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THE FORCING STEINER DOMINATION NUMBER OF A GRAPH

K. RAMALAKSHMI*[1], K. PALANI[2]

[1]Assistant Professor, Department of Mathematics,
Sri Sarada College for Women, Tirunelveli 627 011, Tamil Nadu, India.

[2]Assistant Professor, Department of Mathematics,
A. P. C. Mahalakshmi College, Tuticorin 628 002, Tamil Nadu, India.

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ABSTRACT

In this paper, the forcing steiner domination number of a graph is introduced. Also, this number is found for some standard graphs.

Keywords: *Domination, Steiner number, Steiner domination number and Forcing Steiner domination number.*

1. INTRODUCTION

The concept of domination in graphs was introduced by Ore and Berge [4]. Throughout this paper $G= (V, E)$ denotes a finite undirected simple graph with vertex set V and edge set E. A subset D of $V(G)$ is a dominating set of G if every vertex in $V − D$ is adjacent to at least one vertex in D. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. The concept of Steiner number of a graph was introduced by G.Chatrand and P. Zhang [1]. For a nonempty set W of vertices in a connected graph G, the Steiner distance $d(W)$ of W is the minimum size of a connected sub graph of G containing W. Necessarily each such subgraph is a tree and is called a Steiner tree with respect to W or a Steiner W-tree. The set of all vertices of G that lie in some Steiner W-tree is denoted by $S(W)$. If $S(W) = V$, then W is called a Steiner set for G. A Steiner set with minimum cardinality is the Steiner number of G and is denoted by $s(G)$.

The concept of Steiner domination number of a graph was introduced by J. John *et al.*, [3]. For a connected graph G, a set of vertices W in G is called a Steiner dominating set if W is both a Steiner set and a dominating set. The minimum cardinality of a Steiner dominating set of G is its Steiner domination number and is denoted by $\gamma_s(G)$. A steiner dominating set of cardinality $\gamma_s(G)$ is said to be a γ_s-set.

The concept of Forcing (G, D)-number was introduced by K.Palani and A.Nagarajan [5]. For a connected graph G, let S be a γ_G-set of G. A subset T of S is called a forcing subset for S if S is the unique γ_G-set of G containing T. A forcing subset T of S with minimum cardinality is called a minimum forcing subset for S. The forcing (G,D)-number of S, denoted by $f_{G,D}(S)$, is the cardinality of a minimum forcing subset of S. The forcing (G, D)-number of G is the minimum of $f_{G,D}(S)$, where the minimum is taken over all γ_G-sets S of G and it is denoted by $f_{G,D}(G)$. That is, $f_{G,D}(G) = min\{ f_{G,D}(S) : S \text{ is any } \gamma_G\text{-set of } G\}$.

Theorem 1.1 [3]: *For the complete bipartite graph* $G = K_{m,n}$,

$$s(G) = \gamma_s(G) = \begin{cases} 2 & if\ m = n = 1 \\ n & if\ n \geq 2, m = 1 \\ \min\{m,n\} & if\ m,n \geq 2 \end{cases}$$

Corresponding Author: K. Ramalakshmi*[1]
[1]Assistant Professor, Department of Mathematics,
Sri Sarada College for Women, Tirunelveli 627 011, Tamil Nadu, India.

Theorem 1.2 [8]: *For a Wheel graph* $W_{1,n}$, $n \geq 5$, $\gamma_s(W_{1,n}) = n-2$.

Theorem 1.3 [7]: *For the Wheel* $W_p = K_1 + C_{p-1} (p \geq 5)$, $s(W_p) = p-3$ *and* $f_s(W_p) = p-4$.

Theorem 1.4 [7]: *For the complete bipartite graph* $G = K_{m,n}$ $(m,n \geq 2)$, $f_s(G) = \begin{cases} 0 & if\ m \neq n \\ 1 & if\ m = n \end{cases}$

Theorem 1.5 [3]: *Each extreme vertex of a connected graph G belongs to every minimum Steiner dominating set of G.*

Theorem 1.6 [3]: *For the complete graph* $K_p (p \geq 2), \gamma_s(K_p) = p$.

