaystyle\int_a^b g(y)dy = \int_a^b \frac{\left[2+((y-\mu)/\sigma)^2\right]e^{-(1/2)((y-\mu)/\sigma)^2}}{3\sigma\sqrt{2\pi}}dy$

$$\Phi\left(\frac{b-\mu}{\sigma}\right)-\frac{1}{3}\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)-\left[\Phi\left(\frac{a-\mu}{\sigma}\right)-\frac{1}{3}\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)\right]$$

Let $\displaystyle F(b) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{b} e^{-1/2\left(\frac{y-\mu}{\sigma}\right)^2}dy - \left(\frac{b-\mu}{3\sigma}\right)e^{-1/2\left(\frac{b-\mu}{\sigma}\right)^2} = \Phi\left(\frac{b-\mu}{\sigma}\right)-\frac{1}{3}\left(\frac{b-\mu}{\sigma}\right)\phi\left(\frac{b-\mu}{\sigma}\right)$ and

$$F(a) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{a} e^{-1/2\left(\frac{y-\mu}{\sigma}\right)^2}dy - \left(\frac{a-\mu}{3\sigma}\right)e^{-1/2\left(\frac{a-\mu}{\sigma}\right)^2} = \Phi\left(\frac{a-\mu}{\sigma}\right)-\frac{1}{3}\left(\frac{a-\mu}{\sigma}\right)\phi\left(\frac{a-\mu}{\sigma}\right)$$

Therefore,

$$f(y) = \begin{cases} 0, -\infty < y < a \\ \dfrac{\left[2+((y-\mu)/\sigma)^2\right]e^{-(1/2)((y-\mu)/\sigma)^2}}{3\sigma\sqrt{2\pi}\left[F(b)-F(a)\right]}, a \le y \le b \\ 0, b < y < \infty \end{cases} \tag{2}$$

Where $a \le \mu \le b,$ and $\sigma > 0$ are location and scale parameters, respectively and a and b are the lower and upper truncation points. ϕ and Φ are the probability density and cumulative distribution functions for the standard normal distribution respectively. The various shapes of the, cumulative density function and frequency curves of the distributions are given in Figure 1.

In this paper, we attempt to investigate a simple regression model with non-normal error terms where there is only one independent variable and the regression function is linear. The model can be stated as

$$y_i = \beta_0 + \beta_1 x_i + u_i, i = 1,2,\ldots,n. \tag{3}$$

Where x_i is a single regressor of interest. Y is the dependent variable. The random errors, the u_i 's, are independent and identically distributed. The β's are the unknown regression coefficients. To have an appropriate fitting model, some assumptions describing about the behavior of the errors are needed. It is assumed that the error terms (u_i) are independent and identically distributed (i.i.d.) random variables whose distribution is assumed to follow a two-parameter doubly truncated new symmetric distributions $DTNS\left(0,\sigma^2\right)$.The observations $\left(X_i, Y_i\right)$ pairs where the response variable for the i^{th} observation Y_i also follows a doubly truncated new symmetric distribution and customarily, the non-stochastic variables, X_i 's, are considered to be fixed.

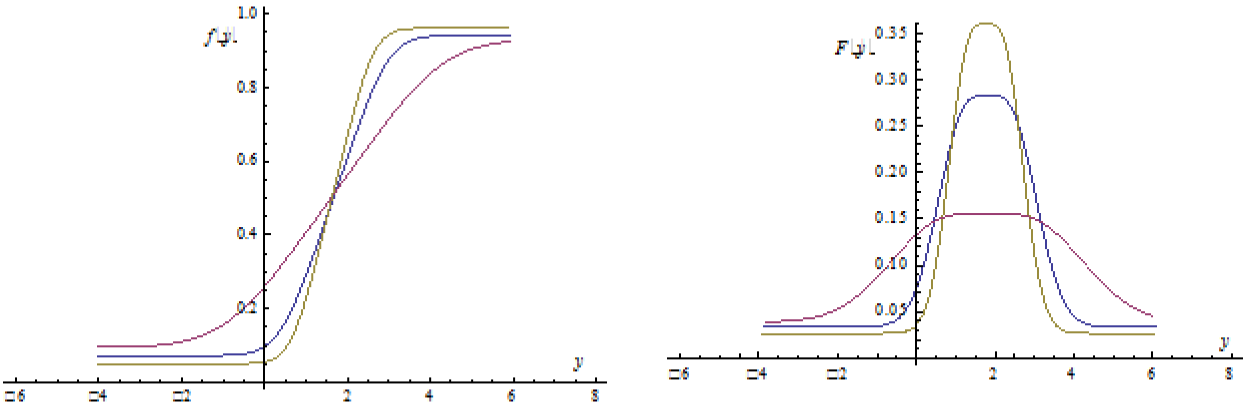

Figure-1: The shape of the frequency curves for the pdf and cumulative distribution function (CDF) of the two parameter DTNS distribution for different values of parameters σ: μ and σ as $(2,0.75) \to green, (2,1) \to blue, and (2,2) \to red,\ for a = -4 \,\&\, b = 6.$

The distributional properties of the DTNSD are:

i. The distribution function of the random variable Y specified by the probability density function (2) is given by

$$F_Y(y) = \int_a^y f(t)dt = \int_a^y \frac{\left[2+\left(\frac{t-\mu}{\sigma}\right)^2\right] e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}}{3\sigma\sqrt{2\pi}\left[F(b)-F(a)\right]} dt \tag{4}$$

ii. The Mean (location of the peak) of the variable is

$$E(y) = \mu + \sigma\lambda(\alpha)$$

$$\text{where } \lambda(\alpha) = \left\{ \frac{\left(\frac{4}{3}+\frac{1}{3}\left(\frac{a-\mu}{\sigma}\right)^2\right)\phi\left(\frac{a-\mu}{\sigma}\right) - \left(\frac{4}{3}+\frac{1}{3}\left(\frac{b-\mu}{\sigma}\right)^2\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{F(b)-F(a)} \right\} \tag{5}$$

If $\mu > 0$ and the truncation is both sides, i.e., $\lambda(\alpha) > 0,$ the mean of the truncated variable is greater than the original mean. $\lambda(\alpha)$ is the mean of the truncated normal distribution.

iii. The Median 'M' of the variable is

$\int_a^M f(y)dy = 1/2$, where M is the median of the distribution

implies

$$\int_a^\mu f(y)dy + \int_\mu^M f(y)dy = 1/2 \tag{6}$$

Solving equation (6) one can set median of the distribution

iv. The Mode of the variable is

The mode $\hat{M} = \begin{cases} a, if \mu < a \\ \mu, if a \le \mu \le b \\ b, if \mu > b \end{cases}$

