International Journal of Mathematical Archive-8(2), 2017, 53-59
Available online through www.ijma.info ISSN 2229 – 5046

A MULTI-SERVER INFINITE CAPACITY MARKOVIAN FEEDBACK QUEUE WITH BALKING, RENEGING AND RETENTION OF RENEGED CUSTOMERS

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(Received On: 28-01-17; Revised & Accepted On: 13-02-17)

ABSTRACT

We consider a multi-server infinite capacity Markovian feedback queue with reneging, balking and retention of reneged customers in which the inter-arrival and service times follow exponential distribution. The reneging times are assumed to be exponentially distributed. After the completion of service, each customer may rejoin the system as a feedback customer for receiving another regular service with a certain probability. A reneged customer can be retained in many cases by employing certain convincing mechanisms to stay in queue for completion of service. Numerical analysis, cost-profit analysis and optimization of the cost function using simulated annealing method are carried out. The steady-state solution of the model is obtained iteratively. Some performance measures are also derived.

Keywords: *retention; reneged customers; queueing system; infinite capacity; feedback; balking; cost-profit analysis; simulated annealing.*

1. INTRODUCTION

Queueing theory is really a fascinating subject, with many applications. Yet, many businesses and organizations still do not understand it or may not be able to use it to the fullest extent. Telecommunication, traffic engineering, factories, shops, offices and hospitals among others can be effectively designed and analyzed using queueing models. Special attention is given to queues with customer impatience in the business world since it negatively affects the revenue generation of a business firm. An impatient customer is one who tends to join the queue only when a short wait is expected and tends to remain in the line if his wait has been sufficiently small. Generally, there are three forms of impatience. The first is balking, deciding not to join the queue at all upon arrival, the second is reneging, the reluctance to remain in the waiting line after joining and waiting, and the third is jockeying between lines when each of a number of parallel service channels has its own queue, see Gross and Harris [7]. Wang *et al.* [19] have made an extensive review on queueing systems with impatient customers. A single server finite capacity Markovian queueing model with retention of reneged customers and balking is proposed for any insurance firm facing the problem of customer impatience (reneging and balking) and implementing different customer retention strategies, Kumar *et al.* [14]. They survey various queueing systems with various dimensions like customer impatience behaviors, solution methods of queueing models with impatient customers and associated optimization aspects.

Feedback in queueing literature represents customer dissatisfaction because of unsuitable quality of service. In case of feedback, after getting partial or incomplete service, customer retries for service. Rework in production operations is an example of feedback. Recently, Kumar and Sharma [10] study multi-server Markovian queueing systems with balking, reneging and retention of reneged customers. Kumar [11] obtains the transient solution of an $M/M/c/N$ queueing system with balking, reneging and retention of reneged customers and performs cost-profit analysis of the model. We extend the work of Sharma and Kumar [13] by considering the infinite capacity case of the queueing system and by adding numerical analysis, the cost-profit analysis and optimization of the cost-profit functions using simulated annealing method. In addition, quality of service measures like average number of customers served, average rate of abandonment, average reneging rate, average retention rate etc. are obtained.

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In this paper, we consider a multi-server, infinite capacity Markovian feedback queueing system with balking, reneging and retention of reneged customers in which the inter-arrival and service times follow exponential distribution. The reneging times are assumed to be exponentially distributed. After the completion of service, each customer may rejoin the system as a feedback customer for receiving another regular service with some probability. A reneged customer can be retained in many cases by employing certain convincing mechanisms to stay in the queue for completion of his service. Thus, a reneged customer can be retained in the queuing system with some probability. This effort to retain the reneging customer in the queue for his service has positive effect on business of any firm.

