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GENERALIZED FRACTIONAL ORDER EOQ MODEL WHEN DEMAND IS STOCK DEPENDENT

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ABSTRACT

In this article, we have developed a fractional order EOQ model, where demand is not assumed to be constant or time dependent. In many classical EOQ model it has been taken that demand may occur as stock dependent. Our objective in this article is to describe a generalized fractional order EOQ model with stock dependent demand.

Keywords*—Fractional differentiation, Fractional Integration, Fractional Differential Equation, Set up Cost, Holding Cost, Economic Order Quantity.*

1. INTRODUCTION

In recent years considerable interest in fractional calculus has been stimulated by the applications it finds in different areas of applied sciences like physics and engineering, possibly including fractal phenomena. Now there are more books of proceedings and special issues of journals published that refer to the applications of fractional calculus in several scientific areas including special functions, control theory, chemical physics, stochastic processes, anomalous diffusion, archeology. Several special issues appeared in the last decade which contain selected and improved papers presented at conferences and advanced schools, concerning various applications of fractional calculus. Already since several years, there exist two international journals devoted almost exclusively to the subject of fractional calculus: Journal of Fractional Calculus (Editor-in-Chief: K.Nishimoto, Japan) started in 1992, and Fractional Calculus and Applied Analysis (Managing Editor: V. Kiryakova, Bulgaria) started in 1998. Recently the new journal Fractional Dynamic Systems has been announced to start in 2010. The authors believe that the volume of research in the area of fractional calculus will continue to grow in the forthcoming years and that it will constitute an important tool in the scientific progress of mankind.

Only recently, fractional calculus was applied to classical EOQ model to generalize this model in operation research. In a previous papers [4-5] we have discussed how the fractional calculus can utilizes to develop the classical EOQ model to generalize EOQ model in operation research. In particular, we have seen fractional calculus has a potentiality to apply this concept in any other EOQ model. In this sense we represent the more generalize EOQ model using the broad concept of fractional calculus where the model is based on stock dependent demand.

Here we have applied the concept of derivative/integrals with an emphasis on Caputo and Riemann-Liouville fractional derivatives [2], [13] and have some interesting results and ideas [23] that demonstrate the generalized EOQ based inventory model. Fractional derivatives and fractional integrals have interesting mathematical properties that may be utilized to develop our motivation. In this article, first we give a short description on general principles, definitions and several features of fractional derivatives/integrals and then we review some of our ideas and findings in exploring potential applications of fractional calculus in inventory control model.

II. A SHORT DESCRIPTION ON FRACTIONAL DIFFERENTIAL CALCULUS

The origin of fractional calculus goes back to Newton and Leibniz in the seventieth century. S.F Lacroix was the first to mention in some two pages a derivative of arbitrary order in a 700 pages text book of 1819.

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He developed the formula for the nth derivative of y= x^m, m is a positive integer,

$$D^n y = \frac{m!}{(m-n)!}\, x^{m-n}$$, where n(≤m) is an integer. (2.1)

Replacing the factorial symbol by the well-known Gamma function, he obtained the formula for the fractional derivative,

$$D^\alpha\left(x^\beta\right) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \tag{2.2}$$

Where α, β are fractional numbers.

In particular he had, $D^{1/2}(x) = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})} x^{1/2} = 2\sqrt{\frac{x}{\pi}}$. (2.3)

Again the normal derivative of a function f is defined as,

$$D^1 f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}, \tag{2.4}$$

And

$$D^2 f\left(x\right) = \lim_{h\to 0} \frac{f^1(x+h)-f^1(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+2h)-f(x+h)+f(x)}{h}.$$

Iterating this operation yields an expression for the nth derivative of a function. As can be easily seen and proved by induction for any natural number n,

$$D^n f\left(x\right) = \lim_{h\to 0} h^{-n} \sum_{r=0}^{n} (-1)^n \binom{n}{r} \text{f(x+(n-r)h)}. \tag{2.5}$$

Where $$\binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{2.6}$$

Or equivalently, $$D^n f(x) = \lim_{h\to 0} h^{-n} \sum_{r=0}^{n} (-1)\binom{n}{r}^r f(x-rh) \tag{2.7}$$

The case of n=0 can be included as well.

The fact that for any natural number n, the calculation of nth derivative is given by an explicit formula (2.5) or (2.7).

Now the generalization of the factorial symbol (!) by the gamma function allows

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)} \tag{2.8}$$

This is also valid for non-integer values of n.

Thus on using of the idea (2.8), fractional derivative leads as the limit of a sum given by

$$D^\alpha f(x) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{r=0}^{n} (-1)^r \frac{\Gamma(\alpha+1)}{\Gamma(r+1)\Gamma(\alpha-r+1)} f(x-rh). \tag{2.9}$$

Provided the limit exists. Using the identity $\left(-1\right)^r \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-r+1)} = \frac{\Gamma(r-\alpha)}{\Gamma(-\alpha)}$ (2.10)

The result (2.9) becomes, $D^\alpha f(x) = \lim_{h\to 0} \frac{h^{-\alpha}}{\Gamma(-\alpha)} \sum_{r=0}^{n} \frac{\Gamma(r-\alpha)}{\Gamma(r+1)} f(x-rh)$ (2.11)

When α is an integer, the result (2.9) reduce to the derivative of integral order n as follows in (2.5).

