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# THE FORCING STEINER DOMINATION NUMBER OF A GRAPH

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***ABSTRACT***

*In this paper, the forcing steiner domination number of a graph is introduced. Also, this number is found for some standard graphs.*

***Keywords:*** *Domination, Steiner number, Steiner domination number and Forcing Steiner domination number.*

## 1. INTRODUCTION

The concept of domination in graphs was introduced by Ore and Berge [4]. Throughout this paper $G= (V, E)$ denotes a finite undirected simple graph with vertex set $V$ and edge set $E$. A subset $D$ of $V(G)$ is a dominating set of $G$ if every vertex in $V - D$ is adjacent to at least one vertex in $D$. The minimum cardinality of a dominating set of $G$ is called the domination number of $G$ and is denoted by $\gamma(G)$. The concept of Steiner number of a graph was introduced by G.Chatrand and P. Zhang [1]. For a nonempty set $W$ of vertices in a connected graph $G$, the Steiner distance $d(W)$ of $W$ is the minimum size of a connected sub graph of $G$ containing $W$. Necessarily each such subgraph is a tree and is called a Steiner tree with respect to $W$ or a Steiner $W$-tree. The set of all vertices of $G$ that lie in some Steiner $W$-tree is denoted by $S(W)$. If $S(W) = V$, then $W$ is called a Steiner set for $G$. A Steiner set with minimum cardinality is the Steiner number of $G$ and is denoted by $s(G)$.

The concept of Steiner domination number of a graph was introduced by J. John *et al.*, [3]. For a connected graph $G$, a set of vertices $W$ in $G$ is called a Steiner dominating set if $W$ is both a Steiner set and a dominating set. The minimum cardinality of a Steiner dominating set of $G$ is its Steiner domination number and is denoted by $\gamma_s(G)$. A steiner dominating set of cardinality $\gamma_s(G)$ is said to be a $\gamma_s$-set.

The concept of Forcing $(G, D)$-number was introduced by K.Palani and A.Nagarajan [5]. For a connected graph $G$, let $S$ be a $\gamma_G$-set of G. A subset $T$ of $S$ is called a forcing subset for $S$ if $S$ is the unique $\gamma_G$-set of $G$ containing $T$. A forcing subset $T$ of $S$ with minimum cardinality is called a minimum forcing subset for $S$. The forcing $(G,D)$-number of $S$, denoted by $f_{G,D}(S)$, is the cardinality of a minimum forcing subset of $S$. The forcing $(G, D)$-number of $G$ is the minimum of $f_{G,D}(S)$, where the minimum is taken over all $\gamma_G$-sets $S$ of $G$ and it is denoted by $f_{G,D}(G)$. That is, $f_{G,D}(G) = min\{ f_{G,D}(S) : S$ is any $\gamma_G$-set of $G\}$.

**Theorem 1.1 [3]:** *For the complete bipartite graph* $G = K_{m,n}$,

$$s(G) = \gamma_s(G) = \begin{cases} 2 & if\ m = n = 1 \\ n & if\ n \geq 2, m = 1 \\ \min\{m, n\} & if\ m, n \geq 2 \end{cases}$$

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**Theorem 1.2 [8]:** *For a Wheel graph* $W_{1,n}$, $n \geq 5$, $\gamma_s(W_{1,n}) = n-2$.

**Theorem 1.3 [7]:** *For the Wheel* $W_p = K_1 + C_{p-1}(p \geq 5)$, $s(W_p) = p-3$ *and* $f_s(W_p) = p-4$.

Theorem 1.4 [7]: *For the complete bipartite graph* $G = K_{m,n}$ $(m, n \geq 2)$, $f_s(G) = \begin{cases} 0 & if\ m \neq n \\ 1 & if\ m = n \end{cases}$

**Theorem 1.5 [3]:** *Each extreme vertex of a connected graph G belongs to every minimum Steiner dominating set of G.*

**Theorem 1.6 [3]:** *For the complete graph* $K_p(p \geq 2), \gamma_s(K_p) = p$.

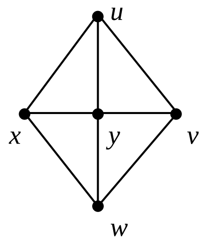
**Theorem 1.7 [6]:** $\gamma_s(P_n) = \begin{cases} \left\lceil \frac{n-4}{3} \right\rceil + 2 & if\ n \geq 5; \\ 2 & if\ n = 2,3\ or\ 4. \end{cases}$