2. LITERATURE REVIEW

The notion of customer impatience appeared in queueing theory in the work of Haight [8]. He studies $M/M/1$ queue with balking in which there is a greatest queue length at which an arrival will not balk. Haight [9] studies queueing with reneging. Ancker and Gafarian [1] study $M/M/1/N$ queueing system with balking and reneging and derive its steady-state solution. Ancker and Gafarian [2] obtain results for a pure balking system (no reneging) by setting the reneging parameterequal to zero. Bhupender Kumar Som [4] studies economic analysis of an input output Markovian queueing system with reverse balking and retention of reneged customers. Kumar and Sharma [15] study queueing with reneging, balking and retention of reneged customers. They also study [16] a Markovian feedback queue with retention of reneged customers and balking. Multi-server queueing systems with customer impatience find their applications in many real life situations such as in hospitals, computer communication, retail stores etc. Montazer-Haghighietal. [12], Abou-El-Ata and Hariri [3] and Zohar *et al.* [20] study and analyze some multi-server queueing systems with balking and reneging. Chauhan and Sharma [5] perform the profit analysis of $M/M/c$ queueing model with balking and reneging. Choudhury and Medhi [6] study multi-server Markovian queueing system with balking and reneging and discuss the design aspects associated with it. Takacs [18] studies queue with feedback to determine the stationary process for the queue size and the first two moments of the distribution function of the total time spent in the system by a customer. Santhakumaran and Thangaraj [17] consider a single-server feedback queue with impatient and feedback customers. They study an $M/M/1$ queueing model for queue length at arrival epochs and obtain results for stationary distribution, mean and variance of queue length.

3. STOCHASTIC QUEUEING MODEL DESCRIPTION

Let us consider a multi-server queueing system, wherein the arrivals occur in accordance with Poisson process with mean arrival rate λ. The inter-arrival times are independently, identically and exponentially distributed with parameter λ. There are c servers and the service times at each server are independently, identically and exponentially distributed with parameter μ. The mean service rate is given by $\mu_n = n\mu \; for \; n \le c \; and \; \mu_n = c\mu \; for \; n > c$. After completion of each service, the customer can either join (feedback) at the end of the queue with probability p_1 or he can leave the system with probability q_1 where $p_1 + q_1 = 1$. The customers, both newly arrived and those that are fedback are served in the order in which they join the tail of original queue. The queue discipline is first come first served, FCFS. We do not distinguish between the regular arrival and feedback arrival. The capacity of the system is taken as infinite. A queue gets developed when the number of customers exceeds the number of servers, that is, when n is greater than c. Each customer upon joining the queue will wait a certain length of time for his service to begin. If it does not begin by then, he reneges i.e., leaves the queue without getting service with probability p_2 or may remain in the queue for his service with probability $q_2 (= 1 - p_2)$. The reneging times follow exponential distribution with parameter ξ. The arriving customer joins the system with certain balking probability p_3 $(= 1 - q_3)$ where q_3 is the probability that the customer may balk. It is assumed that a customer could possibly balk or renege only if the number of customers in the system is greater or equal to the number of servers c.

4. MATHEMATICAL FORMULATION OF THE MODEL

In this section, we present a mathematical model consisting of differential-difference equations. These equations are derived by using the general birth-death arguments based on familiar Markov theory. Define $P_n(t)$, $n > 0$, be the probability that there are n customers in the system at time t. In an infinitesimally small interval $(t, t + \delta)$, $P_n(t + \delta t)$ = Prob {there are n customers in the system at time $(t + \delta t)$}. The Kolmogorov's forward differential-difference equations of the model can be written as:

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu q_1 P_1(t), \tag{1}$$

$$\frac{dP_n(t)}{dt} = -(\lambda + n\mu q_1)P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu q_1 P_{n+1}(t), \qquad 1 \le n \le c-1 \tag{2}$$

$$\frac{dP_c(t)}{dt} = -(q_3\lambda + c\mu q_1)P_c(t) + \lambda P_{c-1}(t) + (c\mu q_1 + \xi p_2)P_{c+1}(t), \tag{3}$$

$$\frac{dP_n(t)}{dt} = -(q_3\lambda + c\mu q_1 + (n-c)\xi p_2)P_n(t) + \lambda q_3 P_{n-1}(t) + (c\mu q_1 + (n+1-c)\xi p_2)P_{n+1}(t), \;\; n \ge c+1 \tag{4}$$

5. STEADY-STATE SOLUTION OF THE MODEL

In this section, we obtain the steady-state solution of the model using iterative method. In steady-state, $\lim_{n\to\infty} P_n(t) = P_n$ and $\lim_{n\to\infty} \frac{dP_n(t)}{dt} = 0$. Therefore, the steady-state equations corresponding to equations (1) to (4) are as follows:

$$0 = -\lambda P_0 + \mu q_1 P_1, \tag{5}$$

$$0 = -(\lambda + n\mu q_1)P_n + \lambda P_{n-1} + (n+1)\mu q_1 P_{n+1}, \quad 1 \le n \le c-1 \tag{6}$$

$$0 = -(q_3\lambda + c\mu q_1)P_c + \lambda P_{c-1} + (c\mu q_1 + \xi p_2)P_{c+1}, \tag{7}$$

$$0 = -(q_3\lambda + c\mu q_1 + (n-c)\xi p_2)P_n + \lambda q_3 P_{n-1} + (c\mu q_1 + (n+1-c)\xi p_2)P_{n+1}, \quad n \ge c+1 \tag{8}$$