At which $$\frac{d^2 \log f(y)}{dy^2} = \frac{d^2 \log}{dy^2} \frac{\left[2+\left(\frac{y-\mu}{\sigma}\right)^2\right] e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}}{[F(b)-F(a)]3\sigma\sqrt{2\pi}} = \frac{d}{dy}\left(\frac{\left(\frac{y-\mu}{\sigma}\right)^3}{\sigma[2+\left(\frac{y-\mu}{\sigma}\right)^2]}\right) = -\frac{6\left(\frac{y-\mu}{\sigma}\right)^2 - \left(\frac{y-\mu}{\sigma}\right)^4}{\sigma[2+\left(\frac{y-\mu}{\sigma}\right)^2]} < 0 \tag{7}$$

v. Variance of the variable is

$$Var(y) = \frac{5\sigma^2}{3} - \sigma^2 \left\{ \frac{\left(\frac{4}{3}+\frac{1}{3}\left(\frac{a-\mu}{\sigma}\right)^2\right)\varphi\left(\frac{a-\mu}{\sigma}\right) - \left(\frac{4}{3}+\frac{1}{3}\left(\frac{b-\mu}{\sigma}\right)^2\right)\varphi\left(\frac{b-\mu}{\sigma}\right)}{F(b)-F(a)} \right\}^2 \tag{8}$$

$$+\frac{\sigma^2}{[F(b)-F(a)]}\left\{\left[\frac{10}{9}\left(\frac{a-\mu}{\sigma}\right)+\frac{1}{3}\left(\frac{a-\mu}{\sigma}\right)^3\right]\varphi\left(\frac{a-\mu}{\sigma}\right) - \left[\frac{10}{9}\left(\frac{b-\mu}{\sigma}\right)+\frac{1}{3}\left(\frac{b-\mu}{\sigma}\right)^3\right]\varphi\left(\frac{b-\mu}{\sigma}\right)\right\}$$

vi. The characteristic function $\phi_y(t)$ of a random variable Y is

$$\phi_y(t) = E\left(e^{ity}\right) = \int_a^b e^{ity} \frac{\left[2+((y-\mu)/\sigma)^2\right]e^{-(1/2)((y-\mu)/\sigma)^2}}{3\sigma\sqrt{2\pi}\left[F(b)-F(a)\right]} dy$$

$$= \frac{e^{\mu it - \frac{1}{2}\sigma^2 t^2}}{F(b)-F(a)} \left\{ \begin{array}{l} \left(1+\frac{(\sigma it)^2}{3}\right)\left[\Phi\left(\frac{b-\mu}{\sigma}-\sigma it\right)-\Phi\left(\frac{a-\mu}{\sigma}-\sigma it\right)\right]+ \\ \frac{1}{3}\left[\left(\frac{a-\mu}{\sigma}+\sigma it\right)\phi\left(\frac{a-\mu}{\sigma}-\sigma it\right)-\frac{1}{3}\left(\frac{b-\mu}{\sigma}+\sigma it\right)\phi\left(\frac{b-\mu}{\sigma}-\sigma it\right)\right] \end{array} \right\} \tag{9}$$

This is the product of the normal characteristic function and a polynomial of even powers of '*t*'

vii. The moment generating function of the distribution is

$$M(t) = \frac{e^{\mu t + \frac{1}{2}\sigma^2 t^2}}{F(b)-F(a)} \left\{ \begin{array}{l} \left(1+\frac{(\sigma t)^2}{3}\right)\left[\Phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)-\Phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)\right]+ \\ \frac{1}{3}\left[\left(\frac{a-\mu}{\sigma}+\sigma t\right)\phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)-\frac{1}{3}\left(\frac{b-\mu}{\sigma}+\sigma t\right)\phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)\right] \end{array} \right\} \tag{10}$$

viii. The cumulant generating function of the distribution is

$g(t;\mu,\sigma^2) = \ln M(t;\mu,\sigma^2)$,

Letting

$$A = \left(1+\frac{(\sigma t)^2}{3}\right)\left[\Phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)-\Phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)\right]+$$
$$\frac{1}{3}\left[\left(\frac{a-\mu}{\sigma}+\sigma t\right)\phi\left(\frac{a-\mu}{\sigma}-\sigma t\right)-\frac{1}{3}\left(\frac{b-\mu}{\sigma}+\sigma t\right)\phi\left(\frac{b-\mu}{\sigma}-\sigma t\right)\right]$$

$$g(t;\mu,\sigma^2) = \ln M(t;\mu,\sigma^2) = \mu t + \frac{1}{2}\sigma^2 t^2 + \ln A - \ln\left(F(b)-F(a)\right) \tag{11}$$

The cumulants k_n, are extracted from the cumulant-generating function via differentiation (at zero) of $g(t)$. The first two cumulants of the distribution are;

$$k_1 = g'(0) = \mu + \sigma\left\{\frac{\left(\frac{4}{3}+\frac{1}{3}\left(\frac{a-\mu}{\sigma}\right)^2\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{4}{3}+\frac{1}{3}\left(\frac{b-\mu}{\sigma}\right)^2\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{F(b)-F(a)}\right\}$$

$$k_2 = g^{''}(0) = \frac{5\sigma^2}{3} - \sigma^2\left\{\frac{\left(\frac{4}{3}+\frac{1}{3}\left(\frac{a-\mu}{\sigma}\right)^2\right)\varphi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{4}{3}+\frac{1}{3}\left(\frac{b-\mu}{\sigma}\right)^2\right)\varphi\left(\frac{b-\mu}{\sigma}\right)}{F(b)-F(a)}\right\}^2 \tag{12}$$

$$+\frac{\sigma^2}{[F(b)-F(a)]}\left\{\left[\frac{10}{9}\left(\frac{a-\mu}{\sigma}\right)+\frac{1}{3}\left(\frac{a-\mu}{\sigma}\right)^3\right]\varphi\left(\frac{a-\mu}{\sigma}\right)-\left[\frac{10}{9}\left(\frac{b-\mu}{\sigma}\right)+\frac{1}{3}\left(\frac{b-\mu}{\sigma}\right)^3\right]\varphi\left(\frac{b-\mu}{\sigma}\right)\right\}$$

ix. The odd central moments of a DTNSD vanish by symmetry, we get

$$\mu_{2n+1} = 0 \tag{13}$$

The $(2n)^{th}$ order central moments of the DTNS distribution is given by

$$\mu_{2n} = \frac{1}{3\sigma\sqrt{2\pi}\left[F(b)-F(a)\right]}\int_a^b \left(y-E(y)\right)^{2n}\left(2+\left(\frac{y-\mu}{\sigma}\right)^2\right)e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}dy$$

Therefore,

$$\mu_2 = \frac{5\sigma^2}{3} - \sigma^2\left\{\frac{\left(\frac{4}{3}+\frac{1}{3}\left(\frac{a-\mu}{\sigma}\right)^2\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{4}{3}+\frac{1}{3}\left(\frac{b-\mu}{\sigma}\right)^2\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{F(b)-F(a)}\right\}^2 + \tag{14}$$

$$\frac{\sigma^2}{[F(b)-F(a)]}\left\{\left[\frac{10}{9}\left(\frac{a-\mu}{\sigma}\right)+\frac{1}{3}\left(\frac{a-\mu}{\sigma}\right)^3\right]\phi\left(\frac{a-\mu}{\sigma}\right)-\left[\frac{10}{9}\left(\frac{b-\mu}{\sigma}\right)+\frac{1}{3}\left(\frac{b-\mu}{\sigma}\right)^3\right]\phi\left(\frac{b-\mu}{\sigma}\right)\right\}$$

