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NUMERICAL SOLUTION OF FUZZY DIFFERENTIAL EQUATIONS BY TAYLOR METHOD

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ABSTRACT

In this paper, we present a representation of fuzzy numbers to the numerical solution of fuzzy initial value differential equations.

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Keywords: *Fuzzy differential equations, Taylor method, Trapezoidal fuzzy number, Parallelogram fuzzy number.*

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1. INTRODUCTION

Fuzziness is a basic type of uncertainty in real world. For a deterministic differential equation, the easiest way to introduce fuzziness is to assume that the initial value is a fuzzy variable. Such an equation was first called fuzzy differential equation by Kandel and Byatt. Fuzzy differential equations and initial value problems were regularly treated by O.Kaleva in [9] and [10], S.Seikkala .The organized paper is as follows: In the first three sections, we recall some concepts and introductory materials to deal with the fuzzy initial value problem. Solving numerically the fuzzy differential equation by Taylor method is discussed in section 4. The proposed algorithm is illustrated by an example in the last section.

2. PRELIMINARY

A trapezoidal and parallelogram fuzzy number u is defined by four real numbers $a < b < c < d$, where the base of the trapezoidal and parallelogram are the interval [a, d] and its vertices at $x = b$, $x = c$. Trapezoidal and parallelogram fuzzy numbers will be written as $u = (a, b, c, d)$.The membership function for the trapezoidal and parallelogram fuzzy number $u = (a, b, c, d)$ is defined as the following:

$$u(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & b \le x \le c \\ \dfrac{x-d}{c-d}, & c \le x \le d \end{cases} \qquad (1)$$

we will have : $u > 0$ if $a > 0$; $u > 0$ if $b > 0$; $u > 0$ if $c > 0$; & $u > 0$ if $d > 0$.

Let us denote R_F by the class of all fuzzy subsets of R (i.e. $u : R \to [0,1]$) satisfying the following properties:

(i) $\forall u \in R_F$, u is normal, i.e. $\exists x_0 \in R$ with $u(x_o) = 1$;
(ii) $\forall u \in R_F$, u is convex fuzzy set (i.e. $u(tx + (1-t)y) \ge \min\{u(x), u(y)\}$, $\forall t \in [0,1]$, $x, y \in R$);
(iii) $\forall u \in R_F$, u is upper semi continuous on R;
(iv) $\{\overline{x \in R; u(x) > 0}\}$ is compact, where $\overline{A}$ denotes the closure of A.

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Then R_F is called the space of fuzzy numbers Obviously $R \subset R_F$. Here
$R \subset R_F$ is understood as $R = \{\chi_{\{x\}}; x \text{ is usual real number}\}$ .We define the r-level set, $x \in R$;

$$[u]_r = \{x \setminus u(x) \geq r\}, \qquad 0 \leq r \leq 1; \tag{2}$$

$$\text{Clearly,} \quad [u]_0 = \{x \setminus u(x) > 0\} \quad \text{is compact,}$$

which is a closed bounded interval and we denote by $[u]_r = [\underline{u}(r), \overline{u}(r)]$. It is clear that the following statements are true,

1. $\underline{u}(r)$ is a bounded left continuous non decreasing function over [0,1],
2. $\overline{u}(r)$ is a bounded right continuous non increasing function over [0,1],
3. $\underline{u}(r) \leq \overline{u}(r)$ for all $r \in (0,1]$, for more details see [2],[3].

Let $D: R_F \times R_F \rightarrow R_+ \cup \{0\}$, $D(u,v) = \text{Sup}_{r\in[0,1]} \max \{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$, be *Hausdorff distance* between fuzzy numbers, where $[u]_r = [\underline{u}(r), \overline{u}(r)]$, $[v]_r = [\underline{v}(r), \overline{v}(r)]$. The following properties are well-known :

$$D(u + w, v + w) = D(u, v), \ \forall\ u,v,w \in R_F,$$
$$D(k.u, k.v) = |k| D(u, v), \ \forall\ k \in R,\ u,v \in R_F,$$
$$D(u + v, w + e) \leq D(u,w) + D(v,e), \ \forall\ u, v, w, e \in R_F$$

and (R_F, D) is a complete metric space.