Again in 1927 Marchaud formulated the fractional derivative of arbitrary order α in the form given by,

$$D^{\alpha} f(x) = \frac{f(x)}{\Gamma(1-\alpha)x^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^x \frac{f(x)-f(t)}{(x-t)^{\alpha+1}} dt, \quad \text{Where } 0 < \alpha < 1 \qquad (2.12)$$

In 1987, Samko et al had shown that (2.12) and (2.9) are equivalent.

Replacing n by (-m) in (2.7), it can be shown that

$$ {}_0D_x^{-m} f(x) = \lim_{h\to 0} h^m \sum_{r=0}^{n} \begin{bmatrix} m \\ r \end{bmatrix} f(x-rh)$$

$$= \frac{1}{\Gamma(m)} \int_0^x (x-t)^{(m-1)} f(t)dt \qquad (2.13)$$

$$\text{Where } \begin{bmatrix} m \\ r \end{bmatrix} = \frac{m(m+1)(m+2)........(m+r+1)}{r!} \qquad (2.14)$$

This observation naturally leads to the idea of generalization of the notations of differentiation and integration by allowing m in (2.13) to be an arbitrary real or even complex number.

2.1. Fractional derivatives and integrals:

The idea of fractional derivative or fractional integral can be described in another different ways.

First, we consider a linear non homogeneous nth order ordinary differential equation,

$$D^n y = f(x), \quad b \le x \le c \qquad (2.1.1)$$

Then $\{1, x, x^2, x^3, ..., x^{n-1}\}$ is a fundamental set the corresponding homogeneous equation $D^n y = 0.$ f(x) is any continuous function in [b, c], then for any a $\in$ (b, c),

$$y(x) = \int_a^x \frac{(x-t)^{(n-1)}}{(n-1)!} f(t)dt \qquad (2.1.2)$$

is the unique solution of the equation (2.1.1) with the initial data $y^{(k)}(a) = 0$, for $0 \le k \le n-1$. Or equivalently,

$$y(x) = {}_aD_x^{-n} f(x) = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t)\,dt \qquad (2.1.3)$$

Replacing n by α, where Re(α)>0 in the above formula (2.1.3),we obtain the Riemann-Liouville definition of fractional integral that was reported by Liouville in 1832 and by Riemann in 1876 as

$${}_aD_x^{-\alpha} f(x) = {}_aJ_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt \qquad (2.1.4)$$

Where ${}_aD_x^{-\alpha} f(x) = {}_aJ_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt$ is the Riemann-Liouville integral operator.

When a = 0, (2.1.4) is the Riemann definition of integral and if a = -∞, (2.1.4) represents Liouville definition. Integral of this type were found to arise in theory of linear ordinary differential equations where they are known as Eulier transform of first kind.

If a = 0 and x > 0, then the Laplace transform solution the initial value problem

$$D^n y(x) = f(x), \quad x>0, \ y^{(k)}(0) = 0, \ 0 \le k \le n-1 \qquad (2.1.5)$$

$$\text{is } \bar{y}(s) = s^{-n} \bar{f}(s) \qquad (2.1.6)$$

Where $\bar{y}(s)$ and $\bar{f}(s)$ are respectively the Laplace transform of the function y(x) and f(x).

The inverse Laplace transform gives the solution of the initial value problem (2.1.5) as

$$y(x) = {}_0D_x^{-n} f(x)$$

Again from (2.1.6) we have $y(x) = L^{-1}\{\bar{y}(s)\}$

$$= L^{-1}\{s^{-n}\bar{f}(s)\}$$

Thus we have $${}_0D_x^{-n} f(x) = L^{-1}\{s^{-n}\bar{f}(s)\} \qquad (2.1.7)$$

i.e $$L^{-1}\{s^{-n}\bar{f}(s)\} = {}_0D_x^{-n} f(x) = \frac{1}{\Gamma(n)}\int_0^x (x-t)^{n-1} f(t)dt \qquad (2.1.8)$$

$$\therefore\ y(x) = {}_0D_x^{-n} f(x) = L^{-1}\{s^{-n}\bar{f}(s)\} = \frac{1}{\Gamma(n)}\int_0^x (x-t)^{n-1} f(t)dt$$

This is the Riemann-Liouville integral formula for an integer n. Replacing n by real α gives the Riemann-Liouville fractional integral (2.1.3) with a = 0.

In complex analysis the Cauchy integral formula for the nth derivative of an analytic function f(z) is given by

$$D^n f(z) = \frac{n!}{2\pi i}\int_C \frac{f(t)}{(t-z)^{n+1}}dt \qquad (2.1.9)$$

Where C is closed contour on which f(z) is analytic, and t=z is any point inside C and t = z is a pole.

If n is replaced by an arbitrary number α and n! by $\Gamma(\alpha+1)$, then a derivative of arbitrary order α can be defined by,

$$D^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_C \frac{f(t)}{(t-z)^{\alpha+1}}dt \qquad (2.1.10)$$

where t=z is no longer a pole but a branch point.