and

$$\mu_4 = 7\sigma^4 + 7\sigma^4\left\{\begin{array}{l}
\frac{1}{[F(b)-F(a)]}\left\{\begin{array}{l}\left[\frac{2}{3}\left(\frac{a-\mu}{\sigma}\right)+\frac{1}{3}\left(\frac{a-\mu}{\sigma}\right)^3+\frac{1}{21}\left(\frac{a-\mu}{\sigma}\right)^5\right]\varphi\left(\frac{a-\mu}{\sigma}\right)- \\ \left[\left[\frac{2}{3}\left(\frac{b-\mu}{\sigma}\right)+\frac{1}{3}\left(\frac{b-\mu}{\sigma}\right)^3+\frac{1}{21}\left(\frac{b-\mu}{\sigma}\right)^5\right]\varphi\left(\frac{b-\mu}{\sigma}\right)\right]\end{array}\right\}-\frac{3}{7}\left[\lambda(\alpha)\right]^4 \\
+\frac{10}{7}\left[\lambda(\alpha)\right]^2+\frac{\left[\lambda(\alpha)\right]^2}{[F(b)-F(a)]}\left\{\left[\begin{array}{l}\frac{20}{21}\left(\frac{a-\mu}{\sigma}\right)+ \\ \frac{2}{7}\left(\frac{a-\mu}{\sigma}\right)^3\end{array}\right]\varphi\left(\frac{a-\mu}{\sigma}\right)-\left[\begin{array}{l}\frac{20}{21}\left(\frac{b-\mu}{\sigma}\right)+ \\ \frac{2}{7}\left(\frac{b-\mu}{\sigma}\right)^3\end{array}\right]\varphi\left(\frac{b-\mu}{\sigma}\right)\right\} \\
-\frac{\lambda(\alpha)}{[F(b)-F(a)]}\left[\begin{array}{l}\left(\begin{array}{l}\frac{16}{7}+\frac{8}{7}\left(\frac{a-\mu}{\sigma}\right)^2+ \\ \frac{4}{21}\left(\frac{a-\mu}{\sigma}\right)^4\end{array}\right)\varphi\left(\frac{a-\mu}{\sigma}\right)- \\ \left(\begin{array}{l}\frac{16}{7}+\frac{8}{7}\left(\frac{b-\mu}{\sigma}\right)^2+ \\ \frac{4}{21}\left(\frac{b-\mu}{\sigma}\right)^4\end{array}\right)\varphi\left(\frac{b-\mu}{\sigma}\right)\end{array}\right]
\end{array}\right\}$$

Where $$\lambda(\alpha)=\left\{\frac{\left(\frac{4}{3}+\frac{1}{3}\left(\frac{a-\mu}{\sigma}\right)^2\right)\phi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{4}{3}+\frac{1}{3}\left(\frac{b-\mu}{\sigma}\right)^2\right)\phi\left(\frac{b-\mu}{\sigma}\right)}{F(b)-F(a)}\right\} \tag{15}$$

Thus, the variance of the distribution is

$$\mu_2=\frac{5\sigma^2}{3}-\sigma^2\left\{\frac{\left(\frac{4}{3}+\frac{1}{3}\left(\frac{a-\mu}{\sigma}\right)^2\right)\varphi\left(\frac{a-\mu}{\sigma}\right)-\left(\frac{4}{3}+\frac{1}{3}\left(\frac{b-\mu}{\sigma}\right)^2\right)\varphi\left(\frac{b-\mu}{\sigma}\right)}{F(b)-F(a)}\right\}^2$$

$$+\frac{\sigma^2}{\left[F(b)-F(a)\right]}\left\{\left[\frac{10}{9}\left(\frac{a-\mu}{\sigma}\right)+\frac{1}{3}\left(\frac{a-\mu}{\sigma}\right)^3\right]\varphi\left(\frac{a-\mu}{\sigma}\right)-\left[\frac{10}{9}\left(\frac{b-\mu}{\sigma}\right)+\frac{1}{3}\left(\frac{b-\mu}{\sigma}\right)^3\right]\varphi\left(\frac{b-\mu}{\sigma}\right)\right\}$$

x. The kurtosis of the distribution is

$$\beta_2=\frac{\mu_4}{\mu_2{}^2}$$

where μ_2 and μ_4 are as given in equations (14) and (15) (16)

3. MAXIMUM LIKELIHOOD METHOD OF ESTMATION FOR MODEL PARAMETERS

Let $y_1, y_{2,\dots,} y_n$ be a sample of size n drawn from a population having a p.d.f of the form given in equation (2), then the likelihood function of the sample is

$$L(y;\beta_0,\beta_1,a,b,\sigma^2)=\prod_{i=1}^{n}\frac{\left[2+\left(\frac{y_i-\beta_0-\beta_1x_i}{\sigma}\right)\right]e^{-\frac{1}{2}\left(\frac{y_i-\beta_0-\beta_1x_i}{\sigma}\right)^2}}{3\sigma\sqrt{2\pi}\left[F(b)-F(a)\right]}I_{[a,b]}(y_i) \tag{17}$$

Where

$$F(b)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{b}e^{-\frac{1}{2}\left(\frac{y_i-\beta_0-\beta_1x_i}{\sigma}\right)^2}dy_i-\left(\frac{b-\beta_0-\beta_1x_i}{3\sigma}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{b-\beta_0-\beta_1x_i}{\sigma}\right)^2}$$
$$=\Phi\left(\frac{b-\beta_0-\beta_1x_i}{\sigma}\right)-\frac{1}{3}\left(\frac{b-\beta_0-\beta_1x_i}{\sigma}\right)\varphi\left(\frac{b-\beta_0-\beta_1x_i}{\sigma}\right)$$

And

$$F(a)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{a}e^{-\frac{1}{2}\left(\frac{y_i-\beta_0-\beta_1x_i}{\sigma}\right)^2}dy_i-\left(\frac{a-\beta_0-\beta_1x_i}{3\sigma}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{a-\beta_0-\beta_1x_i}{\sigma}\right)^2}$$
$$=\Phi\left(\frac{a-\beta_0-\beta_1x_i}{\sigma}\right)-\frac{1}{3}\left(\frac{a-\beta_0-\beta_1x_i}{\sigma}\right)\varphi\left(\frac{a-\beta_0-\beta_1x_i}{\sigma}\right)$$

In standard units of the complete distribution, the truncated points are denoted as

$T_1=\frac{a-\beta_0-\beta_1x_i}{\sigma}$, and $T_2=\frac{b-\beta_0-\beta_1x_i}{\sigma}$

$F(b)$ and $F(a)$ will be simplified as follows

$F(a)=\Phi_1-\frac{T_1}{3}\phi_1$ and $F(b)=\Phi_2-\frac{T_2}{3}\phi_2$

where, $\phi_1=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{a-\beta_0-\beta_1x_i}{\sigma}\right)^2}, \phi_2=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{b-\beta_0-\beta_1x_i}{\sigma}\right)^2}$,

$$\Phi_1 = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{a} e^{-\frac{1}{2}\left(\frac{y_i-\beta_0-\beta_1x_i}{\sigma}\right)^2} dy_i \text{ and } \Phi_2 = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{b} e^{-\frac{1}{2}\left(\frac{y_i-\beta_0-\beta_1x_i}{\sigma}\right)^2} dy_i \quad (18)$$

$\phi(.)$ and $\Phi(.)$ are ordinates and areas of normal distribution respectively.

$$L(y;\beta_0,\beta_1,a,b,\sigma^2) = \frac{1}{\left(3\sigma^3\sqrt{2\pi}\right)^n}\prod_{i=1}^{n}\frac{\left[2\sigma^2+(y_i-\beta_0-\beta_1x_i)^2\right]e^{-\frac{1}{2}\left(\frac{y_i-\beta_0-\beta_1x_i}{\sigma}\right)^2}}{F(b)-F(a)} I_{[a,b]}(y_i) \quad (19)$$