3 FUZZY INITIAL VALUE PROBLEM

Consider a first-order fuzzy initial value differential equation is given by

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [t_0, T] \\ y(t_0) = y_0 \end{cases} \tag{3}$$

where y is a fuzzy function of t, $f(t, y)$ is a fuzzy function of the crisp variable t and the fuzzy variable y , y' is the fuzzy derivative of y and $y(t_0) = y_0$ is the trapezoidal & parallelogram shaped fuzzy number.
We denote the fuzzy function y by $y = [\underline{y}, \overline{y}]$. It means that the r-level set of y(t) for
$t \in [t_0, T]$ is

$$[y(t)]_r = [\underline{y}(t; r), \overline{y}(t; r)], \qquad [y(t_0)]_r = [\underline{y}(t_0; r), \overline{y}(t_0; r)], \ r \in (0,1]$$

we write $f(t, y) = [\underline{f}(t, y), \overline{f}(t, y)]$ and

$$\underline{f}(t, y) = F[t, \underline{y}, \overline{y}], \qquad \overline{f}(t, y) = G[t, \underline{y}, \overline{y}].$$

Because of $y' = f(t, y)$ we have

$$\underline{f}(t, y(t); r) = F[t, \underline{y}(t; r), \overline{y}(t; r)] \tag{4}$$

$$\overline{f}(t, y(t); r) = G[t, \underline{y}(t; r), \overline{y}(t; r)] \tag{5}$$

By using the extension principle, we have the membership function

$$f(t, y(t))(s) = \sup\{y(t)(\tau) \setminus s = f(t, \tau)\}, \ s \in R \tag{6}$$

so fuzzy number $f(t, y(t))$. From this it follows that

$$[f(t, y(t))]_r = [\underline{f}(t, y(t); r), \overline{f}(t, y(t); r)], \qquad r \in (0,1], \tag{7}$$

where

$$\underline{f}(t, y(t); r) = \min\{f(t, u) | u \in [y(t)]_r\} \tag{8}$$

$$\overline{f}(t, y(t); r) = \max\{f(t, u) | u \in [y(t)]_r\}. \tag{9}$$

4 TAYLOR METHOD FOR SOLVING FUZZY DIFFERENTIAL EQUATION

Let $Y = [\underline{Y}, \overline{Y}]$ be the exact solution and $y = [\underline{y}, \overline{y}]$ be the approximated solution of the fuzzy initial value problem (3).

Let $[Y(t)]_r = [\underline{Y}(t;r), \overline{Y}(t;r)]$, $[y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)]$.

Throughout this argument, the value of r is fixed. Then the exact and approximated solution at t_n are respectively denoted by

$$[Y(t_n)]_r = [\underline{Y}(t_n;r), \overline{Y}(t_n;r)], \quad [y(t_n)]_r = [\underline{y}(t_n;r), \overline{y}(t_n;r)] \quad (0 \le n \le N).$$

The grid points at which the solution is calculated are

$$h = \frac{T - t_0}{N}, t_i = t_0 + ih, \quad 0 \le i \le N.$$

Then we obtain,

$$\underline{Y}(t_{n+1};r) = \underline{Y}(t_n;r) \sum_{k=0}^{P} \frac{h^k}{k!} \tag{10}$$

and

$$\overline{Y}(t_{n+1};r) = \overline{Y}(t_n;r) \sum_{k=0}^{P} \frac{h^k}{k!} \tag{11}$$

Also we have

$$\underline{y}(t_{n+1};r) = \underline{y}(t_n;r) \sum_{k=0}^{P} \frac{h^k}{k!} \tag{12}$$

and

$$\overline{y}(t_{n+1};r) = \overline{y}(t_n;r) \sum_{k=0}^{P} \frac{h^k}{k!} \tag{13}$$

Clearly, $\underline{y}(t;r)$ and $\overline{y}(t;r)$ converge to $\underline{Y}(t;r)$ and $\overline{Y}(t;r)$, respectively whenever $h \to 0$ [4].