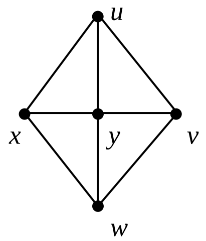
Theorem 1.7 [6]: $\gamma_s(P_n) = \begin{cases} \left\lceil \frac{n-4}{3} \right\rceil + 2 & if\ n \geq 5; \\ 2 & if\ n = 2,3\ or\ 4. \end{cases}$

Theorem 1.8.[6]: *For n > 5,* $\gamma_s(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

2. FORCING STEINER DOMINATION NUMBER

Definition 2.1: Let G be a connected graph and *W* be a minimum steiner dominating set of *G*. A subset *T* of *W* is called a forcing subset for *W* if *W* is the unique minimum steiner dominating set of *G* containing *T*. A forcing subset *T* of *W* with minimum cardinality is called a minimum forcing subset for *W*. The forcing steiner domination number of *W*, denoted by $f\gamma_s(W)$, is the cardinality of a minimum forcing subset of *W*. The forcing steiner domination number of *G* is the minimum of $f\gamma_s(W)$ where the minimum is taken over all steiner dominating sets *W* of *G* and it is denoted by $f\gamma_s(G)$. That is, $f\gamma_s(G)$ = min{ $f\gamma_s(W)$: *W* is any steiner dominating set of *G*}.

Example 2.2: Consider the graph *G* in Figure 2.1. W_1= {*u*, *w*} and W_2 = {*x, v*} are the only two minimum steiner dominating sets of *G*. Forcing subsets of W_1 are {*u*}, {*w*} and {*u, w*}. Therefore, $f\gamma_s(W_1)$ = 1. Similarly, Forcing subsets of W_2 are {*x*}, {*v*} and {*x, v*}. Therefore, $f\gamma_s(W_2)$ = 1. Hence, $f\gamma_s(G)$ = min{1, 1} = 1.



***G*: Figure-2.1**

Remark 2.3: For a connected graph *G*, 0 ≤ $f\gamma_s(G) \leq s(G)$. Here, the lower bound is sharp, since for any complete graph *W* = *V* (*G*) is the unique steiner dominating set. Therefore, *T* = ϕ is a forcing subset for *W* and $f\gamma_s(K_p)$ = 0.

Observation 2.4: Let *G* be a connected graph. Then,

i. From the definition of forcing steiner dominating set, $f\gamma_s(G)$ = 0 if and only if *G* has a unique steiner dominating set.
ii. $f\gamma_s(G)$ =1 if and only if *G* has atleast two steiner dominating sets, one of which has forcing number equal to 1.
iii. $f\gamma_s(G) = \gamma_s(G)$ if and only if every steiner dominating set *W* of *G* has the property, $f\gamma_s(W) = |W| = \gamma_s(G)$.
iv. For a connected graph G, if every minimum steiner set *W* itself is a minimum steiner dominating set then, $f_s(G) = f\gamma_s(G)$.

Theorem 2.5: $f\gamma_s(P_n) = \begin{cases} 0 & if\ n = 2,3\ and\ n \equiv 1(\text{mod}\,3) \\ 1 & otherwise \end{cases}$

Proof: Let $P_n = (v_1, v_2, \ldots, v_n)$.

Case-(i): $n \equiv 0(\text{mod } 3)$

When $n = 3$, there is a unique steiner dominating set and by observation 2.4(i), $f\gamma_s(P_3) = 0$.

If n > 3, P_n has more than one steiner dominating set. Further, for every n, there exists only one steiner dominating set containing v_3. Similar is the case with v_{n-2} also.

Therefore, $f\gamma_s(P_n) = 1$.

Case-(ii): $n \equiv 1(\text{mod } 3)$

In this case, P_n has a unique steiner dominating set.

Therefore, $f\gamma_s(P_3) = 0$.

Case-(iii): $n \equiv 2(\text{mod } 3)$

When $n = 2$, there is a unique steiner dominating set and therefore, $f\gamma_s(P_2) = 0$.

When n > 2, As in case 1, P_n has more than one steiner dominating set. Further, for every n, there exists only one steiner dominating set containing v_2. Similar is the case with v_{n-1} also.

Therefore, $f\gamma_s(P_n) = 1$.

Theorem 2.6: *Let G be any graph with atleast two steiner dominating sets. Suppose G has a steiner dominating set W satisfying the property, "W has a vertex u such that* $u \notin W'$ *for every steiner dominating set* W' *different from W", then* $f\gamma_s(G) = 1$.

Proof: As G has atleast two steiner dominating sets, by Observation 2.4(i), $f\gamma_s(G) \neq 0$. If G satisfies the given condition that is, W has a vertex u such that $u \notin W'$, for every W' different from W, then by the definition of forcing steiner domination number, $f\gamma_s(W) = 1$. Therefore, by observation 2.4(ii), $f\gamma_s(G) = 1$.

Corollary 2.7: Let G be any graph with at least two steiner dominating sets. Suppose G has a steiner dominating set W such that $W \cap W' = \phi$ for every steiner dominating set W' different from W, then, $f\gamma_s(G) = 1$.

Corollary 2.8: Let G be any graph with at least two steiner dominating sets. If pair wise intersection of distinct steiner dominating sets of G is empty, then $f\gamma_s(G) = 1$.