Taking logarithms on both sides of (19), we get

$$l(\beta_0,\beta_1,\sigma^2) = \ln L(y;a,b,\sigma^2) = \frac{-3n}{2}\ln(\sigma^2) - \sum_{i=1}^{n}\ln\left[\frac{1}{\sigma\sqrt{2\pi}}\int_a^b e^{-\frac{1}{2}\left(\frac{y_i-\beta_0-\beta_1x_i}{\sigma}\right)^2}dy_i + \left(\frac{a-\beta_0-\beta_1x_i}{3\sigma}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{a-\beta_0-\beta_1x_i}{\sigma}\right)^2} \left(\frac{b-\beta_0-\beta_1x_i}{3\sigma}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{b-\beta_0-\beta_1x_i}{\sigma}\right)^2}\right]$$

$$+\sum_{i=1}^{n}\ln\left[2\sigma^2+(y_i-\beta_0-\beta_1x_i)^2\right] - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1x_i)^2 \quad (20)$$

To get the Maximum likelihood estimates of the parameters we differentiate equation (20) with respect to β_0, β_1 and σ^2 and equate them to 0 gives:

Therefore,

$$\frac{\partial l(\beta_0,\beta_1,\sigma^2)}{\partial\beta_0} = 0 \text{ this implies}$$

$$-\sum_{i=1}^{n}\frac{\left[(2+T_1^2)\phi_1-(2+T_2^2)\phi_2\right]}{3\sigma[F(b)-F(a)]} - 2\sum_{i=1}^{n}\frac{y_i-\beta_0-\beta_1x_1}{2\sigma^2+(y_i-\beta_0-\beta_1x_i)^2} + \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1x_i) = 0$$

$$-\sum_{i=1}^{n}\frac{\left[\left(2+\left(\frac{a-\beta_0-\beta_1x_i}{\sigma}\right)^2\right)e^{-\frac{1}{2}\left(\frac{a-\beta_0-\beta_1x_i}{\sigma}\right)^2} - \left(2+\left(\frac{b-\beta_0-\beta_1x_i}{\sigma}\right)^2\right)e^{-\frac{1}{2}\left(\frac{b-\beta_0-\beta_1x_i}{\sigma}\right)^2}\right]}{3\sigma[F(b)-F(a)]}$$

$$-2\sum_{i=1}^{n}\frac{y_i-\beta_0-\beta_1x_1}{2\sigma^2+(y_i-\beta_0-\beta_1x_i)^2} + \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1x_i) = 0$$

where

$$F(b)-F(a) = \frac{1}{\sigma\sqrt{2\pi}}\int_a^b e^{-\frac{1}{2}\left(\frac{y_i-\beta_0-\beta_1x_i}{\sigma}\right)^2}dy_i + \left(\frac{a-\beta_0-\beta_1x_i}{3\sigma}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{a-\beta_0-\beta_1x_i}{\sigma}\right)^2} - \left(\frac{b-\beta_0-\beta_1x_i}{3\sigma}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{b-\beta_0-\beta_1x_i}{\sigma}\right)^2} \quad (21)$$

$$\frac{\partial l(\beta_0,\beta_1,\sigma^2)}{\partial\beta_1} = 0 \text{ this implies}$$

$$-\sum_{i=1}^{n}\frac{x_i\left[(2+T_1^2)\phi_1-(2+T_2^2)\phi_2\right]}{3\sigma[F(b)-F(a)]} - 2\sum_{i=1}^{n}\frac{(y_i-\beta_0-\beta_1x_1)x_i}{2\sigma^2+(y_i-\beta_0-\beta_1x_i)^2} + \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i-\beta_0-\beta_1x_i)x_i = 0$$

$$-\sum_{i=1}^{n}\frac{x_i\left[\left(2+\left(\frac{a-\beta_0-\beta_1x_i}{\sigma}\right)^2\right)e^{-\frac{1}{2}\left(\frac{a-\beta_0-\beta_1x_i}{\sigma}\right)^2}-\left(2+\left(\frac{b-\beta_0-\beta_1x_i}{\sigma}\right)^2\right)e^{-\frac{1}{2}\left(\frac{b-\beta_0-\beta_1x_i}{\sigma}\right)^2}\right]}{3\sigma\left[F(b)-F(a)\right]}$$

$$-2\sum_{i=1}^{n}\frac{(y_i-\beta_0-\beta_1x_i)x_i}{2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2}+\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(y_i-\beta_0-\beta_1x_i\right)x_i=0 \qquad (22)$$

$\frac{\partial l\left(\beta_0,\beta_1,\sigma^2\right)}{\partial\sigma^2}=0$ implies

$$-\frac{3n}{2\sigma^2}-\sum_{i=1}^{n}\frac{\left[\left(2T_1+T_1^{\,3}\right)\phi_1-\left(2T_2+T_2^{\,3}\right)\phi_2\right]}{6\sigma^2\left[F(b)-F(a)\right]}+2\sum_{i=1}^{n}\frac{1}{2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2}+\sum_{i=1}^{n}\frac{\left(y_i-\beta_0-\beta_1x_i\right)^2}{2\sigma^4}=0$$

where T_1, T_2, ϕ_1 and ϕ_2 are given in equation (18). (23)

For likelihood equations we couldn't get a closed from solution, but, numerical methods such as Newton-Raphson (NR) iterative method or Fisher scoring method can be used to get the maximum likelihood estimators (MLEs).The usual or standard procedure for implementing this solution is to use the Newton-Raphson method is employed to implement the solution and it is given by

$$\theta^{(n+1)}=\theta^{(n)}-H^{-1}S \qquad (24)$$

where H is the Hessian (second derivative) matrix and S is the first derivative of the log-likelihood function, both evaluated at the current value of the parameter vector. That is,

$$S=\left[s_j\right]=\left[\frac{\partial L}{\partial\theta_j}\right] \text{ and } H=\left[h_{ij}\right]=\left[\frac{\partial^2\ln L}{\partial\theta_i\partial\theta_j{}'}\right] \text{ where } \theta=\left(\beta_0,\beta_1,\sigma^2\right) \qquad (25)$$

The iteration is to be repeated until the sequence $\left\{\hat{\theta}^{(n+1)}\right\}$ thus obtained converges to the desired degree of accuracy. This technique the prior computation of the Hessian matrix and an initial guess $\theta^{(0)}$ for the model parameters β_0,β_1 and σ^2. To derive the Hessian matrix, we need to have the second-order derivatives of the log-likelihood function:

$$\frac{\partial^2 l(\beta_0,\beta_1,\sigma^2)}{\partial\beta_0^{\,2}}=-\sum_{i=1}^{n}\left[\frac{\left(T_1^3\varphi_1-T_2^3\varphi_2\right)\left(F(b)-F(a)\right)-\frac{1}{3\sigma}\left[\left(2+T_1^{\,2}\right)\varphi_1-\left(2+T_2^{\,2}\right)\varphi_2\right]^2}{3\sigma^2\left[F(b)-F(a)\right]^2}\right]$$

$$-2\sum_{i=1}^{n}\frac{-2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2}{\left[2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2\right]^2}-\frac{n}{\sigma^2} \qquad (26)$$

where T_1, T_2, ϕ_1 and ϕ_2 are given in equation (18).

$$\frac{\partial^2 l(\beta_0,\beta_1,\sigma^2)}{\partial\beta_1\partial\beta_0}=\frac{\partial}{\partial\beta_1}\left[-\sum_{i=1}^{n}\frac{\left[\left(2+T_1^{\,2}\right)\phi_1-\left(2+T_2^{\,2}\right)\phi_2\right]}{3\sigma\left[F(b)-F(a)\right]}-2\sum_{i=1}^{n}\frac{y_i-\beta_0-\beta_1x_1}{2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2}+\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(y_i-\beta_0-\beta_1x_i\right)\right]$$