5 NUMERICAL RESULTS

In this section, the exact solutions and approximated solutions are obtained by Taylor method are plotted in Figures 1, 2 , 3 , 4 , 5 & 6.

Example: 5.1

Consider the initial value problem [11]

$$\begin{cases} y'(t) = f(t), & t \in [0,1] \\ y(0) = [(0.8 + 0.125r, 1.1 - 0.1r), (0.8 + 0.125r, 0.95 + 0.125r)]. \end{cases}$$

The exact solution at $t = 1$ is given by

$$Y(1;r) = [(0.8 + 0.125r)e, (1.1 - 0.1r)e], \quad 0 \le r \le 1.$$

and $Y(1;r)=[(0.8+0.125r)e,\ (0.95+0.125r)e]$, $0\le r\le 1$.

Using iterative solution of Taylor method, we have

$$\underline{y}(0;r) = 0.8+0.125r,\ \overline{y}(0;r) = 1.1-0.1r,$$

and $\underline{y}(0;r) = 0.8+0.125r$, $\overline{y}(0;r) = 0.95 + 0.125r$,

and by

$$\underline{y}^{(0)}(t_{i+1};r) = \underline{y}(t_i;r)+h\,\underline{y}(t_i;r)+\frac{h^2}{2}\,\underline{y}(t_i;r)$$

$$\overline{y}^{(0)}(t_{i+1};r) = \overline{y}(t_i;r)+h\,\overline{y}(t_i;r)+\frac{h^2}{2}\,\overline{y}(t_i;r),$$

where $i = 0, 1, \ldots, N-1$ and $h = \frac{1}{N}$. Now, using these equations as an initial guess for following iterative solutions respectively,

$$\underline{y}(t_{i+1};r)= \underline{y}(t_i;r)\ \sum_{k=0}^{P} \frac{h^k}{k!}$$

$$\overline{y}(t_{i+1};r)= \overline{y}(t_i;r)\ \sum_{k=0}^{P} \frac{h^k}{k!}$$

and $j = 1, 2, 3$. Thus, we have $\underline{y}(t_i;r) = \underline{y}^{(3)}(t_i;r)$ and $\overline{y}(t_i;r)= \overline{y}^{(3)}(t_i;r)$, for $i = 1\ldots N$. Therefore, $\underline{Y}(1;r) \approx \underline{y}^{(3)}(1;r)$ and $\overline{Y}(1;r)\approx \overline{y}^{(3)}(1;r)$ are obtained.

Table 2 & 3, shows estimation of approximated solution and error for different values of
$r\in[0,1]$ and h

Table 1: Exact solution

Taylor method		
r	Trapezoidal fuzzy number	Parallelogram fuzzy number
0	2.174625 , 2.990110	2.174625 , 2.582368
0.1	2.208604 , 2.962927	2.208604 , 2.616346
0.2	2.242583 , 2.935744	2.242583 , 2.650325
0.3	2.276561 , 2.908562	2.276561 , 2.684303
0.4	2.310540 , 2.881379	2.310540 , 2.718282
0.5	2.344518 , 2.854196	2.344518 , 2.752260
0.6	2.378497 , 2.827013	2.378497 , 2.786239
0.7	2.412475 , 2.799830	2.412475 , 2.820217
0.8	2.446454 , 2.772647	2.446454 , 2.854196
0.9	2.480432 , 2.745465	2.480432 , 2.888174
1	2.514411 , 2.718282	2.514411 , 2.922153