From equations (5) and (6) we get

$$P_1 = \frac{\lambda}{\mu q_1} P_0 \text{ and } P_2 = \frac{1}{2}\left(\frac{\lambda}{\mu q_1}\right)^2 P_0.$$

Proceeding iteratively in the same way, we get

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu q_1}\right)^n P_0, \quad for\ 1 \le n \le c.$$

We also have from equation (7) and (8)

$$(c\mu q_1 + \xi p_2)P_{c+1} = (q_3\lambda + c\mu q_1)P_c - \lambda P_{c-1},$$

which gives

$$P_{c+1} = \left(\frac{\lambda q_3}{c\mu q_1 + \xi p_2}\right) P_c \text{and} P_{c+2} = \frac{(\lambda q_3)^2}{(c\mu q_1 + \xi p_2)(c\mu q_1 + 2\xi p_2)} P_c.$$

Proceeding iteratively in the same way, we get

$$P_{c+r} = \prod_{k=1}^{r} \frac{\lambda q_3}{c\mu q_1 + k\xi p_2} P_c, \quad r \ge 1,$$

Thus, the steady-state solution of the model is given by:

$$P_n = \begin{cases} \frac{1}{n!}\left(\frac{\lambda}{\mu q_1}\right)^n P_0 & : 1 \le n \le c \\ \prod_{k=c+1}^{n} \frac{\lambda q_3}{c\mu q_1 + (k-c)\xi p_2} \frac{1}{c!}\left(\frac{\lambda}{\mu q_1}\right)^c P_0 & : n \ge c+1. \end{cases}$$

Using the normalization condition $\sum_{n=0}^{\infty} P_n = 1$, we get

$$P_0 = \frac{1}{(1 + M_1 + M_2)},$$

where

$$M_1 = \sum_{n=1}^{c} \frac{1}{n!}\left(\frac{\lambda}{\mu q_1}\right)^n \text{and } M_2 = \sum_{n=c+1}^{\infty} \prod_{k=c+1}^{n} \frac{\lambda q_3}{c\mu q_1 + (k-c)\xi p_2} \frac{1}{c!}\left(\frac{\lambda}{\mu q_1}\right)^c.$$

Hence the steady-state probabilities of the system size are derived explicitly.

6. MEASURES OF PERFORMANCE

- The expected system size (L_s) is:

$$L_s = \sum_{n=1}^{\infty} nP_n$$

$$L_s = \sum_{n=1}^{c} n\frac{1}{n!}\left(\frac{\lambda}{\mu q_1}\right)^n P_0 + \sum_{n=c+1}^{\infty} n \prod_{k=c+1}^{n} \frac{\lambda q_3}{c\mu q_1 + (k-c)\xi p_2} \frac{1}{c!}\left(\frac{\lambda}{\mu q_1}\right)^c P_0.$$

- The expected number of customers served ($E(C.S)$) is:

$$E(C.S) = \sum_{n=1}^{c} n\mu q_1 P_n + \sum_{n=c+1}^{\infty} c\mu q_1 P_n$$

$$E(C.S) = \sum_{n=1}^{c} n\mu q_1 \frac{1}{n!}\left(\frac{\lambda}{\mu q_1}\right)^n P_0 + \sum_{n=c+1}^{\infty} c\mu q_1 \prod_{k=c+1}^{n} \frac{\lambda q_3}{c\mu q_1 + (k-c)\xi p_2} \frac{1}{c!}\left(\frac{\lambda}{\mu q_1}\right)^c P_0.$$

- Rate of abandonment (R_{aband}) is:

$$R_{aband} = \lambda - \sum_{n=1}^{c} n\mu q_1 P_n - \sum_{n=c+1}^{\infty} c\mu q_1 P_n$$

$$R_{aband} = \lambda - \sum_{n=1}^{c} n\mu q_1 \frac{1}{n!}\left(\frac{\lambda}{\mu q_1}\right)^n P_0 - \sum_{n=c+1}^{\infty} c\mu q_1 \prod_{k=c+1}^{n} \frac{\lambda q_3}{c\mu q_1 + (k-c)\xi p_2} \frac{1}{c!}\left(\frac{\lambda}{\mu q_1}\right)^c P_0.$$

