International Journal of Mathematical Archive-8(2), 2017, 95-98
Available online through www.ijma.info ISSN 2229 – 5046

INDEPENDENT VERTEX-EDGE DOMINATION IN GRAPHS

SHIGEHALLI V.S.[1], VIJAYAKUMAR PATIL*[2]

[1]Professor Department of Mathematics,
Rani Channamma University, Belagavi-591156, Karnataka, India.

[2]Research Scholar, Department of Mathematics,
Rani Channamma University, Belagavi-591156, Karnataka, India.

(Received On: 07-01-17; Revised & Accepted On: 01-02-17)

ABSTRACT

Let the vertices and edges of a graph G be called the elements of G. A set X of elements in G is an vertex-edge dominating set of G if every element not in X is either adjacent or incident to at least one element in X. An vertex-edge dominating set X of elements in G is an independent vertex-edge dominating set of G if any two elements in X are neither adjacent nor incident. The independent vertex-edge domination number $\gamma_{ve}^{i}(G)$ of G, is the smallest cardinality of an independent vertex-edge dominating set of G. In this paper, we obtained bounds for $\gamma_{ve}^{i}(G)$.

2000 Mathematics Subject Classification: *05C69.*

Keywords: *domination, vertex-edge domination, independent vertex-edge domination number.*

1. INTRODUCTION

All graphs considered here are simple, finite, connected and nontrivial. Let $G = (V(G), E(G))$ be a graph, where *V(G)* is the vertex set and *E(G)* be the edge set of G. The vertex $v \in V$ is called a *pendant vertex*, if $deg_G(v) = 1$ and an *isolated vertex* $if\ deg_G(v) = 0$, where $deg_G(x)\ is\ the\ degree\ of\ a\ vertex\ x \in V(G)$. A vertex which is adjacent to a pendant vertex is called a *support vertex*. We denote $\delta(G)(\Delta(G))$ as the $minimum\ (maximum)\ degree\ and$ $p = |V(G)|$, $q = |E(G)|$ the *order* and *size* of *G* respectively. A *spanning subgraph* is a subgraph containing all the vertices of *G*. A shortest $u - v\ path$ is often called a *geodesic*. The *diameter* $diam(G)$ of a connected graph *G* is the length of any longest geodesic. The *neighborhood* of a vertex u in *V* is the set *N(u)* consisting of all vertices *v* which are adjacent with *u*. A *claw* is another name for the complete bipartite graph $K_{1,3}$. *A claw-free graph* is a graph that does not have a claw as an induced subgraph.

A subset $D \subseteq V$ is said to be a *dominating set* of G if every vertex $V - D$ is adjacent to at least one vertex in *D*. The minimum cardinality of a minimal dominating set is called the *domination number* $\gamma(G)$ of *G* [2].

A subset *D* of $V(G)$ is an independent set if no two vertices in *D* are adjacent. A dominating set *D* which is also an independent dominating set. The independent domination number $i(G)$ is the minimum cardinality of an independent domination set [2,3].

A set F of edges in a graph $G = (V, E)$ is called an edge dominating set of G if every edge in $E - F$ is adjacent to atleast one edge in F. The edge domination number $\gamma'(G)$ of a graph G is the minimum cardinality of an edge dominating set of G.

Corresponding Author: Vijayakumar Patil*[2]
[2]Research Scholar, Department of Mathematics,
Rani Channamma University, Belagavi-591156, Karnataka, India.

A set X of elements in G is an vertex-edge dominating set of G, if every element not in X is either adjacent or incident to at least one element in X. The vertex-edge domination number $\gamma_{ve}(G)$ is the order of a smallest vertex-edge dominating set of G [6].

In this paper, we define a new parameter as follows:

An vertex-edge dominating set X of elements in G is an independent vertex-edge dominating set of G if any two elements in X are neither adjacent nor incident. The independent vertex-edge domination number $\gamma_{ve}^{i}(G)$ of G is the smallest cardinality of an independent vertex-edge dominating set of G.

2. MAIN RESULTS

Observation: For any graph G, $\gamma_{ve}^{i}(G) = \gamma_i(T(G))$, where T(G) denote the total graph of a graph G.

In next theorem, we compute the independnet vertex-edge domination number of some standard class of graphs.

Theorem 2.1:

(i) For any complete graph K_p; $p \geq 2$, $\gamma_{ve}^{i}(K_p) = \left\lceil \frac{p}{2} \right\rceil$.

(ii) For any cycle C_p; $p \geq 4$, $\gamma_{ve}^{i}(C_p) = \left\lfloor \frac{p}{2} \right\rfloor$.