In (2.1.10) C is no longer appropriate contour, and it is necessary to make a branch cut along the real axis from the point z=x>0 to negative infinity.

Thus we can define a derivative of arbitrary α order by loop integral

$${}_aD_x^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_a^x (t-z)^{-\alpha-1} f(t) \qquad (2.1.11)$$

Where $(t-z)^{-\alpha-1}$ = exp[-(α+1)ln(t-z)] and ln(t-z) is real when t-z>0. Using the classical method of contour integration along the branch cut contour D, it can be shown that

$${}_0D_z^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_D (t-z)^{-\alpha-1} f(t)dt$$

$$= \frac{\Gamma(\alpha+1)}{2\pi i}[1-\exp\{-2\pi i(\alpha+1)\}]\int_0^z (t-z)^{-\alpha-1} f(t)dt$$

$$= \frac{1}{\Gamma(-\alpha)}\int_0^z (t-z)^{-\alpha-1} f(t)dt \qquad (2.1.12)$$

which agrees with Riemann-Liuville definition (2.1.3) with z=x, and a=0, when α is replaced by -α

2.2. Fractional Integration, Fractional Differential Equation using Laplace Transformed Method:

One of the very useful results is formula for Laplace transform of the derivative of an integer order n of a function f(t) is given by

$$L\{f^{(n)}(t)\} = s^n\bar{f}(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) \qquad (2.2.1)$$

$$= s^n \bar{f}(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0) \tag{2.2.2}$$

$$= s^n \bar{f}(s) - \sum_{k=1}^{n} s^{k-1} f^{(n-k)}(0)$$

Where $f^{(n-k)}(0) = c_k$ represents the physically realistic given initial conditions and $\bar{f}(s)$ being the Laplace transform of the function f(t).

Like Laplace transform of integer order derivative, it is easy to shown that the Laplace transform of fractional order derivative is given by

$$L\left\{ {}_0D_t^{\alpha} f(z) \right\} = s^{\alpha} \bar{f}(s) - \sum_{k=0}^{n-1} s^k [{}_0D_t^{\alpha-k-1} f(t)]_{t=0} \tag{2.2.3}$$

$$= s^{\alpha} \bar{f}(s) - \sum_{k=1}^{n} s^{k-1} c_k , \tag{2.2.4}$$

where $n-1 \le \alpha < n$ and $c_k = \left[{}_0D_t^{\alpha-k} f(t) \right]_{t=0}$ (2.2.5)

represents the initial conditions which do not have obvious physical interpretation. Consequently, formula (2.2.4) has limited applicability for finding solutions of initial value problem in differential equations.

We now replace α by an integer-order integral J^n and $D^n f(z) = f(t) \equiv f^{(n)}(t)$ is used to denote the integral order derivative of a function f(t). It turns out that

$$D^n J^n = I, =\text{I}, \quad J^n D^n \neq I. \tag{2.2.6}$$

This simply means that D^n is the left (not the right inverse) of J^n. It also follows in (2.2.9) with α=n that

$$J^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}, \qquad t > 0 \tag{2.2.7}$$

Similarly, D^{α} can also be defined as the left inverse of J^{α}.We define the fractional derivative of order α > 0 with $n-1 \le \alpha < n$ by

$${}_0D_t^{\alpha} f(t) = D^n D^{-(n-\alpha)} f(t)$$

$$= D^n J^{n-\alpha} f(t) \text{f(t)}$$

$$= D^n \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau \,] \tag{2.2.8}$$

On using (2.1.3)

$$\text{Or, } {}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau$$

Where n is an integer and the identity operator 'I' is defined by

$$D^0 f(t) = J^0 f(t) = I, \quad \text{f(t) = f(t), so that} \quad D^{\alpha} J^{\alpha} = I, \ \alpha \ge 0.$$

Due to the lack of physical interpretation of initial data C_k in (2.2.4), Caputo and Mainardi adopted as an alternative new definition of fractional derivative to solve initial value problems. This new definition was originally introduced by Caputo in the form

$${}_0^cD_t^{\alpha} f(t) = J^{n-\alpha} D^n f(t) \text{f(t)}$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau \tag{2.2.9}$$

Where $n-1 \le \alpha < n$ and n is an integer.

It follows from (2.2.8) and (2.2.9) that

$$ {}_0D_t^{\alpha} f(t) = D^n f(t) \neq J^{n-\alpha} D^n f(t) = {}_0^cD_t^{\alpha} f(t) \tag{2.2.10}$$

Unless f(t) and its first (n-1) derivatives vanish at t=0.

Furthermore, it follows (2.2.9) and (2.2.10) that

$$ J^{\alpha}\, {}_0^cD_t^{\alpha} f(t) = J^{n-\alpha} D^n f(t) = J^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0)\frac{t^k}{k!} \tag{2.2.11}$$

This implies that $ {}_0^cD_t^{\alpha} f(t) = {}_0D_t^{\alpha}\left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} f^{(k)}(0)\right]$

$$ = {}_0D_t^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0) \tag{2.2.12}$$

This shows that Caputo's fractional derivative incorporates the initial values $f^{(k)}(0)$, for k =1, 2,…….,n-1.