Where, $T_1^{\,2}=\left(\frac{a-\beta_0-\beta_1x_i}{\sigma}\right)^2$, $T_2^{\,2}=\left(\frac{b-\beta_0-\beta_1x_i}{\sigma}\right)^2$

$$\frac{\partial T_1^{\,2}}{\partial\beta_1}=\frac{-2x_iT_1}{\sigma},\ \frac{\partial T_2^{\,2}}{\partial\beta_1}=\frac{-2x_iT_2}{\sigma},\ \frac{\partial\phi_1}{\partial\beta_1}=\frac{x_iT_1\phi_1}{\sigma}, \text{ and } \frac{\partial\phi_2}{\partial\beta_1}=\frac{x_iT_2\phi_2}{\sigma}$$

$$\frac{\partial}{\partial\beta_1}\left[\left(2+T_1^{\,2}\right)\phi_1\right]=\frac{x_iT_1^{\,3}\phi_1}{\sigma},\ \frac{\partial}{\partial\beta_1}\left[\left(2+T_2^{\,2}\right)\phi_2\right]=\frac{x_iT_2^{\,3}\phi_2}{\sigma},$$

$$\frac{\partial}{\partial\beta_1}\left[F(b)-F(a)\right]=\frac{x_i}{3\sigma}\left[\left(2+T_1^{\,2}\right)\phi_1-\left(2+T_2^{\,2}\right)\phi_2\right]$$

Therefore,

$$\frac{\partial^2 l(\beta_0,\beta_1,\sigma^2)}{\partial\beta_1\partial\beta_0}=-\sum_{i=1}^{n}\left[\frac{\left(T_1^3\varphi_1-T_2^3\varphi_2\right)\left(F(b)-F(a)\right)-\frac{1}{3}\left[\left(2+T_1^2\right)\varphi_1-\left(2+T_2^{\,2}\right)\varphi_2\right]^2}{3\sigma^2\left[F(b)-F(a)\right]^2}\right]x_i$$

$$-2\sum_{i=1}^{n}\frac{\left[-2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2\right]x_i}{\left[2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2\right]^2}-\frac{1}{\sigma^2}\sum_{i=1}^{n}x_i \tag{27}$$

$$\frac{\partial^2 l(\beta_0,\beta_1,\sigma^2)}{\partial\sigma^2\partial\beta_0}=\frac{\partial}{\partial\sigma^2}\left[-\sum_{i=1}^{n}\frac{\left[\left(2+T_1^{\,2}\right)\phi_1-\left(2+T_2^{\,2}\right)\phi_2\right]}{3\sigma\left[F(b)-F(a)\right]}-2\sum_{i=1}^{n}\frac{y_i-\beta_0-\beta_1x_1}{2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2}+\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(y_i-\beta_0-\beta_1x_i\right)\right]$$

$$=-\sum_{i=1}^{n}\frac{1}{6\sigma^3\left[F(b)-F(a)\right]^2}\left\{\begin{array}{l}\left[\left(-2-T_1^{\,2}+T_1^{\,4}\right)\varphi_1-\left(-2-T_2^{\,2}+T_2^{\,4}\right)\varphi_2\right]\\ \left[F(b)-F(a)\right]-\\ \frac{1}{3}\left[\left(2+T_1^{\,2}\right)\varphi_1-\left(2+T_2^{\,2}\right)\varphi_2\right]\\ \left[\left(2T_1+T_1^{\,3}\right)\varphi_1-\left(2T_2+T_2^{\,3}\right)\varphi_2\right]\end{array}\right\} \tag{28}$$

$$4\sum_{i=1}^{n}\frac{y_i-\beta_0-\beta_1x_i}{\left[2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2\right]^2}-\frac{1}{\sigma^4}\sum_{i=1}^{n}\left(y_i-\beta_0-\beta_1x_i\right)$$

$$\frac{\partial^2 l(\beta_0,\beta_1,\sigma^2)}{\partial\beta_0\partial\beta_1}=\frac{\partial}{\partial\beta_0}\left[-\sum_{i=1}^{n}\frac{x_i\left[\left(2+T_1^{\,2}\right)\phi_1-\left(2+T_2^{\,2}\right)\phi_2\right]}{3\sigma\left[F(b)-F(a)\right]}-2\sum_{i=1}^{n}\frac{(y_i-\beta_0-\beta_1x_1)x_i}{2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2}+\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(y_i-\beta_0-\beta_1x_i\right)x_i\right]$$

$$=-\sum_{i=1}^{n}\left[\frac{\left(T_1^3\varphi_1-T_2^3\varphi_2\right)\left(F(b)-F(a)\right)-\frac{1}{3}\left[\left(2+T_1^2\right)\varphi_1-\left(2+T_2^{\,2}\right)\varphi_2\right]^2}{3\sigma^2\left[F(b)-F(a)\right]^2}\right]x_i$$

$$-2\sum_{i=1}^{n}\frac{\left[-2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2\right]x_i}{\left[2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2\right]^2}-\frac{1}{\sigma^2}\sum_{i=1}^{n}x_i \tag{29}$$

$$\frac{\partial^2 l\left(\beta_0,\beta_1,\sigma^2\right)}{\partial\beta_1^{\,2}}=\frac{\partial}{\partial\beta_1}\left[-\sum_{i=1}^{n}\frac{x_i\left[\left(2+T_1^{\,2}\right)\phi_1-\left(2+T_2^{\,2}\right)\phi_2\right]}{3\sigma\left[F(b)-F(a)\right]}-2\sum_{i=1}^{n}\frac{(y_i-\beta_0-\beta_1x_1)x_i}{2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2}+\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(y_i-\beta_0-\beta_1x_i\right)x_i\right]$$

$$=-\sum_{i=1}^{n}\left[\frac{\left(T_1^3\varphi_1-T_2^3\varphi_2\right)\left(F(b)-F(a)\right)-\frac{1}{3}\left[\left(2+T_1^2\right)\varphi_1-\left(2+T_2^{\,2}\right)\varphi_2\right]^2}{3\sigma^2\left[F(b)-F(a)\right]^2}\right]x_i^{\,2}$$

$$-2\sum_{i=1}^{n}\frac{\left[-2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2\right]x_i^2}{\left[2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2\right]^2}-\frac{1}{\sigma^2}\sum_{i=1}^{n}x_i^2 \tag{30}$$

$$\frac{\partial^2 l\left(\beta_0,\beta_1,\sigma^2\right)}{\partial\sigma^2\partial\beta_1}=\frac{\partial}{\partial\sigma^2}\left[\begin{array}{l}-\sum_{i=1}^{n}\frac{x_i\left[\left(2+T_1^2\right)\phi_1-\left(2+T_2^2\right)\phi_2\right]}{3\sigma\left[F(b)-F(a)\right]}-\\ 2\sum_{i=1}^{n}\frac{(y_i-\beta_0-\beta_1x_1)x_i}{2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2}+\frac{1}{\sigma^2}\sum_{i=1}^{n}\left(y_i-\beta_0-\beta_1x_i\right)x_i\end{array}\right]$$