Theorem 2.9: $f\gamma_s(C_n) = \begin{cases} 1 & if\ n = 4\ and\ n \equiv 0(\text{mod}\,3)(n > 3) \\ 2 & otherwise \end{cases}$

Proof: Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$.

Case-(i): $n \equiv 0$ (mod3) and $n > 3$.

The steiner dominating sets of C_n are $W_1 = \{v_1, v_4, \ldots, v_{3(k-1)+1}\}$, $W_2 = \{v_2, v_5, \ldots, v_{3(k-1)+2}\}$ and $W_3 = \{v_3, v_6, \ldots, v_{3k}\}$.

Here, $W_i \cap W_j = \phi$, for $1 \leq i, j \leq 3$.

Therefore, by Corollary 2.7, $f\gamma_s(C_n) = 1$ if $n \equiv 0$ (mod3) and $n > 3$.

Case-(ii): $n \equiv 1$ (mod3).

When $n = 4$, the steiner dominating sets of C_4 are $W_1 = \{v_1, v_3\}$ and $W_2 = \{v_2, v_4\}$.

Here, $W_1 \cap W_2 = \phi$. Therefore, as in case 1, $f\gamma_s(C_4) = 1$.

When $n > 4$, it is easy to observe that any pair of adjacent vertices lie in exactly one steiner dominating set and any single element lie in atleast two steiner dominating sets.

Therefore, $f\gamma_s(C_n) = 2$.

Case-(iii): $n \equiv 2$ (mod3).

When $n = 5$, the cycle is C_5 and it is easy to observe that any pair of adjacent vertices of C_5 lie in exactly one steiner dominating set and any single element lie in atleast two steiner dominating sets.

Therefore, $f\gamma_s(C_5) = 2$.

When $n > 5$, any two vertices u and v with $d(u, v)$ lie in exactly one steiner dominating set and any single element set lie in atleast two steiner dominating sets.

Therefore, $f\gamma_s(C_n) = 2$.

Remark 2.10: *In the above theorem, if n=3, then the cycle is C_3 and it has a unique steiner dominating set. Therefore, by observation 2.4(i), $f\gamma_s(C_3)$ =0.*

Theorem 2.11: *For the wheel $W_p = K_1 + C_{p-1}(p \geq 5)$, $f\gamma_s(W_p) = p - 4$.*

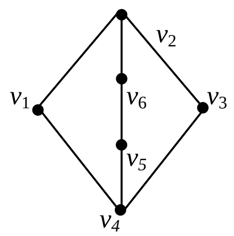
Proof: From Theorem 1.2 and 1.3, it is observed that, for a wheel graph W_p, every steiner set is a steiner dominating set and vice versa. Then, by Observation 2.4(iv), $f_s(W_p) = f\gamma_s(W_p)$. Hence, by Theorem 1.3, $f\gamma_s(W_p) = p - 4$.

Theorem 2.12: *For the complete bipartite graph $G = K_{m,n}(m, n \geq 2)$,* $f\gamma_s(G) = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$

Proof: From Theorem 1.1, it is observed that, for complete bipartite graph every steiner set is a steiner dominating set and vice versa. Therefore, proceeding as in Theorem 2.11 & by Theorem 1.4, $f\gamma_s(G) = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$

Definition 2.13: A vertex v of a graph G is said to be a steiner dominating vertex of G if v belongs to every minimum steiner dominating set of G.

Example 2.14: For the graph G in Figure 2.2 $W_1=\{v_2,v_4,v_5\}$ and $W_2=\{v_2,v_4,v_6\}$ are the minimum steiner dominating sets of G. v_2 and v_4 lie in every steiner dominating set of G. Therefore, v_2 and v_4 are the steiner dominating vertices of G.



***G*: Figure-2.2**

Remark 2.15:

1. Since all the extreme vertices of a graph G belongs to every minimum steiner dominating set of *G*, all the extreme vertices of *G* are steiner dominating vertices of *G*.
2. If *G* has a unique steiner dominating set *W*, then every vertex of *W* is a steiner dominating vertex of *G*.
3. Let $u \in V(G)$ be a steiner dominating vertex of *G*, Suppose *W* is a minimum steiner dominating set of *G* and *T* is a minimum forcing subset of *W*, then $u \notin T$.

Let *G* be a connected graph and let *W* be a minimum steiner dominating set of *G*. Suppose, *T* is one of the minimum forcing subset of *W*. Let $E = W - T$ be the relative complement of *T* in its minimum steiner dominating set *W*. Define ξ = {*E* : *E* is the relative complement of a minimum forcing subset *T* in its minimum steiner dominating set *W* of *G*}.