- Average reneging rate (R_r) is:

$$R_r = \sum_{n=c+1}^{\infty} (n-c)\xi p_2 P_n$$

$$R_r = \sum_{n=c+1}^{\infty}(n-c)\xi p_2 \prod_{k=c+1}^{n} \frac{\lambda q_3}{c\mu q_1+(k-c)\xi p_2} \frac{1}{c!}\left(\frac{\lambda}{\mu q_1}\right)^c P_0.$$

- Average balking rate (R_b) is:

$$R_b = \sum_{n=c+1}^{\infty} \lambda p_3 P_n$$

$$R_b = \sum_{n=c+1}^{\infty} \lambda p_3 \prod_{k=c+1}^{n} \frac{\lambda q_3}{c\mu q_1+(k-c)\xi p_2} \frac{1}{c!}\left(\frac{\lambda}{\mu q_1}\right)^c P_0.$$

- The average retention rate (R_R) is:

$$R_R = \sum_{n=c+1}^{\infty} (n-c)\xi q_2 P_n$$

$$R_R = \sum_{n=c+1}^{\infty}(n-c)\xi q_2 \prod_{k=c+1}^{n} \frac{\lambda q_3}{c\mu q_1+(k-c)\xi p_2} \frac{1}{c!}\left(\frac{\lambda}{\mu q_1}\right)^c P_0.$$

7. NUMERICAL DISCUSSION

In the tables below, we have presented numerical results of all measures of performance. Numerical results are obtained for various service and arrival rates. As can be seen from Table 1, the expected number of customers in the system ($E(C.S)$) and the probability of no customers in the system (P_0) increase with the increasein the service rate, whereas all the other performance measures decrease with the increase in the rate of serviceμ. Here, we are assuming the following constant values for the other parameters, λ = 20; ξ = 0.6; q_3 = 0.7; q_1 = 0.5; p_2 = 0.2; and c = 10.

μ	P_0	L_s	$E(C.S)$	R_r	R_R	R_b
5	0.00035	7.70807	18.54300	0.03490	0.13961	1.8386
6	0.00130	6.53006	19.29150	0.01194	0.04779	0.75293
7	0.00333	5.65325	19.65610	0.00446	0.01786	0.31501
8	0.00676	4.97255	19.83060	0.00178	0.00715	0.13627
9	0.01176	4.43177	19.91450	.00076	0.00304	0.06129
10	0.01833	3.99396	19.95560	0.00034	0.00136	0.02870
11	0.02635	3.63338	19.97620	0.00016	0.00064	0.01398
12	0.03568	3.33181	19.98690	0.00007	0.00031	0.00707
13	0.04610	3.07612	19.99260	0.00004	0.00016	0.00370
14	0.05743	2.85670	19.99570	0.00002	0.00008	0.00200

Table-1: Effect of μ on the various measures of performance

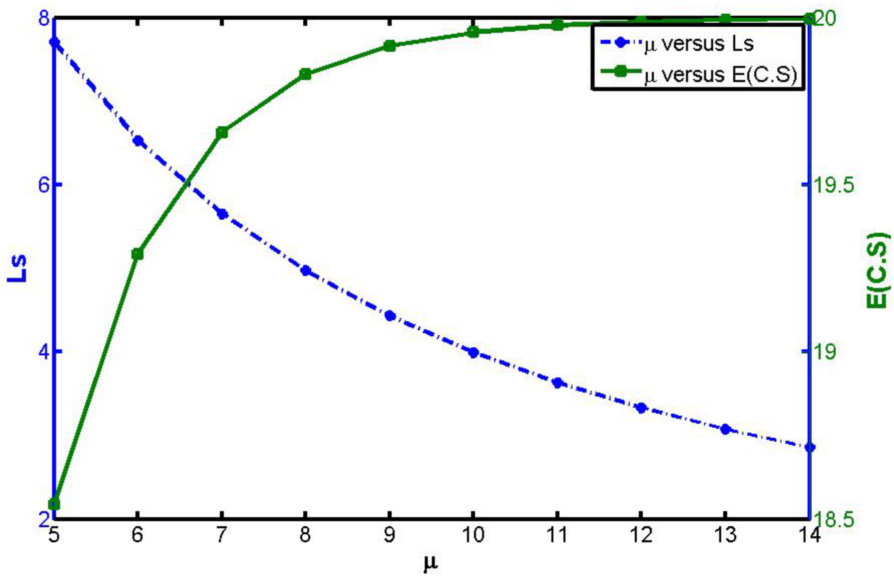

Figure-1: Effect of μ on the expected system size(L_s) and the expected number of customers served, $E(C.S)$