The Laplace transform of Caputo's fractional derivative (2.2.12) gives an interesting formula

$$ L\left\{ {}_0^cD_t^{\alpha} f(t)\right\} = s^{\alpha}\bar{f}(s) - \sum_{k=0}^{n-1} f^{(k)}(0) s^{\alpha-k-1} \tag{2.2.13}$$

transform of $f^{(n)}(t)$ This is a natural generalization of the corresponding well known formula for the Laplace when α=n and can be used to solve the initial value problems in fractional differential equation with physically realistic initial conditions.

2.3 Mittag-Leffler function:

The one of the very important function, used in fractional calculus known as Mittag-Leffler function [16], is the generalization of the exponential function e^z. One parameter Mittag-Leffler function is denoted by $E_{\alpha}(z)$ and is defined by the infinite series,

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \tag{2.3.1}$$

The two parameter function of this type, which plays a very important role in solving the fractional differential equations is defined by the infinite series,

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \text{ where } \alpha > 0,\ \beta > 0 \tag{2.3.2}$$

It follows from the definition (2.3.2) that

$$ E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z \tag{2.3.3}$$

$$ E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{1}{z}(e^z - 1) \tag{2.3.4}$$

$$ E_{1,3}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+2)!} = \frac{1}{z^2}\sum_{k=0}^{\infty} \frac{z^{k+2}}{(k+2)!} = \frac{1}{z^2}(e^z - z - 1) \tag{2.3.5}$$

And in general $$ E_{1,m}(z) = \frac{1}{z^{m-1}}\left\{ e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!}\right\} \tag{2.3.6}$$

The hyperbolic sine and cosine are also particular cases of the Mittag-Leffler function (2.3.2) as given by

$$E_{2,1}\left(z^2\right)=\sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma(2k+1)}=\sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!}=\cosh(z) \tag{2.3.7}$$

$$E_{2,2}\left(z^2\right)=\sum_{k=0}^{\infty}\frac{z^{2k}}{\Gamma(2k+2)}=\frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!}=\sinh(z) \tag{2.3.8}$$

Also we can show that

$$E_{\frac{1}{2},1}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\frac{k}{2}+1)}=e^{z^2}erfc(-z) \tag{2.3.9}$$

Where erfc(-z)is the complement of error function defined by

$$\text{erfc(z)}=\frac{2}{\sqrt{\pi}}\int_{z}^{\infty}e^{-t^2}dt\quad . \tag{2.3.10}$$

For β=1, we obtain the Mittag-Leffler function in one parameter**:**

$$E_{\alpha,1}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+1)}\equiv E_{\alpha}(z) \tag{2.3.11}$$

III. BASIC CONCEPT ON CLASSICAL EOQ MODEL

The order quantity means the quantity produced or procured in one production cycle or order cycle (the time period between placement of two successive orders (or production) is referred to as an order cycle (or production cycle). This is also termed re-order quantity when the size of order increases, the order costs (cost of purchasing, inspection, etc.) will decrease whereas the inventory carrying costs will increase. Thus in the production or purchasing case, there are two opposite costs, one encourages the increase in the order size and the other discourages. Economic order quantity (EOQ) is that size of order which minimizes total annual costs of carrying inventory and cost of ordering.

Notations and Assumptions:

D	Demand rate
Q	Order quantity
U	Per unit cost
C_1	Holding cost per unit
C_3	Set up cost
q(t)	Stock level
T	Ordering interval

In classical EOQ based inventory model, we already have

$$\left.\begin{array}{l}\dfrac{dq(t)}{dt}=-D, \quad for \quad 0\le t\le T\\ 0 \qquad\qquad otherwise\end{array}\right\} \tag{3.1}$$

With the initial condition q(0) = Q and with the boundary condition q(T) = 0.

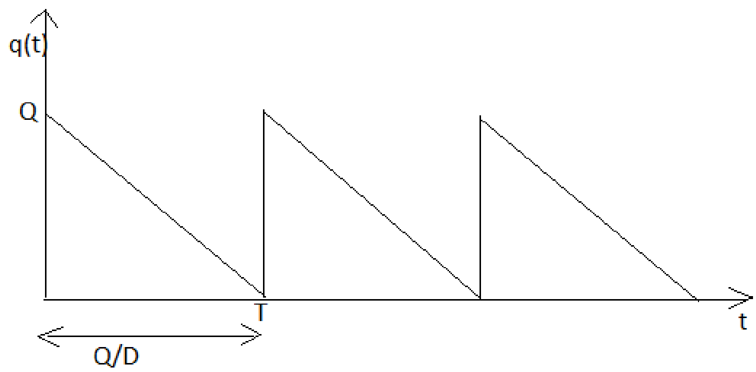

Figure-1.1: Development of inventory level over time.