Table 2: Approximated solution for different values of r & *h*

Approximated solution						
Taylor method						
	Trapezoidal fuzzy number			Parallelogram fuzzy number		
h / *r*	0.03	0.02	0.01	0.03	0.02	0.01
0	2.152675 , 2.959927	2.174484 , 2.989915	2.174577 , 2.990042	2.152675 , 2.556300	2.174484 , 2.582200	2.174577 , 2.582309
0.1	2.186309 , 2.933018	2.208460 , 2.962736	2.208555 , 2.962861	2.186309 , 2.589936	2.208460 , 2.616176	2.208555 , 2.616288
0.2	2.219945 , 2.906110	2.242436 , 2.935553	2.242532 , 2.935677	2.219945 , 2.623572	2.242436 , 2.650152	2.242532 , 2.650266
0.3	2.253581 , 2.879201	2.276413 , 2.908374	2.276511 , 2.908496	2.253581 , 2.657207	2.276413 , 2.684129	2.276511 , 2.684242
0.4	2.287217 , 2.852293	2.310389 , 2.881191	2.310488 , 2.881313	2.287217 , 2.690843	2.310389 , 2.718105	2.310488 , 2.718220
0.5	2.320852 , 2.825385	2.344366 , 2.854010	2.344465 , 2.854133	2.320852 , 2.724478	2.344366 , 2.752081	2.344465 , 2.752199
0.6	2.354487 , 2.798477	2.378343 , 2.826830	2.378443 , 2.826948	2.354487 , 2.758114	2.378343 , 2.786058	2.378443 , 2.786175
0.7	2.388123 , 2.771568	2.412318 , 2.799649	2.412420 , 2.799768	2.388123 , 2.791749	2.412318 , 2.820035	2.412420 , 2.820152
0.8	2.421759 , 2.744660	2.446294 , 2.772467	2.446398 , 2.772586	2.421759 , 2.825385	2.446294 , 2.854010	2.446398 , 2.854133
0.9	2.455394 , 2.717751	2.480272 , 2.745286	2.480376 , 2.745403	2.455394 , 2.859020	2.480272 , 2.887987	2.480376 , 2.888110
1	2.489029 , 2.690843	2.514247 , 2.718105	2.514354 , 2.718220	2.489029 , 2.892656	2.514247 , 2.921963	2.514354 , 2.922087

Table 3: **Error for different values of r & *h***

Taylor method						
	Trapezoidal fuzzy number			Parallelogram fuzzy number		
r \ *h*	0.03	0.02	0.01	0.03	0.02	0.01
0	0.052133	0.000336	0.000116	0.048018	0.000309	0.000107
0.1	0.052204	0.000335	0.000115	0.048705	0.000314	0.000107
0.2	0.052272	0.000338	0.000118	0.049391	0.000320	0.000110
0.3	0.052341	0.000336	0.000116	0.050076	0.000322	0.000111
0.4	0.052409	0.000339	0.000118	0.050762	0.000328	0.000114
0.5	0.052477	0.000338	0.000116	0.051448	0.000331	0.000114
0.6	0.052546	0.000337	0.000119	0.052135	0.000335	0.000118
0.7	0.052614	0.000338	0.000117	0.052820	0.000339	0.000120
0.8	0.052682	0.000340	0.000117	0.053506	0.000346	0.000119
0.9	0.052752	0.000339	0.000118	0.054192	0.000347	0.000120
1	0.052821	0.000341	0.000119	0.054879	0.000354	0.000123