(iii) For any path P_p; $p \geq 2$, $\gamma_{ve}^{i}(P_p) = \left\lfloor \frac{p}{2} \right\rfloor$.

(iv) For any complete bipartite graph K_{p_1,p_2}, $\gamma_{ve}^{i}(K_{p_1,p_2}) = p_1$, $1 \leq p_1 \leq p_2$.

(v) For any wheel W_p; $p \geq 4$, $\gamma_{ve}^{i}(W_p) = \left\lfloor \frac{p}{2} \right\rfloor$

In the following theorem, a relation between $\gamma_{ve}(G)$ and $\gamma_{ve}^{i}(G)$ is obtained.

Theorem 2.2: *For any graph* G, $\gamma_{ve}(G) \leq \gamma_{ve}^{i}(G)$.

Proof: It is known that, for any graph G, $\gamma(G) \leq \gamma_i(G)$. Therefore similarly $\gamma_{ve}(G) \leq \gamma_{ve}^{i}(G)$.

Next we obtain the relation between $\gamma_i(G)$, $\gamma_i'(G)$ and $\gamma_{ve}^{i}(G)$.

Theorem 2.3: *For any graph* G

$$\frac{\gamma_i(G)+\gamma_i'(G)}{2} \leq \gamma_{ve}^{i}(G) \leq \gamma_i(G) + \gamma_i'(G).$$

Proof. First we establish the lower bound. D and F be the minimum independent dominating and independent edge dominating sets of G respectively. Let $X = D \cup F$ be a minimum independent vertex-edge dominating set of G. For each edge $e = uv$ in F, choose a vertex u or v but not both which are independent. Let F' be the collection of such vertices. Clearly $D \cup F'$ is an independent dominating set of G. Therefore

$$\gamma_i(G) \leq |D \cup F'| = |D \cup F| = \gamma_{ve}^{i}(G). \tag{2.1}$$

Now for each vertex u in D, choose exactly one edge incident with u which is independent. Let D' be the collection of such edges. Clearly $D' \cup F$ is an independent edge dominating set of G. Therefore

$$\gamma_i'(G) \leq |D \cup F'| = |D \cup F| = \gamma_{ve}^{i}(G). \tag{2.2}$$

From (2.1) and (2.2) it follows that $\frac{\gamma_i(G)+\gamma_i'(G)}{2} \leq \gamma_{ve}^{i}(G)$.

Now for the upper bound, let D and F be the minimum independent dominating and independent edge dominating sets of G respectively. Then $D \cup F$ is an independent vertex-edge dominating set. Thus $\gamma_{ve}^{i}(G) \leq |D \cup F| = \gamma_i(G) + \gamma_i'(G)$.

Hence, $\gamma_{ve}^{i}(G) \leq \gamma_i(G) + \gamma_i'(G)$.

Theorem 2.4: *If G is a claw-free graph, then $\gamma_{ve}^{i}(G) \leq \gamma_i(G) + \gamma_i^{'}(L(G))$.*

Proof: In [1], it is proved that, if G is a claw free-graph, then $\gamma(G) = \gamma_i(G)$ and $\gamma(L(G)) = \gamma_i(L(G))$. Therefore using Theorem 2.3, we get the required result.

Theorem A[6]: For any connected graph G of order p

$$\frac{p+q}{2\Delta(G)+1} \leq \gamma_{ve}(G).$$

Now we establish another lower bound for $\gamma_{ve}^{i}(G)$.

Theorem 2.5: *For any graph G*

$$\frac{p+q}{2\Delta(G)+1} \leq \gamma_{ve}^{i}(G).$$

Proof: The proof follows from Theorem A and the fact that $\gamma_{ve}(G) \leq \gamma_{ve}^{i}(G)$.

In a graph G if $deg(v) = 1$, then v is called a pendant vertex of G

Lemma 1: Let G be a connected graph of order $p \geq 3$. Then there exist two nonadjacent vertices u and v having a common neighbor w such that $G - \{u, v\}$ is connected.

Proof: Let T denote a spanning tree of G. If T has exactly one nonpendant vertex, then the removal of any two pendant vertices u and v of T, results in a connected graph. Suppose T has at least two nonpendant vertices. Then there exist at least two nonpendant vertices each of which is adjacent to exactly one nonpendant vertex. If w is adjacent to at least two pendant vertices u and v, then removal of u and v results in a connected graph. If w is adjacent to exactly one pendant vertex u, then removal of w and u results in a connected graph. This completes the proof.