By solving the equation (3.1), we have q(t) = Q - Dt, for $0 \le t \le T$ (3.2)

And on using the boundary condition q(T) = 0, we have Q = DT. (3.3)

Holding cost, HC(T)= $C_1\int_{t=0}^{T} q(t)dt = C_1\int_{t=0}^{T}(Q-Dt)dt = C_1[Qt-\frac{Dt^2}{2}]_{t=0}^{T} = C_1(QT-\frac{DT^2}{2}) = \frac{C_1DT^2}{2}$ (3.4)
[On using (3.3)]

Total cost, TC(T) = Purchasing cost(PC) + Holding cost(HC) + Set up cost(SC)

$$= UQ + \frac{C_1DT^2}{2} + C_3. \tag{3.5}$$

Total average cost over [0, T] is given by

$$TAC(T) = \frac{1}{T}[UQ + \frac{C_1DT^2}{2} + C_3]$$

$$= \frac{UQ}{T} + \frac{C_1DT}{2} + \frac{C_3}{T} \tag{3.6}$$

Then the classical EOQ model is

$$\text{Min } TAC(T) = UD + \frac{C_1DT}{2} + \frac{C_3}{T} \tag{3.7}$$

Subject to, T > 0.

Solving (3.7) we can show that TAC(T) will be minimum for

$$T^* = \sqrt{\frac{2C_3D}{C_1}} \tag{3.8}$$

and

$$TAC^*(T^*) = UD + \sqrt{2C_1C_3D}\ . \tag{3.9}$$

IV. GENERALIZED EOQ MODEL WITH STOCK DEPENDENT DEMAND

We now generalize our discussion by accepting the equation (3.1) as a differential equation of fractional order instead of the linear order. i.e we here consider that demand(D) varies in fractional order say α, here instantaneous inventory level

$$\left.\begin{array}{ll} \frac{d^\alpha q(t)}{dt^\alpha} = -D, & for \quad 0 \le t \le T \\ 0 & otherwise \end{array}\right\} \tag{4.1}$$

where D =aq(t) + b; a, b are constants.

Then we have the equation (4.1) as

$$\frac{d^\alpha q}{dt^\alpha} = -(aq+b) \tag{4.2}$$

Taking Laplace transformation of the above equation (4.2) we have,

$$s^\alpha \bar{q}(s) - s^{\alpha-0-1}q(0) = -a\bar{q}(s) - \frac{b}{s}$$

$\bar{q}(s)$ being Laplace transformation of q(t)

i, e $\bar{q}(s) = L\{q(t)\}$

$$\therefore (s^\alpha + a)\bar{q}(s) = Qs^{\alpha-1} - \frac{b}{s}$$

$$or, \bar{q}(s) = Q\frac{s^{\alpha-1}}{s^{\alpha}+a} - \frac{bs^{-1}}{s^{\alpha}+a}$$

$$\therefore q(t) = L^{-1}\{\bar{q}(s)\} = QE_{\alpha,1}(-at^{\alpha}) - bt^{\alpha}E_{\alpha,\alpha+1}(-at^{\alpha}) \quad (4.3)$$

[Where $E_{\alpha,\beta}(z)$ the Mittag-Leffler function is is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\alpha}\frac{z^k}{\Gamma(\alpha k+\beta)} \quad \text{and}$$

$$L\{t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\pm at^{\alpha})\} = \frac{k!s^{\alpha-\beta}}{(s^{\alpha}\mp a)} \quad \text{where } \left[\operatorname{Re}(s)\rangle|a|^{\frac{1}{\alpha}}\right]$$

When t = T, q(t) = 0

Then $QE_{\alpha,1}(-aT^{\alpha}) = bt^{\alpha}E_{\alpha,\alpha+1}(-aT^{\alpha})$ (4.4)

Again when α = 1

$$q(t) = QE_{1,1}(-at) - btE_{1,2}(-at)$$

$$= Qe^{-at} - bt\frac{1}{(-at)}(e^{-at}-1)$$

$$= Qe^{-at} + \frac{b}{a}(e^{-at}-1)$$

$$= \left(Q+\frac{b}{a}\right)e^{-at} - \frac{b}{a} \quad (4.5)$$

And $Qe^{-aT} = bT\left(-\frac{1}{aT}\right)(e^{-aT}-1)$ (4.6)

$$\therefore Q = \frac{b}{a}(e^{aT}-1)$$

Case -(i): for α = 1, β = 1

Holding cost H.C = $C_1\int_0^T q(t)dt$

Where $q(t) = \left(Q+\frac{b}{a}\right)e^{-aT} - \frac{b}{a}$

Then H.C $= C_1\int_0^T\left[\left(Q+\frac{b}{a}\right)e^{-at} - \frac{b}{a}\right]dt$

$$= C_1\left[\left(Q+\frac{b}{a}\right)\left(\frac{1-e^{-aT}}{a}\right) - \frac{b}{a}T\right]$$

$$= \frac{C_1}{a}\left[\frac{b}{a}e^{aT}\left(1-e^{-aT}\right) - bT\right]$$

$$= \frac{C_1}{a}\left[\frac{b}{a}\left(e^{aT}-1\right) - bT\right]$$

$$= \frac{C_1 b}{a^2}\left[e^{aT}-1-aT\right] \quad (4.7)$$

Case -(ii): for 0 < α ≤ 1 and β = 1

Holding cost $H.C(T) = C_1\int_0^T q(t)dt$

Where q(t) is given by (2)