Graphical Representation of exact and approximated solution

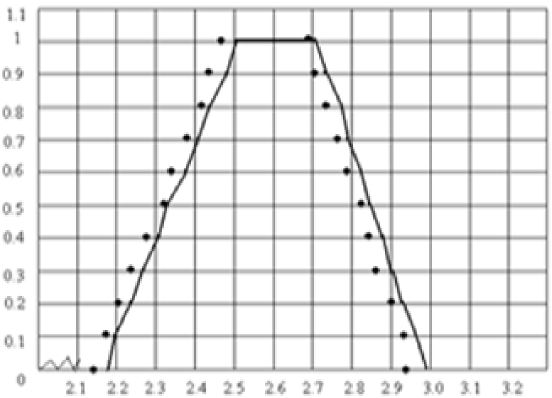

Figure 1 : h = 0.03

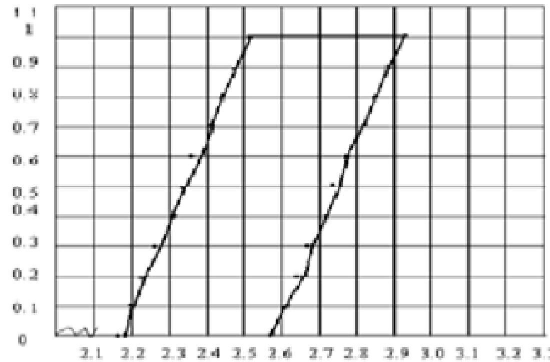

Figure 4 : h = 0.02

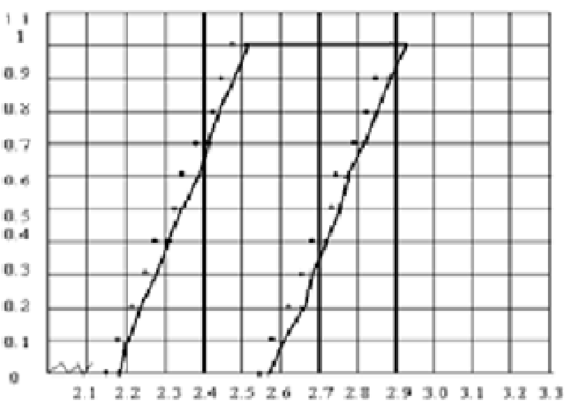

Figure 2 : h = 0.03

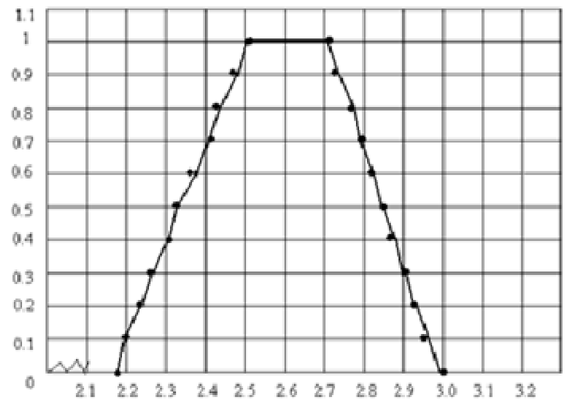

Figure 5 : h = 0.01

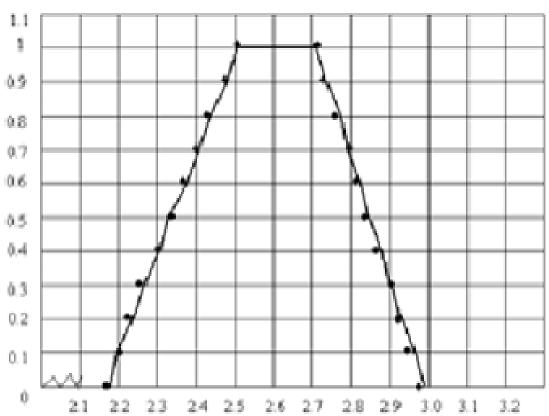

Figure 3 : h = 0.02

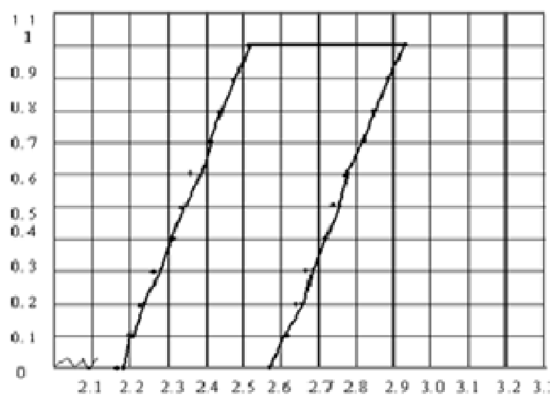

Figure 6 : h = 0.01

CONCLUSION:

By minimizing the step size *h*, the solution by exact method and Taylor method almost coincides. Comparing trapezoidal and parallelogram fuzzy number, parallelogram fuzzy number is quickly converges than trapezoidal fuzzy number

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