$$=-\sum_{i=1}^{n}\frac{1}{6\sigma^3\left[F(b)-F(a)\right]^2}\left\{\begin{array}{l}\left[\left(-2-T_1^2+T_1^4\right)\varphi_1-\left(-2-T_2^2+T_2^4\right)\varphi_2\right]\\ \left[F(b)-F(a)\right]-\\ \frac{1}{3}\left[\left(2+T_1^2\right)\varphi_1-\left(2+T_2^2\right)\varphi_2\right]\\ \left[\left(2T_1+T_1^3\right)\varphi_1-\left(2T_2+T_2^3\right)\varphi_2\right]\end{array}\right\}x_i \tag{31}$$

$$+4\sum_{i=1}^{n}\frac{\left(y_i-\beta_0-\beta_1x_i\right)x_i}{\left[2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2\right]^2}-\frac{1}{\sigma^4}\sum_{i=1}^{n}\left(y_i-\beta_0-\beta_1x_i\right)x_i$$

$$\frac{\partial^2 l\left(\beta_0,\beta_1,\sigma^2\right)}{\partial\beta_0\partial\sigma^2}=\frac{\partial}{\partial\beta_0}\left[\begin{array}{l}-\frac{3n}{2\sigma^2}-\sum_{i=1}^{n}\frac{\left[\left(2T_1+T_1^3\right)\phi_1-\left(2T_2+T_2^3\right)\phi_2\right]}{6\sigma^2\left[F(b)-F(a)\right]}+\\ 2\sum_{i=1}^{n}\frac{1}{2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2}+\sum_{i=1}^{n}\frac{\left(y_i-\beta_0-\beta_1x_i\right)^2}{2\sigma^4}\end{array}\right]$$

$$=-\sum_{i=1}^{n}\frac{1}{6\sigma^3\left[F(b)-F(a)\right]^2}\left\{\begin{array}{l}\left[\left(-2-T_1^2+T_1^4\right)\phi_1-\left(-2-T_2^2+T_2^4\right)\phi_2\right]\\ \left[F(b)-F(a)\right]-\\ \frac{1}{3}\left[\left(2+T_1^2\right)\phi_1-\left(2+T_2^2\right)\phi_2\right]\\ \left[\left(2T_1+T_1^3\right)\phi_1-\left(2T_2+T_2^3\right)\phi_2\right]\end{array}\right\}$$

$$+4\sum_{i=1}^{n}\frac{y_i-\beta_0-\beta_1x_i}{\left[2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2\right]^2}-\frac{1}{\sigma^4}\sum_{i=1}^{n}\left(y_i-\beta_0-\beta_1x_i\right) \tag{32}$$

$$\frac{\partial^2 l\left(\beta_0,\beta_1,\sigma^2\right)}{\partial\beta_1\partial\sigma^2}=\frac{\partial}{\partial\beta_1}\left[\begin{array}{l}-\frac{3n}{2\sigma^2}-\sum_{i=1}^{n}\frac{\left[\left(2T_1+T_1^3\right)\phi_1-\left(2T_2+T_2^3\right)\phi_2\right]}{6\sigma^2\left[F(b)-F(a)\right]}+\\ 2\sum_{i=1}^{n}\frac{1}{2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2}+\sum_{i=1}^{n}\frac{\left(y_i-\beta_0-\beta_1x_i\right)^2}{2\sigma^4}\end{array}\right]$$

$$=-\sum_{i=1}^{n}\frac{1}{6\sigma^3\left[F(b)-F(a)\right]^2}\left\{\begin{array}{l}\left[\left(-2-T_1^2+T_1^4\right)\phi_1-\left(-2-T_2^2+T_2^4\right)\phi_2\right]\\ \left[F(b)-F(a)\right]-\\ \frac{1}{3}\left[\left(2+T_1^2\right)\phi_1-\left(2+T_2^2\right)\phi_2\right]\\ \left[\left(2T_1+T_1^3\right)\phi_1-\left(2T_2+T_2^3\right)\phi_2\right]\end{array}\right\}x_i \tag{33}$$

$$+4\sum_{i=1}^{n}\frac{\left(y_i-\beta_0-\beta_1x_i\right)x_i}{\left[2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2\right]^2}-\frac{1}{\sigma^4}\sum_{i=1}^{n}\left(y_i-\beta_0-\beta_1x_i\right)x_i$$

$$\frac{\partial^2 l\left(\beta_0,\beta_1,\sigma^2\right)}{\left(\partial\sigma^2\right)^2}=\frac{3n}{2\sigma^4}-$$

$$\sum_{i=1}^{n}\frac{\begin{bmatrix}\frac{1}{2}\left[\left(-2T_1-T_1^3+T_1^5\right)\varphi_1-\left(-2T_2-T_2^3+T_2^5\right)\varphi_2\right]\left[F(b)-F(a)\right]-\\ \left[F(b)-F(a)+\frac{1}{6}\left[\left(2T_1+T_1^3\right)\varphi_1-\left(2T_2+T_2^3\right)\varphi_2\right]\right]\left[\left(2T_1+T_1^3\right)\varphi_1-\left(2T_2+T_2^3\right)\varphi_2\right]\end{bmatrix}}{6\sigma^4\left[F(b)-F(a)\right]^2}$$

$$-4\sum_{i=1}^{n}\frac{1}{\left[2\sigma^2+\left(y_i-\beta_0-\beta_1x_i\right)^2\right]^2}-\frac{1}{\sigma^6}\sum_{i=1}^{n}\left(y_i-\beta_0-\beta_1x_i\right)^2 \tag{34}$$

Thus, the Hessian matrix is the following:

$$H=\begin{pmatrix}\frac{\partial^2\ell}{\partial\beta_0^2} & \frac{\partial^2\ell}{\partial\beta_1\partial\beta_0} & \frac{\partial^2\ell}{\partial\sigma^2\partial\beta_0}\\ \frac{\partial^2\ell}{\partial\beta_0\partial\beta_1} & \frac{\partial^2\ell}{\partial\beta_1^2} & \frac{\partial^2\ell}{\partial\sigma^2\partial\beta_1}\\ \frac{\partial^2\ell}{\partial\beta_0\partial\sigma^2} & \frac{\partial^2\ell}{\partial\beta_1\partial\sigma^2} & \frac{\partial^2\ell}{\left(\partial\sigma^2\right)^2}\end{pmatrix} \tag{35}$$

4. SIMULATION AND RESULTS

In this section we conduct simulation studies to investigate the properties of Maximum likelihood estimators of the simple linear model with the proposed Truncated New Symmetric distribution errors having probability density function given in equation (2). The probability integral transform property (PIT) is a universally applicable way of generating data set with a given distribution.

The CDF given equation (4) is used to generate random numbers Y from the two parameter doubly truncated new symmetric distribution. The data sets were generated from the model using Wolfram Mathematica 10.4. The method is used to generate a data from a Uniform distribution. If d is distributed uniformly on $(0,1)$, then $F^{-1}(d)$ will have the two parameter doubly truncated new symmetric distribution. Accordingly, random numbers **y** is generated from the DTNS distribution, and the corresponding algorithmic representation is easily obtained by:

Step-1: Given $\mu=2,\sigma=1,a=1,b=4$, generate $d_i\sim U(0,1)$, $i=1,2,\ldots,n$,

Step-2: Solve $\frac{1}{\sigma\sqrt{2\pi}[F(b)-F(a)]}\int_a^y e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}dt-\frac{(y-\mu)e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}}{3\sigma\sqrt{2\pi}[F(b)-F(a)]}=d_i$. The solution for a random variable $Y_i, i=1,2,\ldots,n$, follows the standard doubly truncated new symmetric distribution.