$$= C_1 \int_0^T \left[Q E_{\alpha,1}(-at^\alpha) - bt^\alpha E_{\alpha,\alpha+1}(-at^\alpha) \right] dt$$

$$= C_1 \left[QTE_{\alpha,2}(-aT^\alpha) - bT^{\alpha+1} E_{\alpha,\alpha+2}(-aT^\alpha) \right]$$

$$= C_1 \left[\frac{bT^{\alpha+1} E_{\alpha,2}(-aT^\alpha) E_{\alpha,\alpha+1}(-aT^\alpha)}{E_{\alpha,1}(-aT^\alpha)} - bT^{\alpha+1} E_{\alpha,\alpha+2}(-aT^\alpha) \right] \tag{4.8}$$

For α = 1

$$\text{H.C(T)} = C_1 \left[QTE_{1,2}(-aT) - bT^2 E_{1,3}(-aT) \right]$$

$$= C_1 \left[\frac{b}{a}\left(e^{aT} - 1\right) T \left\{ \frac{e^{-aT} - 1}{(-aT)} \right\} - bT^2 \frac{\left(e^{-aT} - 1 + aT\right)}{\left(-aT\right)^2} \right]$$

$$= C_1 \left[\frac{b}{a^2}\left(e^{aT} - 1\right)\left(1 - e^{-aT}\right) - \frac{b}{a^2}\left(e^{-aT} - 1 + aT\right) \right]$$

$$= C_1 \frac{b}{a^2}\left[e^{aT} - 1 - aT \right]$$

Which is true as (4.7)

Case-(iii): For α =1 and for any β

Holding cost (H.C) $= C_1 D^{-\beta} q(t)$

Where $$q(t) = \left(Q + \frac{b}{a} \right) e^{-aT} - \frac{b}{a}$$

Now $$L\left\{ D^{-\beta} q(t) \right\} = L\left\{ D^{-\beta} \left(Q + \frac{b}{a} \right) e^{-aT} - \frac{b}{a} \right\}$$

$$= \left(Q + \frac{b}{a} \right) \frac{s^{-\beta}}{s+a} - \frac{b}{a} \frac{1}{s^{\beta+1}}$$

$$\therefore D^{-\beta} q(t) = L^{-1} \left\{ \left(Q + \frac{b}{a} \right) \frac{s^{-\beta}}{s+a} - \frac{b}{a} \frac{1}{s^{\beta+1}} \right\}$$

$$= \left(Q + \frac{b}{a} \right) t^\beta E_{1,\beta+1}(-aT) - \frac{b}{a} \frac{t^\beta}{\Gamma(\beta+1)}$$

Then for t =T

Holding cost $$= \left(Q + \frac{b}{a} \right) T^\beta E_{1,\beta+1}(-aT) - \frac{b}{a} \frac{T^\beta}{\Gamma(\beta+1)}$$

$$= C_1 \left\{ \frac{b}{a} e^{aT} T^\beta E_{1,\beta+1}(-aT) - \frac{b}{a} \frac{T^\beta}{\Gamma(\beta+1)} \right\}$$

$$= C_1 \frac{b}{a} T^\beta \left[e^{aT} E_{1,\beta+1}(-aT) - \frac{1}{\Gamma(\beta+1)} \right] \tag{4.9}$$

Case-(iv): Now for any 0 < α ≤ 1 and β,

Holding cost (H.C) $= C_1 D^{-\beta} q(t)$

Where $$q(t) = QE_{\alpha,1}(-at^\alpha) - bt^\alpha E_{\alpha,\alpha+1}(-at^\alpha) \tag{4.10}$$

Now $$L\{D^{-\beta}q(t)\} = L\{D^{-\beta}\{QE_{\alpha,1}(-at^{\alpha}) - bt^{\alpha}E_{\alpha,\alpha+1}(-at^{\alpha})\}\}$$
$$= Qs^{-\beta}\frac{s^{\alpha-1}}{s^{\alpha}+a} - bs^{-\beta}\frac{s^{-1}}{s^{\alpha}+a}$$
$$= Q\frac{s^{\alpha-\beta-1}}{s^{\alpha}+a} - bs^{-\beta}\frac{s^{-\beta-1}}{s^{\alpha}+a}$$

$$\therefore\ D^{-\beta}q(t) = L^{-1}\{Q\frac{s^{\alpha-\beta-1}}{s^{\alpha}+a} - bs^{-\beta}\frac{s^{-\beta-1}}{s^{\alpha}+a}\}$$
$$= Qt^{\beta}E_{\alpha,\beta+1}(-at^{\alpha}) - bt^{\alpha+\beta}E_{\alpha,\alpha+\beta+1}(-at^{\alpha})$$