**Theorem 1.8.[6]:** *For n > 5,* $\gamma_s(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

**2. FORCING STEINER DOMINATION NUMBER**

**Definition 2.1:** Let G be a connected graph and *W* be a minimum steiner dominating set of *G*. A subset *T* of *W* is called a forcing subset for *W* if *W* is the unique minimum steiner dominating set of *G* containing *T*. A forcing subset *T* of *W* with minimum cardinality is called a minimum forcing subset for *W*. The forcing steiner domination number of *W*, denoted by $f\gamma_s(W)$, is the cardinality of a minimum forcing subset of *W*. The forcing steiner domination number of *G* is the minimum of $f\gamma_s(W)$ where the minimum is taken over all steiner dominating sets *W* of *G* and it is denoted by $f\gamma_s(G)$. That is, $f\gamma_s(G)$ = min{ $f\gamma_s(W)$ : *W* is any steiner dominating set of *G*}.

**Example 2.2:** Consider the graph *G* in Figure 2.1. $W_1$= {*u*, *w*} and $W_2$ = {*x*, *v*} are the only two minimum steiner dominating sets of *G*. Forcing subsets of $W_1$ are {*u*}, {*w*} and {*u*, *w*}. Therefore, $f\gamma_s(W_1)$ = 1. Similarly, Forcing subsets of $W_2$ are {*x*}, {*v*} and {*x*, *v*}. Therefore, $f\gamma_s(W_2)$ = 1. Hence, $f\gamma_s(G)$ = min{1, 1} = 1.




***G*: Figure-2.1**

**Remark 2.3:** For a connected graph *G*, 0 ≤ $f\gamma_s(G) \leq s(G)$. Here, the lower bound is sharp, since for any complete graph *W* = *V* (*G*) is the unique steiner dominating set. Therefore, *T* = $\phi$ is a forcing subset for *W* and $f\gamma_s(K_p)$ = 0.

**Observation 2.4:** Let *G* be a connected graph. Then,

i. From the definition of forcing steiner dominating set, $f\gamma_s(G)$ = 0 if and only if *G* has a unique steiner dominating set.
ii. $f\gamma_s(G)$ =1 if and only if *G* has atleast two steiner dominating sets, one of which has forcing number equal to 1.
iii. $f\gamma_s(G) = \gamma_s(G)$ if and only if every steiner dominating set *W* of *G* has the property, $f\gamma_s(W) = |W| = \gamma_s(G)$.
iv. For a connected graph G, if every minimum steiner set *W* itself is a minimum steiner dominating set then, $f_s(G) = f\gamma_s(G)$.

**Theorem 2.5:** $f\gamma_s(P_n) = \begin{cases} 0 & if\ n = 2,3\ and\ n \equiv 1(\bmod 3) \\ 1 & otherwise \end{cases}$

**Proof:** Let $P_n = (v_1, v_2, \ldots, v_n)$.

**Case-(i):** $n \equiv 0(\text{mod } 3)$

When $n = 3$, there is a unique steiner dominating set and by observation 2.4(i), $f\gamma_s(P_3) = 0$.

If n > 3, $P_n$ has more than one steiner dominating set. Further, for every $n$, there exists only one steiner dominating set containing $v_3$. Similar is the case with $v_{n-2}$ also.

Therefore, $f\gamma_s(P_n) = 1$.

**Case-(ii):** $n \equiv 1(\text{mod } 3)$

In this case, $P_n$ has a unique steiner dominating set.

Therefore, $f\gamma_s(P_3) = 0$.

**Case-(iii):** $n \equiv 2(\text{mod } 3)$

When $n = 2$, there is a unique steiner dominating set and therefore, $f\gamma_s(P_2) = 0$.

When n > 2, As in case 1, $P_n$ has more than one steiner dominating set. Further, for every $n$, there exists only one steiner dominating set containing $v_2$. Similar is the case with $v_{n-1}$ also.

Therefore, $f\gamma_s(P_n) = 1$.

**Theorem 2.6:** *Let G be any graph with atleast two steiner dominating sets. Suppose G has a steiner dominating set W satisfying the property, "W has a vertex u such that $u \notin W'$ for every steiner dominating set $W'$ different from W", then $f\gamma_s(G) = 1$.*

**Proof:** As $G$ has atleast two steiner dominating sets, by Observation 2.4(i), $f\gamma_s(G) \neq 0$. If $G$ satisfies the given condition that is, $W$ has a vertex $u$ such that $u \notin W'$, for every $W'$ different from $W$, then by the definition of forcing steiner domination number, $f\gamma_s(W) = 1$. Therefore, by observation 2.4(ii), $f\gamma_s(G) = 1$.