Step-3: Generating X_i from a uniform distribution $U(0,1)$ and used it as explanatory variable x_i, $i=1,2,\ldots,n$, for the linear model. For illustrative purposes, we can get a set of simulated data (x_i,y_i) for sample sizes of $n=100,500,1000,3000,5000,10000.$

Table-1: Summary of simulations for the ML estimation of the regression model

Sample size (n)	Parameter	Estimate	Standard error	Wald 95% Confidence Limits		Chi-Square	Pr>Chi-Square	Log-Likelihood
				Lower	Upper			
100	β_0	0.7293	0.0039	0.7216	0.7370	34324.9	<.0001	249.9619
	β_1	2.6230	0.0069	2.6095	2.6366	143971	<.0001	
	σ	0.0199	0.0014	0.0173	0.0228			
500	β_0	0.7267	0.0017	0.7233	0.7300	178690	<.0001	1262.9126
	β_1	2.6231	0.0030	2.6172	2.6290	755229	<.0001	
	σ	0.0194	0.0006	0.0182	0.0206			
1000	β_0	0.7260	0.0012	0.7236	0.7284	352561	<.0001	2529.1812
	β_1	2.6238	0.0022	2.6196	2.6281	1472578	<.0001	
	σ	0.0193	0.0004	0.0185	0.0202			
3000	β_0	0.7264	0.0007	0.7250	.0.7278	1031271	<.0001	7598.5318
	β_1	2.6213	0.0012	2.6189	2.6237	4556373	<.0001	
	σ	0.0192	0.0002	0.0187	0.0197			
5000	β_0	0.7241	0.0006	0.7230	0.7252	1722832	<.0001	12600.0290
	β_1	2.6269	0.0010	2.6250	2.6287	7632647	<.0001	
	σ	0.0195	0.0002	0.0191	0.0199			
10000	β_0	0.7257	0.0004	0.7250	0.7265	3590270	<.0001	25348.1979
	β_1	2.6238	0.0007	2.6225	2.6251	1.567E7	<.0001	
	σ	0.0192	0.0001	0.0189	0.0195			

The value of $\hat{\beta}$ for a sample size of 10,000 that maximizes the likelihood function is thus,

$$\hat{\beta} = \left(\hat{\beta}_0, \hat{\beta}_1, \hat{\sigma}^2\right) = \left(0.7257, 2.6238, 0.0192\right)' \tag{36}$$

The Table 1 reveals that as the sample size increases, the precision of estimates increases and the value of the log likelihood increases accordingly. It can also be verified that the Hessian matrix (34) evaluated at $\hat{\beta}_0 = 0.7257$; $\hat{\beta}_1 = 2.6238$; and $\hat{\sigma}^2 = 0.0192$, is a negative definite matrix. The ML method provides good estimates of the underlying model to obtain the regression coefficients. The fitted simple linear model with doubly truncated new symmetric errors for a sample size of 10,000, is:

$$\hat{Y} = 0.7257 + 2.6238X \tag{37}$$

The estimated covariance matrix is:

	β_0	β_1	σ
β_0	1.4669E-7	-2.198E-7	-5.88E-21
β_1	-2.198E-7	4.3946E-7	1.097E-20
σ	-5.88E-21	1.097E-20	1.8398E-8

(38)

Standard errors of the estimates were obtained by the square root of the diagonal elements of the inverse of the Hessian of the log-likelihood function. Thus, the estimate standard errors are

$$s.e.(\hat{\beta}_0) = 0.0004 \quad \text{and} \quad s.e.(\hat{\beta}_1) = 0.0007 \tag{39}$$

Table-2: ML estimation output for the simulation data.

Criterion	DF	Value	Value/DF
Deviance	9998	3.6797	0.0004
Scaled Deviance	9998	10000	1.0002
Pearson Chi-Square	9998	3.6797	0.0004
Scaled Pearson X^2	9998	10000	1.0002
Log Likelihood		25348.1979	
Algorithm converged			

Parameter	DF	Estimate	Standard Error	Wald 95% Confidence Limits		Chi-Square	Pr> Chi-Sq
β_0	1	0.7257	0.0004	0.7250	0.7265	3590270	<.0001
β	1	2.6238	0.0007	2.6225	2.6251	1.567E7	<.0001
σ	1	0.0192	0.0001	0.0189	0.0195		

Here the scale parameter is estimated using maximum likelihood method.

Scaled deviance and Pearson's chi-square statistic are helpful in evaluating the goodness of fit of a given generalized linear model. The scaled deviance is defined to be twice the difference between the maximum achievable log likelihood and the log likelihood at the ML estimates of the regression parameters. Table 2shows that the value of the deviance divided by its degree of freedom is less than one. Scaled deviance is approximately equal to its degrees of freedom is a possible indication of a good model fit.

5. LEAST SQUARE METHOD OF ESTIMATION FOR MODEL PARAMETERS

The ordinary least square (OLS) is the most commonly used method for estimating the unknown regression coefficients is a standard linear regression model. When the errors are normally distributed ordinary least squares (OLS) is an important procedure for solving regression problems. Even though small departures from normality of the error terms do not affect the regression coefficients greatly, errors with a heavier or lighter tailed distribution can result in extreme observations and can significantly affect the estimated OLS regression coefficients. The OLS estimates of β_0 and β_1 in the linear model are the values which minimize

$$SS = \sum_{i=1}^{n}\left(y_i - \beta_0 - \beta_1 x_i\right)^2 \tag{40}$$

And provide the OLS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ as

$$\hat{\beta}_1 = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{n}(x_i - \overline{x})^2} = \frac{S_{xy}}{S_{xx}}$$

And

$$\hat{\beta}_0 = \overline{y} - \hat{\beta}_1 \overline{x} \tag{41}$$

where $\overline{x} = \frac{\sum_{i=1}^{n} x_i}{n}$ and $\overline{y} = \frac{\sum_{i=1}^{n} y_i}{n}$.

The values $\hat{\beta}_0$ and $\hat{\beta}_1$ are called the least squares estimates of β_0 and β_1, respectively.

Table-3: OLS estimation output for the simulation data Nonlinear OLS summary of residual errors

Equation	DF model	DF error	SSE	MSE	Root MSE	R-square	Adjusted R-square
y	2	9998	3.6797	0.00368	0.0192	0.9994	0.9994

DF degrees of freedom, SSE sum of the squared errors.

Parameter	Estimate	Approximate standard error	t value	Approx Pr> |t|
β_0	0.725707	0.000383	1894.61	<.0001
β_1	2.623794	0.00663	3957.54	<.0001

7. COMPARISON OF THE REGRESSION MODEL WITH DTNS DISTRIBUTED ERRORS WITH THE REGRESSION MODELS HAVING NS DISTRIBUTED ERRORS AND GAUSSIAN DISTRIBUTED ERRORS

The simulated data are used to compare the performance of the linear model with doubly truncated new symmetric error terms with that of the linear model with new symmetric and normal error terms. AIC and BIC with model diagnostics root mean square error (RMSE) are obtained using linear regression models with normal, new symmetric and doubly truncated new symmetric distributed error terms as shown in Table 4. Simulated data using various sample sizes are used to compare the performance of AIC and BIC with model diagnostics root mean square error (RMSE).

Table-4: Summary for information criteria and model diagnostics for normal, new symmetric and doubly truncated new symmetric error model.