Then for t=T Holding cost = $C_1[QT^{\beta}E_{\alpha,\beta+1}(-aT^{\alpha}) - bT^{\alpha+\beta}E_{\alpha,\alpha+\beta+1}(-aT^{\alpha})]$
$$= C_1[\frac{bT^{\alpha}E_{\alpha,\alpha+1}(-aT^{\alpha})}{E_{\alpha,1}(-aT^{\alpha})}T^{\beta}E_{\alpha,\beta+1}(-aT^{\alpha}) - bT^{\alpha+\beta}E_{\alpha,\alpha+\beta+1}(-aT^{\alpha})]$$
$$= C_1bT^{\alpha+\beta}[\frac{E_{\alpha,\alpha+1}(-aT^{\alpha})}{E_{\alpha,1}(-aT^{\alpha})}E_{\alpha,\beta+1}(-aT^{\alpha}) - E_{\alpha,\alpha+\beta+1}(-aT^{\alpha})] \quad (4.11)$$

Then for β=1 and any α , we have from (4.11)

HC(T)= $C_1bT^{\alpha+1}[\frac{E_{\alpha,\alpha+1}(-aT^{\alpha})}{E_{\alpha,1}(-aT^{\alpha})}E_{\alpha,2}(-aT^{\alpha}) - E_{\alpha,\alpha+2}(-aT^{\alpha})]$, which is same as (4.8)

Again for α=1 and any we have β from (4.11)

HC(T) = $$C_1bT^{\beta+1}\left[\frac{E_{1,2}(-aT)}{E_{1,1}(-aT)}E_{1,\beta+1}(-aT) - E_{1,\beta+2}(-aT)\right] \quad (4.12)$$

Now using $E_{1,2}(-aT) = \frac{e^{-aT}-1}{-aT}$ and $E_{1,1}(-aT) = e^{-aT}$ and
$$E_{1,\beta+2}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+\beta+2)} = \frac{1}{z}\sum_{k=0}^{\infty}\frac{z^{k+1}}{\Gamma(k+1+\beta+1)}$$
$$= \frac{1}{z}\sum_{p=1}^{\infty}\frac{z^p}{\Gamma(p+\beta+1)} = \frac{1}{z}[\sum_{p=0}^{\infty}\frac{z^p}{\Gamma(p+\beta+1)} - \frac{1}{\Gamma(\beta+1)}]$$
$$= \frac{1}{z}[E_{1,\beta+1}(z) - \frac{1}{\Gamma(\beta+1)}]$$

We have from (4.11)

HC(T) = $C_1\frac{b}{a}T^{\beta}[e^{aT}E_{1,\beta+1}(-aT) - \frac{1}{\Gamma(\beta+1)}]$ which is same as (4.9)

4.2. Generalized Total Average Cost:

Total cost(TC)=Purchasing cost(PC)+Holding cost(HC)+Set up cost(SC).

Total Average Cost (TAC) = $\frac{1}{T}$ [Total Cost(TC)]

For α=1 and β=1, Average Cost $TAC^*_{1,1}(T^*) = \frac{1}{T}[UQ + HC_{1,1}(T) + C_3]$
$$= \frac{1}{T}[U\frac{b}{a}(e^{aT}-1) + \frac{bC_1}{a^2}(e^{aT}-1-aT) + C_3]$$

$$=\frac{e^{aT}}{T}(\frac{Ub}{a}+\frac{bC_1}{a^2})+(C_3-\frac{Ub}{a}-\frac{bC_1}{a^2})-\frac{bC_1}{a}$$

$$=M_1+F_1\frac{e^{aT}}{T}+\frac{G_1}{T} \tag{4.2.1}$$

Where $M_1=-\frac{bC_1}{a}$, $F_1=\frac{Ub}{a}+\frac{bC_1}{a^2}$ & $G_1=C_3-\frac{Ub}{a}-\frac{bC_1}{a^2}$

Here the EOQ model is,

$$\text{Min } TAC_{1,1}(T)=M_1+F_1\frac{e^{aT}}{T}+\frac{G_1}{T} \tag{4.2.2}$$

subject to T ≥0,

Again for any α and β=1,

$$HC_{\alpha,1}(T)=C_1bT^{\alpha+1}[\frac{E_{\alpha,2}(-aT^{\alpha})E_{\alpha,\alpha+1}(-aT^{\alpha})}{E_{\alpha,1}(-aT^{\alpha})}-E_{\alpha,\alpha+2}(-aT^{\alpha})]$$

$$\therefore TC_{\alpha,1}(T)=\text{UQ}+HC_{\alpha,1}(T)+C_3,$$

where $\text{Q}=\frac{bT^{\alpha}E_{\alpha,\alpha+1}(-aT^{\alpha})}{E_{\alpha+1}(-aT^{\alpha})}$

$$\therefore TC_{\alpha,1}(T)=\text{U}\frac{bT^{\alpha}E_{\alpha,\alpha+1}(-aT^{\alpha})}{E_{\alpha+1}(-aT^{\alpha})}+$$

$$C_1bT^{\alpha+1}[\frac{E_{\alpha,2}(-aT^{\alpha})E_{\alpha,\alpha+1}(-aT^{\alpha})}{E_{\alpha,1}(-aT^{\alpha})}-E_{\alpha,\alpha+2}(-aT^{\alpha})]+C_3$$