Theorem 2.6: *For any connected graph G of order $p \geq 2$ and $\delta(G) \geq 2$, $\gamma_{ve}^{i}(G) \leq \left\lceil \frac{p}{2} \right\rceil$.*

Proof: We prove the result by induction on p.

If $p = 3$ or 4, then the result can be verified. Assume the result is true for all connected graphs G with $\delta(G) \geq 2$ and $p-1$ vertices. Let G_1 be a connected graph with $\delta(G) \geq 2$ and p vertices. Let u and v denote two nonadjacent vertices having a common neighbor w such that $G = G_1 - \{uv\}$ is connected. Let X be a minimum independent vertex-edge dominating set of G, then either $X \cup \{w\}$ or $X \cup \{uv\}$ is an vertex-edge dominating set of G_1.

Thus $\gamma_{ve}^{i}(G_1) \leq |X| + 1 \leq \left\lceil \frac{p-1}{2} \right\rceil + 1 \leq \left\lceil \frac{p}{2} \right\rceil$.

In the following theorem we give characterization of graphs in which $\gamma_{ve}(G) = \gamma_{ve}^{i}(G)$.

Theorem 2.7: *For any graph G of order $p \geq 2$, $\gamma_{ve}(G) = \gamma_{ve}^{i}(G)$. if and only if every vertex-edge dominating set of G is independent.*

Proof: Let $\gamma_{ve}(G) = \gamma_{ve}^{i}(G)$. If possible, suppose every vertex-edge dominating set of G is not independent. Then there exists at least one vertex-edge dominating set X of G such that at least two elements of X are either adjacent or incident. Therefore $\gamma_{ve}(G) < \gamma_{ve}^{i}(G)$, a contradiction. Sufficiency is obvious.

Theorem 2.8: *Let G be a connected graph of order at least 2. Then $\gamma(G) = \gamma_{ve}^{i}(G)$ if and only if any two adjacent vertices form a minimal dominating set.*

Proof: Let G be a connected graph and let a set consisting of any two vertices of G form a minimal dominating set of G, Also by the fact that a set consisting of any two adjacent vertices of G forms a minimal dominating set of G if and only if G is isomorphic to the complete $k-$partite graph $K_{p_1, p_2, \cdots, p_k}$; $p_1 \geq 2$ for each $i \in \{1, 2, \cdots, k\}$ with the partite sets of sizes $p_1, p_2, \cdots, p_k$.

$G = K_{p_1, p_2, \cdots, p_k}$. Hence $\gamma(G) = \gamma_{ve}^i(G)$.

Conversely, let $\gamma(G) = \gamma_{ve}^i(G)$ and any two adjacent do not form a minimal dominating set. Let $D = \{v_1, v_2, \cdots, v_k\}$ be the set of all maximal independent vertex set of G. Then $\gamma(G) = |D|$. Let $D' = \{e_1, e_2, \cdots, e_s\}$ be the maximal independent set of edges of G. Then $\gamma_{ve}^i(G) \le |D \cup D'| > \gamma(G)$, a contradiction. Hence any two adjacent vertices of G form a minimal dominating set of G or $G = K_{p_1, p_2, \cdots, p_k}$.

This completes the proof.

We need the following definition for our next results.

Definition: A graph G is $k-$partite, $k \ge 1$, if it is possible to partition $V(G)$ into $k-$ subsets $V_1, V_2, \cdots V_k$ called partite sets, such that every element of $E(G)$ joins a vertex of V_i to a vertex of V_j, $i \ne j$.

Theorem 2.9: *For any connected graph G of order at least 2, $\gamma_i(G) = \gamma_{ve}^i(G)$, if and only if $G = K_{p_1, p_2, \cdots, p_k}$.*

Proof: Since every independent dominating set is a dominating set, therefore the proof of the following theorem is similar to Theorem 2.8.

Theorem 2.10: *Let G be any connected graph of order at least 2. Then $\gamma^i(G) = \gamma_{ve}^i(G)$, if and only if G is $k-$ partite graph.*

Proof: The proof is similar to the proof of Theorem 2.8.

Finally we prove Nordhaus-Gaddum type results for $\gamma_{ve}^i(G)$

Theorem 2.11: *Let a graph G and its complement $\overline{G}$ be connected with $\delta(G) \ge 2$. Then*

(i) $\gamma_{ve}^i(G) + \gamma_{ve}^i(\bar{G}) \le 2\left\lceil\frac{p}{2}\right\rceil$

(ii) $\gamma_{ve}^i(G) \cdot \gamma_{ve}^i(\bar{G}) \le \left\lceil\frac{p}{2}\right\rceil^2$

Proof: The result follows from Theorem 2.6.

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Source of support: Nil, Conflict of interest: None Declared.