**Corollary 2.7:** Let $G$ be any graph with at least two steiner dominating sets. Suppose $G$ has a steiner dominating set $W$ such that $W \cap W' = \phi$ for every steiner dominating set $W'$ different from $W$, then, $f\gamma_s(G) = 1$.

**Corollary 2.8:** Let $G$ be any graph with at least two steiner dominating sets. If pair wise intersection of distinct steiner dominating sets of $G$ is empty, then $f\gamma_s(G) = 1$.

**Theorem 2.9:** $f\gamma_s(C_n) = \begin{cases} 1 & if\ n = 4\ and\ n \equiv 0(\bmod 3)(n > 3) \\ 2 & otherwise \end{cases}$

**Proof:** Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$.

**Case-(i):** $n \equiv 0$ (mod3) and $n > 3$.

The steiner dominating sets of $C_n$ are $W_1 = \{v_1, v_4, \ldots, v_{3(k-1)+1}\}$, $W_2 = \{v_2, v_5, \ldots, v_{3(k-1)+2}\}$ and $W_3 = \{v_3, v_6, \ldots, v_{3k}\}$.

Here, $W_i \cap W_j = \phi$, for $1 \leq i, j \leq 3$.

Therefore, by Corollary 2.7, $f\gamma_s(C_n) = 1$ if $n \equiv 0$ (mod3) and $n > 3$.

**Case-(ii):** $n \equiv 1$ (mod3).

When $n = 4$, the steiner dominating sets of $C_4$ are $W_1 = \{v_1, v_3\}$ and $W_2 = \{v_2, v_4\}$.

Here, $W_1 \cap W_2 = \phi$. Therefore, as in case 1, $f\gamma_s(C_4) = 1$.

When $n > 4$, it is easy to observe that any pair of adjacent vertices lie in exactly one steiner dominating set and any single element lie in atleast two steiner dominating sets.

Therefore, $f\gamma_s(C_n) = 2$.

**Case-(iii):** $n \equiv 2$ (mod3).

When $n = 5$, the cycle is $C_5$ and it is easy to observe that any pair of adjacent vertices of $C_5$ lie in exactly one steiner dominating set and any single element lie in atleast two steiner dominating sets.

Therefore, $f\gamma_s(C_5) = 2$.

When $n > 5$, any two vertices $u$ and $v$ with $d(u, v)$ lie in exactly one steiner dominating set and any single element set lie in atleast two steiner dominating sets.

Therefore, $f\gamma_s(C_n) = 2$.

**Remark 2.10:** *In the above theorem, if n=3, then the cycle is $C_3$ and it has a unique steiner dominating set. Therefore, by observation 2.4(i),* $f\gamma_s(C_3) = 0$.

**Theorem 2.11:** *For the wheel* $W_p = K_1 + C_{p-1} (p \geq 5)$, $f\gamma_s(W_p) = p - 4$.

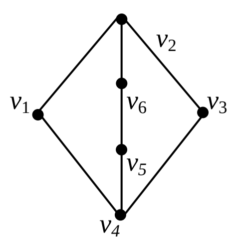
**Proof:** From Theorem 1.2 and 1.3, it is observed that, for a wheel graph $W_p$, every steiner set is a steiner dominating set and vice versa. Then, by Observation 2.4(iv), $f_s(W_p) = f\gamma_s(W_p)$. Hence, by Theorem 1.3, $f\gamma_s(W_p) = p - 4$.

**Theorem 2.12:** *For the complete bipartite graph* $G = K_{m,n}(m, n \geq 2)$, $f\gamma_s(G) = \begin{cases} 0 & if\ m \neq n \\ 1 & if\ m = n. \end{cases}$

**Proof:** From Theorem 1.1, it is observed that, for complete bipartite graph every steiner set is a steiner dominating set and vice versa. Therefore, proceeding as in Theorem 2.11 & by Theorem 1.4, $f\gamma_s(G) = \begin{cases} 0 & if\ m \neq n \\ 1 & if\ m = n. \end{cases}$

**Definition 2.13:** A vertex $v$ of a graph $G$ is said to be a steiner dominating vertex of $G$ if $v$ belongs to every minimum steiner dominating set of $G$.

**Example 2.14:** For the graph $G$ in Figure 2.2 $W_1=\{v_2,v_4,v_5\}$ and $W_2=\{v_2,v_4,v_6\}$ are the minimum steiner dominating sets of $G$. $v_2$ and $v_4$ lie in every steiner dominating set of $G$. Therefore, $v_2$ and $v_4$ are the steiner dominating vertices of $G$.