Then the model in this case will be

$$\text{Min } TAC_{\alpha,1}(T)=U\frac{bE_{\alpha,\alpha+1}(-aT^{\alpha})}{E_{\alpha+1}(-aT^{\alpha})}T^{\alpha-1}+C_1b[\frac{E_{\alpha,2}(-aT^{\alpha})E_{\alpha,\alpha+1}(-aT^{\alpha})}{E_{\alpha,1}(-aT^{\alpha})}-E_{\alpha,\alpha+2}(-aT^{\alpha})]T^{\alpha}+\frac{C_3}{T},$$

Subject to T≥0 (4.2.3)

Again for α=1 and any β, $HC_{1,\beta}(T)=C_1\frac{b}{a}\left[e^{aT}E_{1,\beta+1}(-aT)-\frac{1}{\Gamma(\beta+1)}\right]T^{\beta}$

$$\therefore TC_{1,\beta}(T)=\text{UQ}+HC_{1,\beta}(T)+C_3,\text{ where } Q=\frac{b}{a}(e^{aT}-1)$$

$$\therefore TC_{1,\beta}(T)=U\frac{b}{a}(e^{aT}-1)+C_1\frac{b}{a}T^{\beta}\left[e^{aT}E_{1,\beta+1}(-aT)-\frac{1}{\Gamma(\beta+1)}\right]+C_3$$

$$\therefore TAC_{1,\beta}(T)=\left(C_3-\frac{Ub}{a}\right)\frac{1}{T}+\frac{Ub}{a}\frac{e^{aT}}{T}+C_1\frac{b}{a}\left[e^{aT}E_{1,\beta+1}(-aT)-\frac{1}{\Gamma(\beta+1)}\right]T^{\beta-1}$$

Then the model in this case will be

$$\text{Min } TAC_{1,\beta}(T)=\left(C_3-\frac{Ub}{a}\right)\frac{1}{T}+\frac{Ub}{a}\frac{e^{aT}}{T}+C_1\frac{b}{a}\left[e^{aT}E_{1,\beta+1}(-aT)-\frac{1}{\Gamma(\beta+1)}\right]T^{\beta-1}$$

Subject to T ≥ 0 (4.2.4)

Now for any α and β, $HC_{\alpha,\beta}(T)=C_1bT^{\alpha+\beta}\left[\frac{E_{\alpha,\alpha+1}(-aT^{\alpha})E_{\alpha,\beta+1}(-aT^{\alpha})}{E_{\alpha+1}(-aT^{\alpha})}-E_{\alpha,\alpha+\beta+1}(-aT^{\alpha})\right]$

$$\therefore TC_{\alpha,\beta}(T)=\text{UQ}+HC_{\alpha,\beta}(T)+C_3,\quad \text{where Q}=\frac{bT^{\alpha}E_{\alpha,\alpha+1}(-aT^{\alpha})}{E_{\alpha+1}(-aT^{\alpha})}$$

$$\therefore TAC_{\alpha,\beta}(T) = \frac{UbE_{\alpha,\alpha+1}(-aT^{\alpha})}{E_{\alpha+1}(-aT^{\alpha})}T^{\alpha-1} + C_1 b[\frac{E_{\alpha,\alpha+1}(-aT^{\alpha})E_{\alpha,\beta+1}(-aT^{\alpha})}{E_{\alpha+1}(-aT^{\alpha})} - E_{\alpha,\alpha+\beta+1}(-aT^{\alpha})]T^{\alpha+\beta-1} + \frac{C_3}{T}$$

Then here the model will be

$$\text{Min } TAC_{\alpha,\beta}(T) = \frac{UbE_{\alpha,\alpha+1}(-aT^{\alpha})}{E_{\alpha+1}(-aT^{\alpha})}T^{\alpha-1} + C_1 b[\frac{E_{\alpha,\alpha+1}(-aT^{\alpha})E_{\alpha,\beta+1}(-aT^{\alpha})}{E_{\alpha+1}(-aT^{\alpha})} - E_{\alpha,\alpha+\beta+1}(-aT^{\alpha})]T^{\alpha+\beta-1} + \frac{C_3}{T}$$

Subject to $T \geq 0$ (4.2.5)

The models given in (4.2.2), (4.2.3), (4.2.4), (4.2.5) are highly complicated form cannot be optimized analytically by any ordinary optimization method. But it can be optimized numerically by taking some particular values of α and β in fractional form say $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$.

V. CONCLUSION

Many classical EOQ models developed so far in field of inventory management are based on the basic assumption that the demand is constant or time dependent (linearly) or may be stock dependent. Presently fractional order EOQ model has also been developed by taking demand is constant or time dependent. In the present article we have introduced a much more generalized EOQ model with fractional order where demand is dependent upon the stock only (linearly). Minimization of the average system cost is adopted as the criterion of optimization. This optimization requires us to solve highly non linear algebraic expression. These expressions being not amenable to any known analytical treatment. These can be optimized by using numerical method with taking proper numerical values of the constants. In future, the analytical method of optimization of these complicated expressions may be considerable.

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