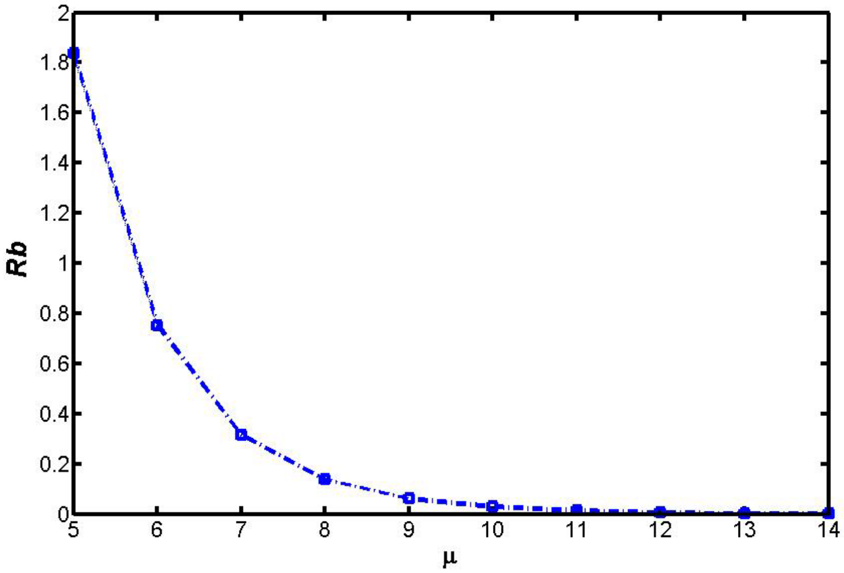

Figure-2: Effect of μ on the average rate of balking, R_b.

From Table 2, we see that with the increase in the rate of arrival λ, the size of the system increases rapidly. As a result both the reneging rate and the balking rate increase due to the long queue. On the other hand, the expected number of customers served does not make a significant increase since it is mainly dependent on the service rate than the arrival rate of the system. Here also we assume that $\mu = 3$; $\xi = 0.6$; $q_3 = 0.7$; $q_1 = 0.5$; $p_2 = 0.2$; and $c = 10$. The effect of the rate of service on the expected number of customers served and the length of the systemis shown graphically in Figure 1. The average rate of balking is also shown to be a decreasing function of μ in Figure 2.

λ	L_s	$E(C.S)$	R_r	R_b
20	14.6181	14.4435	0.5986	10.2383
21	16.6069	14.6308	0.8223	11.8778
22	19.2171	14.7745	1.1241	13.4622
23	22.5778	14.8761	1.5192	14.9103
24	26.7316	14.9403	2.0125	16.1625
25	31.5873	14.9753	2.5924	17.2078
26	36.9449	14.9913	3.2340	18.0868
27	42.5868	14.9974	3.9106	18.8630
28	48.3563	14.9993	4.6028	19.5898
29	54.1719	14.9999	5.3006	20.2976

Table-2: Effect of λ on the various measures of performance

8. COST-PROFIT ANALYSIS AND OPTIMIZATION INVESTIGATION

Simulated annealing is a probabilistic technique for approximating the global optimum of a given function. The name and inspiration came from annealing in metallurgy, a technique involving heating and controlled cooling of a material to increase the size of its crystals and reduce their defects. Simulated annealing interprets slow cooling as a slow decrease in the probability of accepting worse solutions as it explores the solution space. A brief description of the algorithm is given below.

The Simulated annealing algorithm

Step-1: Initialize the iteration count $i = 1$ with an initial design X_1 and a high value of the temperature T (in our case X_i is replaced by the service rate μ_i.

Step-2: Generate a new design point X_{i+1} randomly in the vicinity of the current design point X_i and the difference in function values $\Delta f = f(X_{i+1}) - f(X_i)$, where f can be taken as the total expected cost for our purpose.

Step-3: If Δf is negative, X_{i+1} is taken as the next design point and go to *step 5*.