Theorem 2.16: *Let G be a connected graph. Then,* $\bigcap_{E\in\xi} E$ *is the set of all steiner dominating vertices of G.*

Proof: Let *S* be the set of all steiner dominating vertices of *G*.

To Prove: $S = \bigcap_{E\in\xi} E$

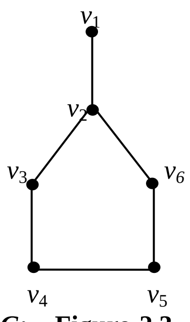
Let $v \in S$. By Definition 2.13, *v* is in every minimum steiner dominating set of *G*. Let *W* be a minimum steiner dominating set of *G* and *T* be a minimum forcing subset of *W*. Then, $v \in W$. Then by Remark 2.15(3), $v \notin T$. So, $v \in E = W - T$. Hence, $v \in E$ for every $E \in \xi$. That is, $v \in \bigcap_{E\in\xi} E$.

Conversely, let $v \in \bigcap_{E\in\xi} E$. Then, $v \in E = W - T$, where *T* is a minimum forcing subset of the minimum steiner dominating set *W*. Therefore, $v \in W$ for every steiner dominating set W of G. Therefore, $v \in S$.

Hence, $\bigcap_{E\in\xi} E$ is the set of all steiner dominating vertices of *G*.

Remark 2.17:

1. Let *W* be a minimum steiner dominating set of a graph *G* and let *T* be a minimum forcing subset of *W*. If *S* is the set of all steiner dominating vertices of *G*, then, $S \cap T = \phi$. For, by Definition 2.13, $u \in S$ if and only if $u \notin T$.
2. The above result holds even if *G* has a unique minimum steiner dominating set. For, $T = \phi$ for the unique minimum steiner dominating set.
3. Let *S* be the set of all steiner dominating vertices of a graph *G*. Then by (1), $f\gamma_s(G) \le \gamma_s(G) - |S|$.
4. The above inequality is strict. For example, Consider the graph *G* in Figure 2.3, $W_1=\{v_1,v_4,v_5\}$, $W_2=\{v_1,v_3,v_5\}$ and $W_3 = \{v_1, v_4, v_6\}$ are the distinct minimum steiner dominating sets of *G*. Therefore, $\gamma_s(G) = 3$. $f\gamma_s(W_1) = 2$ and $f\gamma_s(W_2) = f\gamma_s(W_3) = 1$. Therefore, $f\gamma_s(G)$ = min{ $f\gamma_s(W)$: *W* is a minimum steiner dominating set of *G*}=1. Here, $S = \{v_1\}$. Therefore, $\gamma_s(G) - |S| = 3-1 = 2$. Hence, $f\gamma_s(G) < \gamma_s(G) - |S|$.



***G*: Figure-2.3**

5. By Remark 2.15(1), $f\gamma_s(G) \le \gamma_s(G) - k$ where *k* is the number of extreme vertices of *G*.

Observation 2.18: For a complete graph $G = K_p$, $f\gamma_s(G) = 0$ and $|S| = p$.

For, K_p has unique minimum steiner dominating set and so $f\gamma_s(G) = 0$. By Remark 2.15(1), $S = V(G)$ and $|S| = p$.

Remark 2.19: From the above observation, we get that $f\gamma_s(K_p) = \gamma_s(K_p) - |S|$. Hence the upper bound in Remark 2.17(3) is sharp as well.

REFERENCES

1. G. Chatrand and P. Zhang, *The Steiner number of a graph*, Discrete Math., 242(2002), 41–54.
2. T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in graphs*, Marcel Decker, Inc., New York, 1998.
3. J. John, G. Edwin and P. Paul Sudhahar, *The Steiner domination number of a graph*, International Journal of Mathematics and Computer Applications Research, 3(3) (2013), 37–42.
4. Ore and Berge, *Theory of Graphs*, American Mathematical Society, Colloquium Publications XXXVIII, 1962.
5. K. Palani and A. Nagarajan, *Forcing (G, D)-Number of a Graph*, International Journal of Mathematical Combinatorics, Vol 3(2011), pp.82-87.
6. K. Ramalakshmi, K. Palani and T. Tamil Chelvam, *Steiner Domination Number of Graphs*, Proceedings of International Conference on Recent Trends in Mathematical Modelling, ISBN 13-978-93-82592-00-06, pp.128-134.
7. A.P.SanthaKumaran and J.John, *The forcing steiner number of a graph*, Discussiones Mathematicae, Graph Theory 31(2011), 171-181.
8. S.K.Vaidya and R.N.Mehta, *Steiner domination number of some wheel related graphs*, International journal of Mathematics and soft computing, Vol.5, No.2 (2015), 15-19.

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