Sample Size	Normal			New symmetric			DTNSD		
	AIC	BIC	RMSE	AIC	BIC	RMSE	AIC	BIC	RMSE
100	-144.7349	-142.6541	0.48019	-299.9583	-297.8775	0.22098	-779.7115	-777.6307	0.02007
500	-741.3906	-739.3746	0.47550	-1646.5457	-1644.529	0.19233	-3940.7638	-3938.7478	0.01939
1000	-1574.2537	-1572.2457	0.45470	-3356.1916	-3354.1836	0.18654	-7892.2394	-7890.2314	0.01931
3000	-4881.6706	-4879.6680	0.44311	-9500.2233	-9498.2207	0.20521	-23706.695	-23704.692	0.01923
5000	-8103.8095	-8101.8079	0.44460	-16214.227	-16212.225	0.19758	-39385.443	-39383.442	0.01947
10000	-16166.232	-16164.231	0.44557	-32635.715	-32633.714	0.19556	-79071.167	-79069.166	0.01918

The model with the smallest AIC or BIC among all competing models is deemed the best model where it can be seen that the DTNS distribution provides the better fit to the data. It is observed that the information criteria and model diagnostics for linear model with doubly truncated new symmetrically distributed error terms consistently performed better for all of the sample sizes. The Figures 2, 3, and 4 shows the comparison of three models with respect to AIC, BIC, and RMSE using different sample sizes, respectively.

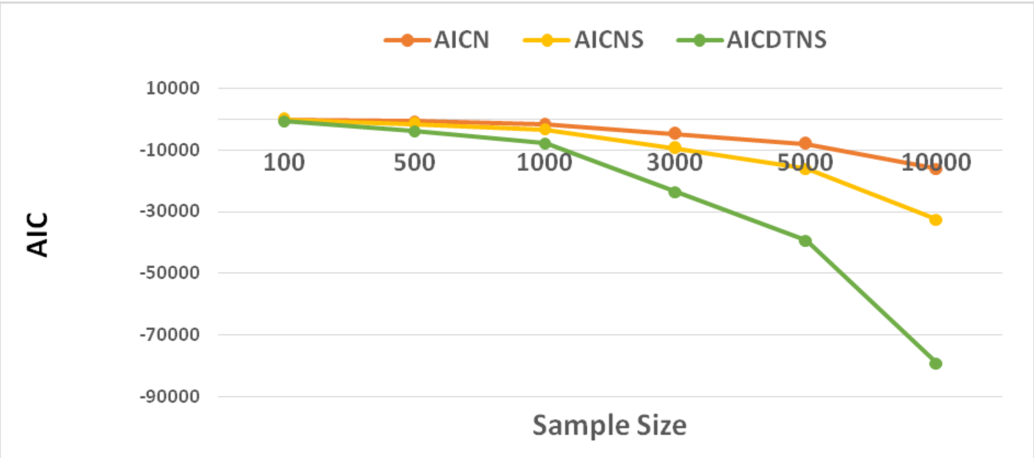

Figure-2: Comparison of DTNS Linear Model versus NS and N Linear Model using AIC

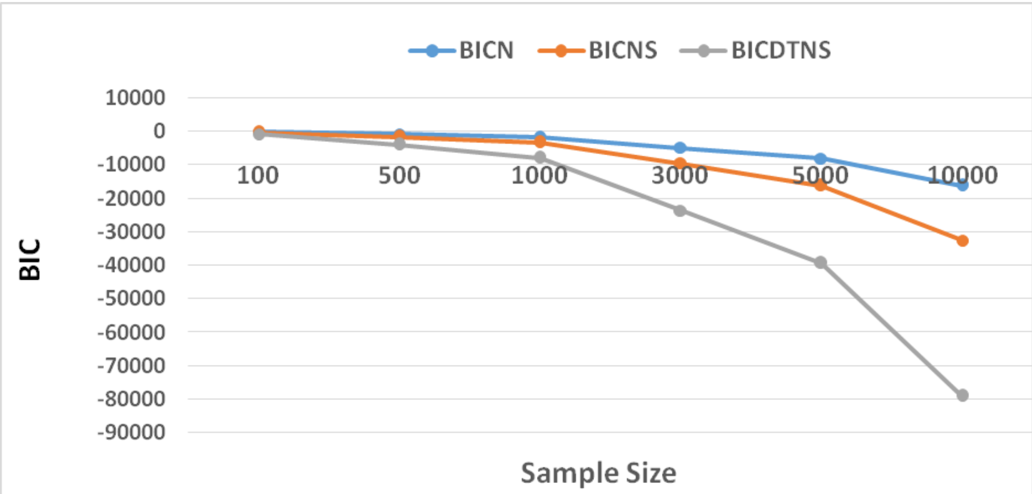

Figure-3: Comparison of DTNS Linear Model versus NS and N Linear Model using BIC

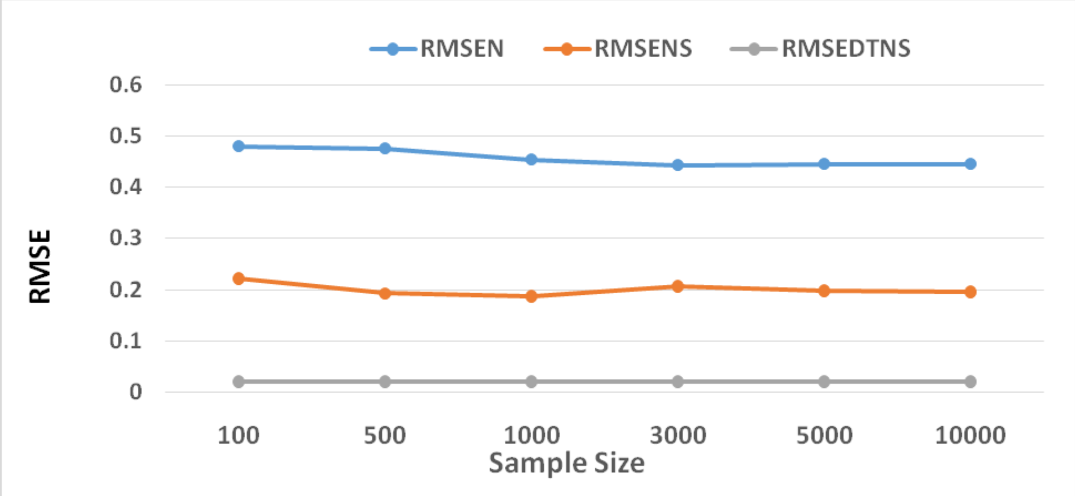

Figure-4: Comparison of DTNS Linear Model versus NS and N Linear Model using RMSE

From Figures 2, 3, and 4 it is observed that the linear model for DTNS gives better results.

8. CONCLUSIONS

This paper addresses the regression analysis with doubly truncated new symmetric distributed errors. This model is a deviation from the classical regression model. This regression model is useful when the response variable or the error term follows a platykurtic distribution having constrained tails. The truncation of the error term distribution has significant influence on the model parameters. The Maximum likelihood estimators of the parameters are obtained using Newton–Raphson (NR) iterative numerical method. A simulation study is carried to obtain the properties of the model parameters as well as its performance. The simulation results revealed that the MLE estimators are superior than OLS estimators for the model. A comparative study of the proposed model with that of new symmetric distributed errors and Gaussian errors revealed that the former gives a better fit than the other two models with respect to AIC, BIC, and RMSE. This regression model can also extended to multivariate regression model with doubly truncated new symmetric distributed errors which will be taken elsewhere.

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