***G*: Figure-2.2**

**Remark 2.15:**

1. Since all the extreme vertices of a graph G belongs to every minimum steiner dominating set of *G*, all the extreme vertices of *G* are steiner dominating vertices of *G*.
2. If *G* has a unique steiner dominating set *W*, then every vertex of *W* is a steiner dominating vertex of *G*.
3. Let $u \in V(G)$ be a steiner dominating vertex of *G*, Suppose *W* is a minimum steiner dominating set of *G* and *T* is a minimum forcing subset of *W*, then $u \notin T$.

Let *G* be a connected graph and let *W* be a minimum steiner dominating set of *G*. Suppose, *T* is one of the minimum forcing subset of *W*. Let $E = W - T$ be the relative complement of *T* in its minimum steiner dominating set *W*. Define $\xi = \{E$ : *E* is the relative complement of a minimum forcing subset *T* in its minimum steiner dominating set *W* of $G\}$.

**Theorem 2.16:** *Let G be a connected graph. Then,* $\bigcap_{E\in\xi} E$ *is the set of all steiner dominating vertices of G.*

**Proof:** Let *S* be the set of all steiner dominating vertices of *G*.

To Prove: $S = \bigcap_{E\in\xi} E$

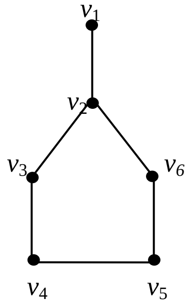
Let $v \in S$. By Definition 2.13, *v* is in every minimum steiner dominating set of *G*. Let *W* be a minimum steiner dominating set of *G* and *T* be a minimum forcing subset of *W*. Then, $v \in W$. Then by Remark 2.15(3), $v \notin T$. So, $v \in E = W - T$. Hence, $v \in E$ for every $E \in \xi$. That is, $v \in \bigcap_{E\in\xi} E$.

Conversely, let $v \in \bigcap_{E\in\xi} E$. Then, $v \in E = W - T$, where *T* is a minimum forcing subset of the minimum steiner dominating set *W*. Therefore, $v \in W$ for every steiner dominating set W of G. Therefore, $v \in S$.

Hence, $\bigcap_{E\in\xi} E$ is the set of all steiner dominating vertices of *G*.

**Remark 2.17:**

1. Let *W* be a minimum steiner dominating set of a graph *G* and let *T* be a minimum forcing subset of *W*. If *S* is the set of all steiner dominating vertices of *G*, then, $S \cap T = \phi$. For, by Definition 2.13, $u \in S$ if and only if $u \notin T$.
2. The above result holds even if *G* has a unique minimum steiner dominating set. For, $T = \phi$ for the unique minimum steiner dominating set.
3. Let *S* be the set of all steiner dominating vertices of a graph *G*. Then by (1), $f\gamma_s(G) \le \gamma_s(G) - |S|$.
4. The above inequality is strict. For example, Consider the graph *G* in Figure 2.3, $W_1 = \{v_1, v_4, v_5\}$, $W_2 = \{v_1, v_3, v_5\}$ and $W_3 = \{v_1, v_4, v_6\}$ are the distinct minimum steiner dominating sets of *G*. Therefore, $\gamma_s(G) = 3$. $f\gamma_s(W_1) = 2$ and $f\gamma_s(W_2) = f\gamma_s(W_3) = 1$. Therefore, $f\gamma_s(G) = \min\{f\gamma_s(W)$ : *W* is a minimum steiner dominating set of $G\} = 1$. Here, $S = \{v_1\}$. Therefore, $\gamma_s(G) - |S| = 3 - 1 = 2$. Hence, $f\gamma_s(G) < \gamma_s(G) - |S|$.




***G*: Figure-2.3**

5. By Remark 2.15(1), $f\gamma_s(G) \le \gamma_s(G) - k$ where *k* is the number of extreme vertices of *G*.

**Observation 2.18:** For a complete graph $G = K_p$, $f\gamma_s(G) = 0$ and $|S| = p$.

For, $K_p$ has unique minimum steiner dominating set and so $f\gamma_s(G) = 0$. By Remark 2.15(1), $S = V(G)$ and $|S| = p$.

**Remark 2.19:** From the above observation, we get that $f\gamma_s(K_p) = \gamma_s(K_p) - |S|$. Hence the upper bound in Remark 2.17(3) is sharp as well.

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