Step-4: If Δf is positive, X_{i+1} is taken as the next design point with a probability $(\frac{-\Delta f}{kT})$, where k is the Boltzmann constant and can be chosen as 1 for simplicity. Go to *step 5*.

Step-5: Terminate the procedure if the current value of the temperature or the changes in the function value Δf are observed to be sufficiently small. Else *step 6*.

Step-6: Decrease the temperature and go to *step 2*.

The cost analysis of the model is presented by developing cost functions. We develop the total expected cost function (TEC) as:

$$TEC = C_s \mu q_1 + C_h L_s + C_r R_r + C_t R_R + C_b R_b;$$

the total expected revenue (TER) and total expected profit (TEP) functions are given respectively as:

$$TER = R \times E(C.S) \text{ and } TEP = TER - TEC$$

Assuming the values of the constant coefficients in the cost functions as $C_s = 6, C_h = 25, C_r = 40, C_t = 25, C_b = 35$ and $R = 50$ and varying the rate ofservice μ, we see from Table 3 that the minimum value of the total expectedcost function and the maximum value of the total expected profit function is at avalue close to $\mu = 17$. This is in agreement with the optimum value of μ that is computed through simulated annealing method by using the Mathematica software. The result from the simulated annealing method gives the optimum value of the service rate μ approximately 18.3. Corresponding to this optimum μ the minimumtotal expected cost is approximately 109.5 and the maximum total expected profitisapproximately 890.4. Figure 3 shows the corresponding graphical representation.

μ	TEC	TER	TEP
12	119.554	999.345	879.791
13	116.038	999.629	883.591
14	113.491	999.784	886.293
15	111.701	999.871	888.170
16	110.520	999.921	889.401
17	109.835	999.951	890.116
18	109.563	999.969	890.406
19	109.636	999.980	890.344
20	110.003	999.987	889.984
21	110.621	999.991	889.370
22	111.456	999.994	888.538
23	112.479	999.996	887.517

Table-3: Effect of μ on the total expected cost, revenue and profit functions

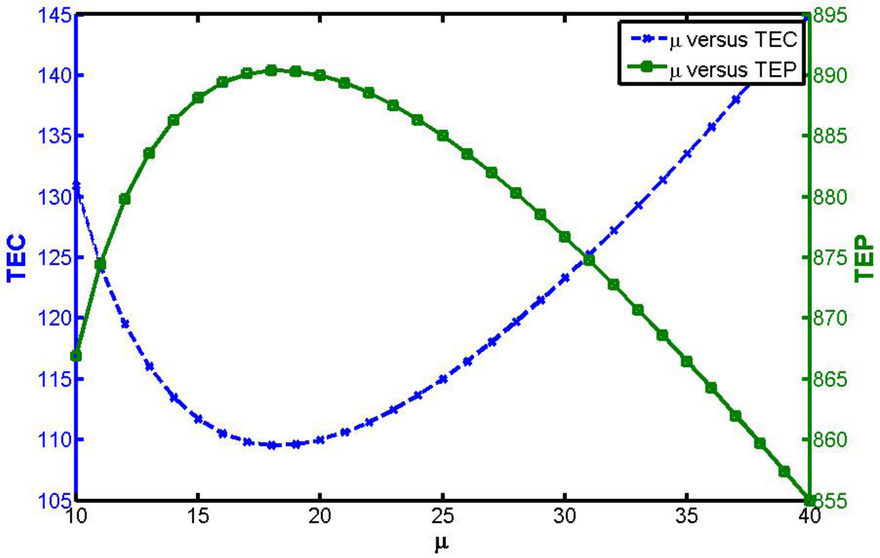

Figure-3: Effect of μ on the total expected cost and total expected profit functions

9. CONCLUSION

In this paper, we study an infinite capacity multi-server Markovian feedback queuing model with balking, reneging and retention of reneged customers. The steady-state solution of the model is obtained and some quality of service measures are also derived. The model results may be useful in modeling various production and service processes involving feedback and impatient customers. The model can also be solved in transient state to get time-dependent results.

The cost-profit analysis of the model is performed and the impact of various rates of service on the total expected cost of the system is studied. Tabular descriptions are also presented to show these effects. The optimization of the model is performed using the simulated annealing method in order to minimize the total expected cost of the system with respect to the service rate. The analysis carried out in this paper is very useful to any business firm, as it helps to choose the best service rate thatis used to optimize the profit of the firm. The model can further be optimized for other parameters available.

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Source of support: Nil, Conflict of interest: None Declared.